PSG COLLEGE OF ARTS & SCIENCE
(AUTONOMOUS)

MSc DEGREE EXAMINATION MAY 2024
(Third Semester)

Branch – **MATHEMATICS**

FUNCTIONAL ANALYSIS

Time: Three Hours | Maximum: 50 Marks

SECTION-A (5 Marks)
Answer **ALL** questions
ALL questions carry **EQUAL** marks (5 x 1 = 5)

1 Which of the following is the definition for zero operator?
(i) $0x = x$ (ii) $0x = 0$
(iii) $Ix = x$ (iv) $Ix = 0$

2 $\|x+y\|^2 + \|x-y\|^2 = \ldots\ldots\ldots$
(i) $\|x\|^2 + \|y^2\|$ (ii) $2\|x\|^2$
(iii) $2\|y^2\|$ (iv) $2(\|x\|^2 + \|y^2\|)$

3 If $\rho(x+y) \le \rho(x) + \rho(y)$, then ρ is called ......
(i) reflexive (ii) chain
(iii) subadditive (iv) positive homogeneous

4 X is said to be ...... in a normed space Z if X is isomorphic with a subspace Z.
(i) embeddable (ii) complete
(iii) adjoint (iv) reflexive

5 Which of the following is the condition for the contraction ?
(i) $d(Tx,Ty) \le d(x,y)$ (ii) $d(Tx,Ty) \ge \alpha d(x,y)$
(iii) $d(Tx,Ty) \le \alpha d(x,y)$ (iv) $d(Tx,Ty) \le \alpha d(Tx,y)$

SECTION - B (15 Marks)
Answer **ALL** Questions
ALL Questions Carry **EQUAL** Marks (5 x 3 = 15)

6 a Prove that on a finite dimensional vector space X, any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$.

OR

b Let T be a bounded linear operator. Then prove that the null space $N(T)$ is closed.

7 a If in an inner product space, $x_n \to x$ and $y_n \to y$, then prove that $\langle x_n, x\rangle \to \langle x, y\rangle$.

OR

b Prove that an orthonormal set is linearly independent.

8 a If $\langle v_1, w\rangle = \langle v_2, w\rangle$ for all w in an inner product space X, then prove that $v_1 = v_2$. Also prove that $\langle v_1, w\rangle = 0$ for all $w \in X$ implies $v_1 = 0$.

OR

b Prove that in every Hilbert space $H \ne \{0\}$, there exists a total orthonormal set.

9 a Prove that every Hilbert space H is reflexive.

OR

b Let $T : D(T) \to Y$ be a bounded linear operator with domain $D(T) \subset X$, where X and Y are normed spaces. Then prove that if T is closed and Y is complete, then $D(T)$ is a closed subset of X.

Cont...

10 a Let X and Y be normed spaces and $T: X \to Y$ a linear operator. Then prove that T is compact if and only if it maps every bounded sequence (x_n) in X onto a sequence $T(x_n)$ in Y which has a convergent subsequence.

OR

b Let $T: X \to X$ be a compact linear operator and $S: X \to Y$ a bounded linear operator on a normed space X. Then prove that TS and ST are compact.

SECTION -C (30 Marks)

Answer **ALL** questions
ALL questions carry **EQUAL** Marks (5 x 6 = 30)

11 a Prove that every finite dimensional subspace Y of a normed space X is complete. Also Prove that every finite dimensional normed space is complete.

OR

b Let $T: D(T) \to Y$ be a bounded linear operator, where $D(T)$ lies in a normed space X and Y is Banach space. Then prove that T has an extension $\tilde{T}: \overline{D(T)} \to Y$ where $\tilde{T}$ is a bounded linear operator of norm $\|\tilde{T}\| = \|T\|$.

12 a State and Prove
i)Schwarz inequality
ii) Triangle inequality

OR

b i) If Y is a closed subspace of a Hilbert space H, then prove that $Y = Y^{\perp\perp}$
ii)Prove that for any subset $M \neq \phi$ of a Hilbert space H, the span of M is dense in H if and only if $M^{\perp} = \{0\}$.

13 a State and prove Riesz's theorem on functionals on Hilbert space.

OR

b Let $T: H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then prove that the Hilbert-adjoint operator T^* of T exists, is unique and is a bounded linear operator with norm $\|T^*\| = \|T\|$.

14 a Let (x_n) be a weakly convergent sequence in a normed space X say, $x_n \xrightarrow{w} x$.
Then prove that
i) The weak limit x of (x_n) is unique.
ii) every subsequence of (x_n) converges weakly to x.
iii) The sequence $(\|x_n\|)$ is bounded.

OR

b Let $T: X \to Y$ be a bounded linear operator, where X and Y are normed spaces. Then prove that the adjoint operator $T^{\times}$ is linear and bounded and $\|T^{\times}\| = \|T\|$.

15 a Let $T: X \to Y$ be linear operator. Prove that if T is compact, then so is its adjoint operator $T^{\times}: Y' \to X'$; here X and Y are normed spaces and X' and Y' the dual spaces of X and Y.

OR

b Let X and Y be normed spaces and $T: X \to Y$ a compact linear operator. Suppose that (x_n) in X is weakly convergent, say, $x_n \xrightarrow{w} x$ .Then prove that $T(x_n)$ is strongly convergent in Y and has the limit $y = Tx$.

END

Z-Z-Z