PSG COLLEGE OF ARTS & SCIENCE
(AUTONOMOUS)

MSc DEGREE EXAMINATION MAY 2023
(Fourth Semester)

Branch – **MATHEMATICS**

CONTROL THEORY

Time: Three Hours   Maximum: 50 Marks

SECTION-A (5 Marks)
Answer **ALL** questions
ALL questions carry **EQUAL** marks (5 x 1 = 5)

1. Let X be a real Banach space, $M \subset X$ a nonempty closed bounded convex subset and $F: M \to M$ be compact. Then F has a fixed point. This is a statement of
   (i) Leray-Schauder Theorem (ii) Banach fixed point theorem
   (iii) Brouwer fixed point theorem (iv) Schauder theorem

2. If rank B=n, then the system $\dot{x} = Ax + Bu$ is ----
   (i) stable (ii) unstable
   (iii) controllable (iv) completely controllable

3. The stability of the system $\dot{x} = Ax$, when $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ is
   (i) unstable (ii) stable
   (iii) asymptotically stable (iv) none of these

4. The pair (H, A) is detectable if and only if the pair $(A^*, -H^*)$ is ----
   (i) stabilizable (ii) controllable
   (iii) asymptotically stable (iv) unstable

5. When will you say that the extremum of Hamiltonian to be minimum with respect to $u(t)$
   (i) $(H_u)_{m\times m}$ is positive definite (ii) $(H_{uu})_{m\times m}$ is positive definite
   (iii) $(H_{uu})_{m\times m}$ is negative definite (iv) $(H_u)_{m\times m}$ is negative definite

SECTION - B (15 Marks)
Answer **ALL** Questions
ALL Questions Carry **EQUAL** Marks (5 x 3 = 15)

6 a Solve the initial value problem $\dot{x} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x, x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

OR

b The observed linear system $\dot{x} = A(t)x, y(t) = H(t)x(t)$ is observable on $[0, T]$ if there are distinct points $s_1, s_2, .., s_k \varepsilon [0, T]$ such that

$$rank V(s_1, s_2, .., s_k, 0) = n, \text{ where } V(s_1, s_2, .., s_k, 0) = \begin{pmatrix} H(s_1)X(s_1, 0) \\ H(s_2)X(s_2, 0) \\ \vdots \\ H(s_k)X(s_k, 0) \end{pmatrix}.$$

7 a The system $\dot{x} = A(t)x + B(t)u, x\varepsilon R^n, u\varepsilon R^m$ and $A(t)$ and $B(t)$ are $n \times n$ *and* $n \times m$ continuous matrices on $[0, T]$ respectively is controllable on $[0, T]$ if and only if controllability Grammian
$M(0, T) = \int_0^T X(T, t)B(t)B^*(t)X^*(T, t)dt$ is positive definite.

OR

b Consider the system governed by the equations
$\dot{x_1} = -3x_1 - 2x_2 + u_1$; $\dot{x_2} = x_1 + u_2$.

8 a If all the characteristic roots of A have negative real parts and B(t) satisfies the condition $\lim_{t\to\infty} \|B(t)\| = 0$ then prove that all the solutions of the system $\dot{x} = Ax + B(t)x$ tend to zero as $t \to \infty$.

OR

b Determine the stability or instability of the system $\dot{x} = Ax$ when
$$A = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{bmatrix}.$$

9 a The pair $(A + BK, B)$ is controllable if and only if the pair (A, B) is controllable.

OR

b The time invariant system $\dot{x} = A(t)x + B(t)u$, $x\varepsilon R^n$, $u\varepsilon R^m$ is controllable if and only if there exist $m \times n$ matrices K_1, K_2 for which the matrix $I - e^{-(A+BK_2)T}e^{(A+BK_1)T}$ is invertible.

10 a If $u(t) = -R^{-1}(t)B^*(t)K(t)x(t)$ then prove J attains a local minimum.

