20MAP19

PSG COLLEGE OF ARTS & SCIENCE
(AUTONOMOUS)

MSc DEGREE EXAMINATION MAY 2022
(Fourth Semester)

Branch – **MATHEMATICS**

CONTROL THEORY

Time: Three Hours Maximum: 75 Marks

SECTION-A (10 Marks)
Answer **ALL** questions
ALL questions carry **EQUAL** marks (10 x 1 = 10)

1. Let X be a banach space and T be an operator such that $T: X \to X$.
   $Tx = x^2 \to$ Fixed point is ______
   (i) 0 (ii) 1 (iii) 2 (iv) 3

2. The norm of a continuous $n \times m$ continuous matrix valued function $D: J \to R^n \times R^m$ is defined by
   (i) $\|D(t)\| = \max_i \sum_{j=1}^{m} \max |d_{ij}(t)|$ (ii) $\|D(t)\| = \max_i \sum_{j=1}^{m} \min |d_{ij}(t)|$
   (iii) $\|D(t)\| = \min_i \sum_{j=1}^{m} \max |d_{ij}(t)|$ (iv) $\|D(t)\| = \min_i \sum_{j=1}^{m} \min |d_{ij}(t)|$

3. Choose the correct option.
   (i) $\det[X(t)] = \exp[\int_0^t Tr[A(s)]ds]$ (ii) $\det[X(t)] = \exp[\int_0^T Tr[A(s)]ds]$
   (iii) $\det[X(t)] = \exp[\int_0^T [A(s)]ds]$ (iv) $\det[X(t)] = \exp[\int_0^t [A(s)]ds]$

4. An $m \times n$ matrix function $S(t)$ with entries in $\mathcal{L}_m^2[0,T]$ is a steering function for $\dot{x} = A(t)x + B(t)u$ on $[0,T]$ if
   (i) $\int_0^T X(T,t)B(t)S(t)dt = 0$ (ii) $\int_0^T X(T,t)B(t)S(t)dt = I$
   (iii) $\int_0^t X(T,t)B(t)S(t)dt = I$ (iv) $\int_0^t X(T,t)B(t)S(t)dt = 0$

5. If A is a constant matix in $\dot{x} = A(t)x(t)$, then it is called an autonomous system and in that case stability is equivalent to
   (i) stability (ii) uniformasymptotic stability
   (iii) uniform stability (iv) asymptotic stability

6. The solution $\phi(t)$ is called ______ if it is stable and there exists a constant $\sigma > 0$ such that $x(t) - \phi(t) \to 0$ as $t \to \infty$ whenever $\|x(0) - \phi(0)\| \leq \sigma$.
   (i) unstable (ii) asymptotically stable
   (iii) uniformly stable (iv) stable

7. The linear time invariant control system $\dot{x} = Ax + Bu$ is _____ if there exists an $m \times n$ matrix K such that $A + BK$ is a stability matrix.
   (i) closed loop system (ii) open loop system
   (iii) controllable (iv) stabilizable

8. A system $\dot{x} = Ax + 0u$ with A stability matrix but clearly not
   (i) observable (ii) stability
   (iii) controllable (iv) stabilizable

Cont...

9. The optimal control $u(t)$ is given by
   (i) $u(t) = -R^{-1}(t)B^*(t)K(t)x(t)$ (ii) $u(t) = R^{-1}(t)B^*(t)K(t)\dot{x}(t)$
   (iii) $u(t) = -R^{-1}(t)B(t)K(t)x(t)$ (iv) $u(t) = R^{-1}(t)B(t)K(t)\dot{x}(t)$

10. The negative real parts of the eigen values of G implies that the system $\dot{x}(t) = Gx(t)$ is _____
    (i) stable (ii) unstable
    (iii) controllable (iv) stabilizable

SECTION - B (35 Marks)

Answer **ALL** questions

ALL questions carry **EQUAL** Marks (5 x 7 = 35)

11 a Solve the initial value problem $\dot{x} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

OR

b Prove that there exists a reconstruction kernel $R(t)$ on $[0,T]$ if and only if $\dot{x} = A(t)x$, $y(t) = H(t)x(t)$ is observable on $[0,T]$.

12 a Prove that the system $\dot{x} = A(t)x + B(t)u$ is controllable on $[0,T]$ if and only if for each vector $x_1 \in R^n$ there is a control $u \in L_m^2[0,T]$ which steers 0 to x_1 during $[0,T]$.

OR

b Verify the controllability of the system $\frac{d^4x}{dt^4} + \frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$ and $y = \dot{x}$.

13 a State and prove Gronwall's Inequality.

OR

b Verify the stability of the equation of the motion of simple pendulum with damping given by $\ddot{\theta} + \frac{k}{m}\dot{\theta} + \frac{g}{l}\sin\theta = 0$.

14 a **i)** Define controllable subspace **ii)** Prove that $C(A,B)$ is an invariant subspace of the matrix A.

OR

b Prove that (A,B) is stabilizable, and (H,A) is detectable, the matrices K, L of dimensions $m \times n$ and $n \times r$ respectively such that the matrix $\begin{pmatrix} A & Bk \\ LH & A + Bk - LH \end{pmatrix}$ is a stability matrix.

15 a Prove that if $K(t)$ is the solution of the Ricatti equation and $K(T) = F$, then $K(t)$ is symmetric for all $t \in [0,T]$ ie) $K(t) = K^*(t)$.

OR

b Find the optimal control u for the system $\dot{x} = x + u$, which minimize the cost functional, $J = \frac{1}{2}\int_0^1 (x^2 + u^2)\, dt$.

SECTION -C (30 Marks)

Answerany **THREE** questions

ALL questions carry **EQUAL** Marks (3 x 10 = 30)

16 Prove that the constant coefficient system $\dot{x} = Ax$ and $y = Hx$ is observable on an arbitrary interval $[0, T]$ iff for some k, $0 < k \le n$ the rank of the observability matrix is $rank \begin{pmatrix} H \\ HA \\ \cdot \\ \cdot \\ HA^{k-1} \end{pmatrix} = n.$

17 Prove that the system $\dot{x} = A(t)x + B(t)u$ is controllable on $[0, T]$ if and only if thecontrollability grammian$M(0,T) = \int_0^T X(T,t)B(t)B^*(t)X^*(T,t)dt$ is positive definite.

18 Let all the solutions of $\dot{x} = A(t)x$be bounded. Let **i)** $\int_0^\infty \|B(s)\| ds < \infty$ **ii)** $\lim_{t\to\infty} \int_0^t Tr[A(s)]ds > -\infty$. Then prove that all the solutions of $\dot{x} = A(t)x + B(t)x$ are bounded.

19 Consider the invertible pendulum, whose linearized equation of motion take the form $\ddot{x} + x = u$ Show that the given system is stabilizableby using Bass's Method.

20 Find the optimal control for the first order system, $\dot{x}(t) = x(t) + u$, $x(0) = x_0$ and the cost functional $J = \int_0^1 \left[3x^2(t) + u^2(t)\right]dt$

Z-Z-Z END