22MAP209

PSG COLLEGE OF ARTS & SCIENCE
(AUTONOMOUS)

MSc DEGREE EXAMINATION MAY 2023
(Second Semester)

Branch – **MATHEMATICS**

MEASURE THEORY

Time: Three Hours — Maximum: 50 Marks

SECTION-A (5 Marks)

Answer **ALL** questions
ALL questions carry **EQUAL** marks (5 x 1 = 5)

1 $m^*(\phi)=$........
(i) 0 (ii) 2
(iii) 1 (iv) ϕ

2 A non-negative finite valued function $\phi(x)$, taking only a finite number of different values, is called.........
(i) measurable (ii) continuous
(iii) Riemann integrable (iv) simple

3 A ring is called .... if it is closed under the formation of countable unions.
(i) $\mu-ring$ (ii) hereditary
(iii) $\sigma-ring$ (iv) outer measure

4 $\|f\|_p=$.......
(i) $(\int|f|^p\,d\mu)$ (ii) $(\int|f|^p\,d\mu)^{1/p}$
(iii) $(\int|f|\,d\mu)^{1/p}$ (iv) $(\int|f|\,d\mu)$

5 A is a .... with respect to ν if it is a positive set with respect to $-\nu$.
(i) measurable set (ii) positive set
(iii) countable set (iv) negative set

SECTION - B (15 Marks)

Answer **ALL** Questions
ALL Questions Carry **EQUAL** Marks (5 x 3 = 15)

6 a Prove that every interval is measurable.
OR
b If f is a measurable function and f = g a.e , then prove that g is measurable.

7 a State and prove Lebesgue's Monotone Convergence theorem.
OR
b Let f be an integrable function. Then prove that af is integrable and $\int af\,dx=a\int f\,dx$.

8 a Prove that the completion of a σ-finite measure is σ-finite.
OR
b If f is a measurable function and f = g a.e (μ) ,where μ is a complete measure, then prove that g is measurable.

Cont...

9 a State and prove Minkowski's inequality.

OR

b State and prove Jensen's inequality.

10 a Prove that a countable union of sets positive with respect to a signed measure ν is a positive set.

OR

b Write a note on Jordan decomposition.

SECTION -C (30 Marks)

Answer **ALL** questions

ALL questions carry **EQUAL** Marks (5 x 6 = 30)

11 a Prove that the outer measure of an interval equals its length.

OR

b Let c be any real number and let f and g be real valued measurable functions defined on the same measurable set E .Then prove that f+c, cf , f+g , f-g and fg are also measurable.

12 a State and prove Fatou's lemma.

OR

b Let f be a bounded function defined on the finite interval $[a,b]$, then prove that f is Riemann integrable over $[a,b]$ if and only if it continuous a.e.

13 a If μ is a measure on a σ-ring S, then prove that the class $\overline{S}$ of sets of the form $E\Delta N$ for any sets E, N such that $E \in S$ while N is contained in some set in S of zero measure, is a σ-ring and the set function $\overline{\mu}$ defined by $\overline{\mu}(E\Delta N) = \mu(E)$ is a complete measure on $\overline{S}$.

OR

b Prove that the measurability of a function f is equivalent to

(i) $\forall\alpha$, $\{f(x) \geq \alpha\} \in S$ (ii) $\forall\alpha$, $\{f(x) < \alpha\} \in S$ (iii) $\forall\alpha$, $\{f(x) \leq \alpha\} \in S$

where S is a σ algebra.

14 a i) Let $f, g \in L^p(\mu)$ and let a, b be constants. Then prove that $af + bg \in L^p(\mu)$ (2)

ii) State and prove Holder's inequality. (4)

OR

b If $1 \leq p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ such that $\| f_n - f_m \|_p \to 0$ as $n, m \to \infty$, then prove that there exists a function f and a subsequence $\{n_i\}$ such that $\lim f_{n_i} = f$ a.e. Also prove that $f \in L^p(\mu)$ and $\| f_n - f \|_p = 0$

15 a State and prove Radon-Nikodym theorem.

OR

b Let ν be a signed measure on $[[X, S]]$. Let $E \in S$ and $\nu(E) > 0$. Then prove that there exists A, a set positive with respect to ν , such that $A \subset E$ and $\nu(A) > 0$.

Z-Z-Z END