22MAP420

PSG COLLEGE OF ARTS & SCIENCE
(AUTONOMOUS)

MSc DEGREE EXAMINATION MAY 2024
(Fourth Semester)

Branch – **MATHEMATICS**

CONTROL THEORY

Time: Three Hours Maximum: 50 Marks

SECTION-A (5 Marks)
Answer **ALL** questions
ALL questions carry **EQUAL** marks (5 x 1 = 5)

1 The system $\dot{x} = A(t)x, y(t) = H(t)x(t)$ is observable on an interval [0,T] if $y(t) = H(t)x(t) = 0,\ t \in [0,T]$ implies ________ .
(i) $x(0) = 0$ (ii) $x_0 = 0$
(iii) $x(t) = 0$ (iv) all of the above

2 If rank B is ________ , then the system $\dot{x} = Ax + Bu$ is controllable.
(i) equal to n (ii) less than n
(iii) greater than n (iv) not equal to n

3 If $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ then the system $\dot{x} = Ax$ is ________ .
(i) stable (ii) unstable
(iii) asymptotically stable (iv) bounded

4 The pair (A, B) is controllable if and only if ________ is controllable for arbitrary scalar λ.
(i) $(A + \lambda I, B)$ (ii) $(B + \lambda I, B)$
(iii) $(A - \lambda I, B)$ (iv) $(B - \lambda I, B)$

5 Given the linear system $\dot{x} = A(t)x(t) + B(t)u(t)$ and the cost functional $J = \frac{1}{2}x^*(T)Fx(T) + \frac{1}{2}\int_0^T [x^*(t)Q(t)x(t) + u^*(t)R(t)u(t)]dt$, there exists an optimal control of the form $u(t) = -R^{-1}(t)B^*(t)K(t)x(t)$ where $K(t)$ is the solution of the matrix Riccati equation with ________ .
(i) $K(T) = B$ (ii) $K(T) = x$
(iii) $K(T) = u$ (iv) $K(T) = F$

SECTION - B (15 Marks)
Answer **ALL** Questions
ALL Questions Carry **EQUAL** Marks (5 x 3 = 15)

6 a State and prove Banach fixed point theorem.

OR

b Consider the system $\dot{x} = A(t)x + f(t,x)$ with observation $y = H(t)x$ and if there exists a constant $c > 0$ such that $\det W(t,\theta) \geq c$ then prove that the system is globally observable at t .

7 a Show that the system $\dot{x} = A(t)x + B(t)u$ is controllable on $[0, T]$ if and only if $M(0,T) = \int_0^T X(T,t)B(t)B^*(t)X^*(T,t)dt$ is positive definite.

OR

b Verify the controllability of the system
$\dot{x}_1 = -x_1 + x_2 + (\cos t)u_1 + (\sin t)u_2 + 10x_1/(1 + x_1{}^2 + x_2{}^2 + u_1{}^2)$,
$\dot{x}_2 = -x_1 - x_2 - (\sin t)u_1 + (\cos t)u_2 + x_2/(1 + x_2{}^2 + u_2{}^2 + t)$.

8 a Prove that the system $\dot{x} = Ax$ is (asymptotically) stable if all the eigenvalues of A have negative real parts.

OR

b State and prove the Gronwall's inequality.

Cont...

9 a If the system $\dot{x} = Ax + Bu,\ x \in R^n,\ u \in R^m$ is countable, the prove that it is stabilizable.

OR

b Show that the pair (H, A) is detectable if and only if the pair $(A^*, -H^*)$ is stabilizable.

10 a If $K(t)$ is the solution of the Riccati equation $\dot{K} + K(t)A(t) + A^*(t)K(t) - K(t)S(t)K(t) + Q(t) = 0$ and if $K(t) = F$, then prove that $K(t)$ is symmetric for all $t \in [0, T]$.

OR

b Find the optimal control u for the system $\dot{x}_1(t) = x_2(t), \dot{x}_2(t) = u(t)$ and the cost functional $J = \frac{1}{2}\int_0^\infty [x_1^2(t) + 2bx_1(t)x_2(t) + ax_2^2(t) + u^2(t)]dt$ where we assume that $a - b^2 > 0$.

<u>SECTION -C (30 Marks)</u>

Answer **ALL** questions

ALL questions carry **EQUAL** Marks (5 x 6 = 30)

11 a Solve the initial value problem $\dot{x} = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} x,\ x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

OR

b Verify the observability of the second order equation $t^2\ddot{x} + t\dot{x} - x = 0$ for the observer $y = \dot{x}$.

12 a Find the control u which steers the system $\ddot{x} - x = u$ from $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

OR

b Assume that the continuous function f satisfies the condition $\lim_{|(x,u)| \to \infty} \frac{|f(t,x,u)|}{|(x,u)|} = 0$ uniformly for $t \in I$. If the system $\dot{x} = A(t)x + B(t)u$ is completely controllable, then prove that the system $\dot{x} = A(t)x(t) + B(t)u(t) + f(t, x(t), u(t))$ is completely controllable.

13 a Let all the solutions of the equation $\dot{x}(t) = A(t)x(t)$ be bounded. Let

(i) $\int_0^\infty \|B(s)\|\, ds < \infty$,

(ii) $\lim_{t\to\infty} \int_0^t Tr[A(s)]\, ds > -\infty$.

Then prove that all the solutions of $\dot{x} = A(t)x + B(t)x$ are bounded.

OR

b Show that the equilibrium solution $x(t) = 0$ of the following system $\dot{x}_1(t) = x_2 - x_1(x_1^2 + x_2^2),\ \ \dot{x}_2(t) = x_2 - x_1(x_1^2 + x_2^2)$ is asymptotically stable.

14 a Stabilize the system $\ddot{x} - x = u$ by Bass method.

OR

b Prove that the control problem $x(0) = x_0,\ x(T) = x_1$ for the system $\dot{x} = Ax + Bu,\ x \in R^n,\ u \in R^m$ is solvable if and only if $x_1 - e^{AT}x_0 \in C(A, B)$.

15 a Find the optimal control u for the second order system

$\dot{x}_1(t) = x_2(t), \dot{x}_2(t) = u(t)$ and the cost functional

$$J = \frac{1}{2}[x_1^2(3) + 2x_2^2(3)] + \frac{1}{2}\int_0^3 \left[2x_1^2(t) + 4x_2^2(t) + 2x_1(t)x_2(t) + \frac{1}{2}u^2(t)\right] dt.$$

OR

b Consider the nonlinear system $\dot{x} = A(t)x(t) + B(t)u(t) + f(t, x(t))$, with the cost functional $J = \frac{1}{2}x^*(T)Fx(T) + \frac{1}{2}\int_0^T [x^*(t)Q(t)x(t) + u^*(t)R(t)u(t)]dt$ exists and if $\|f(t,x) - f(t,y)\| \le a\|x - y\|$ where a is a positive constant. Show that the optimal control is given by $u(x(t), t) = -R^{-1}(t)B^*(t)K(t)x(t) - R^{-1}(t)B^*(t)h(t,x)$ where $K(t)$ satisfies the Riccati equation and $\dot{h}(t,x) = -[A^*(t) - K(t)R^{-1}(t)B^*(t)]h(t,x) - K(t)f(t,x(t))$, $h(T,x) = 0$.

7-7-7. END