OR

b. If $x(t)$ and $p(t)$ are the solutions of the canonical equations $\begin{bmatrix}\dot{x}(t)\\ \dot{p}(t)\end{bmatrix} = \begin{bmatrix}A(t) & -S(t)\\ -Q(t) & -A^*(t)\end{bmatrix}\begin{bmatrix}x(t)\\ p(t)\end{bmatrix}$ and if $p(t) = K(t)x(t)$ for all $t\varepsilon[0, T]$ and all $x(t)$, then prove $K(t)$ must satisfies the equation $\dot{K}(t) + K(t)A(t) + A^*(t)K(t) - K(t)S(t)K(t) + Q(t) = 0$.

SECTION -C (30 Marks)

Answer **ALL** questions

ALL questions carry **EQUAL** Marks (5 x 6 = 30)

11 a Solve the initial value problem

$$\dot{x}(t) = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & -2\\ 3 & 2 & 1\end{bmatrix}x(t) + \begin{bmatrix}0\\ 0\\ e^t\cos 2t\end{bmatrix}, x(0) = \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}, x(t) = (x_1(t), x_2(t), x_3(t))^*.$$

OR

b Let $A(t)$ be an $n \times n$ matrix that is continuous on a closed bounded interval J and let $f\varepsilon L_n^2(J)$. Given $t_0\varepsilon J$ and $x_0\varepsilon R^n$ there exists a unique solution $x(t)$ of $\dot{x}(t) = A)t)x(t) + f(t)$ on the interval J with $x(t_0) = x_0$.

12 a. Assume that the continuous function f satisfies the condition

$$\lim_{|(x,u)|\to\infty}\frac{|f(t,x,u)|}{|(x,u)|} = 0$$

Uniformly for $t\varepsilon I$. If system $\dot{x} = A(t)x + B(t)u$ is completely controllable, then prove the system $\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t, x(t), u(t)$ is completely controllable.

OR

b Verify the controllability of the system

$$\dot{x}_1 = -x_1 + x_2 + (cost)u_1 + (sint)u_2 + \frac{10x_1}{(1+x_1{}^2+x_2{}^2+u_1{}^2)},$$

$$\dot{x}_2 = -x_1 - x_2 + (cost)u_2 - (sint)u_1 + \frac{x_2}{(1+x_2{}^2+u_2{}^2+t)}$$

13 a State and prove Gronwall's Inequality.

OR

b Suppose that the function $\frac{g(x)}{\|x\|}$ is a continuous function of x which tends to zero for x=0. Then the solution $x(t) = 0$ of $\dot{x} = Ax + g(x)$ is asymptotically stable if the solution $x(t) = 0$ of the linearized equation $\dot{x} = Ax$ is asymptotically stable.

14 a If the system $\dot{x} = A(t)x + B(t)u$, $x\varepsilon R^n$, $u\varepsilon R^m$ is controllable, then prove it is stabilizable.

OR

b Let C(A,B) have dimension $k \le n$ and let P be any nonsingular matrix such that the vectors in its first k rows $p_1, p_2, .. p_k$ form a basis for C(A,B). Then verify the change of variable $x = Py$ carries into $\dot{y} = \hat{A}y + \hat{B}u$ which has decomposition $\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}A_1 & A_3\\ 0 & A_2\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} + \begin{bmatrix}B_1\\ 0\end{bmatrix}u$. Moreover, the k-dimensional system $y_1 = A_1y_1 + B_1u$ is controllable.

15 a Find the optimal control u for the nonlinear scalar system $\dot{x} = -\frac{1}{2}x + u + \frac{1}{50}tan^{-1}x$ with cost functional $J = \frac{1}{2}\int_0^1[2x^2 + u^2]dt$.

OR

b. Find the optimal control u for the second order system $\dot{x_1}(t) = x_2(t)$; $\dot{x_2}(t) = u(t)$ with cost functional functional

$$J = \frac{1}{2}[x_1^2(3) + 2x_2^2(3)] + \frac{1}{2}\int_0^3\left[2x_1^2(t) + 4x_2^2(t) + 2x_1(t)x_2(t) + \frac{1}{2}u^2(t)\right]dt.$$

Z-Z-Z END