Exam Date & Time: 26-Sep-2020 (02:00 PM - 05:30 PM)

PSG COLLEGE OF ARTS AND SCIENCE

Note: Writing 3hrs: Checking & Inserting Image : 30mins

MSc DEGREE EXAMINATION MAY 2020
(Fourth Semester)

Branch - MATHEMATICS
OPERATOR THEORY [18MAP14]

Marks: 75 **Duration: 210 mins.**

SECTION - A

Answer all the questions.

1) For T and T^* be an operator on a Hilbert space H. Which one is true? (1)
(i) $(T^*)^* = T^{**}$
(ii) $(\alpha T)^* = \alpha^{**} T^*$
(iii) $T^* = T$
(iv) $(\alpha T)^* = \bar{\alpha} T^*$

2) If there exits an operator A such that $A_1 \geq A_2 \geq \ldots \geq A_n \geq \ldots \geq A$, then the name the sequence $\{A_n\}$ is ______. (1)
(i) bounded monotone decreasing
(ii) bounded monotone increasing
(iii) oscillatory
(iv) compact

3) What is an unitary operator? (1)
(i) $U.U^* = U$
(ii) $U^*.U = I$
(iii) $U.U^* = U^*.U$
(iv) $U.U^* = U^*$

4) An operator T on a Hilbert space H is invertible operator, if there exits an operator S is ______. (1)
(i) $ST = H$
(ii) $TS = H^{-1}$
(iii) $ST = TS = I$
(iv) $TS = ST = S^{-1}.T^{-1}$

5) (1)

What is the numerical range W(T) of an operator T on a Hilbert space H?
(i) $\{(Tx,x) : \|x\|=1\}$
(ii) $\{(Tx,x) : \|x\|\neq 1\}$
(iii) $\{(Tx,x) : \|x\|<1\}$
(iv) $\{(Tx,x) : \|x\|>1\}$

6) Identify the spectrum of T.
(i) $\{\lambda\in\ : T-\lambda$ is invertible$\}$
(ii) $\{\lambda\in\ : T-\lambda$ is not invertible$\}$ (1)
(iii) $\{\lambda\in\ : T+\lambda$ is invertible$\}$
(iv) $\{\lambda\in\ : T+\lambda$ is not invertible$\}$

7) Which one is paranormal operator?
(i) $\|T^2x\|\leq\|Tx\|^2, x\in H$
(ii) $\|T^2x\|=\|Tx\|^2, x\in H$
(iii) $\|T^2x\|\geq\|Tx\|^2, x\in H$ (1)
(iv) $\|T^2x\|\leq\|xT^2\|$

8) An operator is convexiod if and only if T-λ is spectraloid for all ____ number λ.
(i) Real
(ii) Irrational
(iii) Rational (1)
(iv) Complex

9) Which one is p-hypernormal?
(i) $(T^*T)^p\leq(TT^*)^p$
(ii) $(T^*T)^p\geq(TT^*)^p$ (1)
(iii) $(T^*T)^p=(TT^*)^p$
(iv) $(T^*T)^p\neq(TT^*)^p$

10) An operator T belongs to class A if
(i) $|T^2|\geq|T^2|$
(ii) $\|T^2\|\leq\|T\|^2$
(iii) $|T^2|\geq|T|^2$ (1)
(iv) $|T^2|\leq|T|^2$

SECTION - B

Answer all the questions.

11) a) Show that for any linear operator T on a Hilbert space H, the following statements are mutually equivalent:
(i) T is bounded
(ii) T is continuous on the whole space H (5)
(iii) T is continuous on some point x_0 on H.

[OR]
b) State and prove Generalized Schwarz inequality. (5)

12) (5)

a) Let U be a partial isometry operator on a Hilbert space H with the initial space M and the final space N. Then prove that
(i) $UP_M=U$ and $U^*U=P_M$ and (ii) N is a closed subspace of H.

[OR]

b) Assume A and B be normal operators. If AX=XB holds for some operator X, them rove that $A^*X=XB^*$. (5)

13) a) If T is an operator such that $\|I-T\|<1$, then prove that T is invertible. (5)

[OR]

b) If T is a normal operator, then prove that T is normaloid, i.e., $\|T\|=r(T)$. (5)

14) a) If an operator T is convexoid such that both $\sigma(T)$ and $\sigma(ReT)$ are connected, then prove that $Re\sigma(T)=\sigma(ReT)$. (5)

[OR]

b) State and prove Lowher-Heinz inequality. (5)

15) a) Prove that every log-hyponormal operator is a class A operator and also prove that every class A operator is a paranormal operator. (5)

[OR]

b) Let $T=U|T|$ be the polar decomposition of a log-hyponormal operator. Then prove that $\tilde{T}_{s,t}=|T|^s U|T|^t$ is $\frac{\min\{s,t\}}{s+t}$ - hyponormal for any s>0 and t>0. (5)

SECTION - C

Answer all the questions.

16) a) Assume P_1 and P_2 be two projections onto M_1 and M_2 respectively. The prove that (i) $P=P_1+P_2$ is a projection if and only if $M_1 \perp M_2$.
(ii) If $P=P_1+P_2$ is a projection, then P is the projection on to $M_1 \oplus M_2$. (8)

[OR]

b) If T is an operator on a Hilbert space H over the complex scalars C, then prove the following
(i) T is normal iff $\|Tx\|=\|T^*x\|$ for all $x \in H$.
(ii) T is self-adjoint iff (Tx,x) is real for all $x \in H$. (8)
(iii) T is unitary iff $\|Tx\|=\|T^*x\|=\|x\|$ for all $x \in H$.
(iv) T is hyponormal iff $\|Tx\| \geq \|T^*x\|$ for all $x \in H$.

17) a) Let $T=U|T|$ be the polar decomposition of an operator T on a Hilbert space H. Then prove that (i) $N(|T|)=N(T)$ (ii) $|T^*|^q=U|T|^qU^*$ for any positive number q. (8)

[OR]

b) State and prove polar decomposition theorem. (8)

18) State and prove spectral mapping theorem. (8)

a) [OR] b) Assess the characterizations of normaloid operators. (8)

19) a) Prove that the relations self-adjoint $\subseteq$ Normal $\subseteq$ Quasinormal $\subseteq$ Subnormal Hyponormal $\subseteq$ Paranormal $\subseteq$ Normaloid $\subseteq$ Spectraloid. (8)

[OR] b) State and prove generalized Furuta inequality. (8)

20) a) Let $T=U|T|$ be p-hyponormal for $P>0$ and U be unitary. Then prove that

(i) $\tilde{T}=|T|^{1/2}U|T|^{1/2}$ is $\left(P+\frac{1}{2}\right)$-hyponormal if $0<P<\frac{1}{2}$.

(ii) $\tilde{T}=|T|^{1/2}U|T|^{1/2}$ is hyponormal if $\frac{1}{2}\leq P<1$. (8)

[OR] b) Prove that for each $k>0$, an operator T is absolute k-para normal if and only if $T^{*}|T|^{2k}T-(k+1)\lambda^{k}|T|^{2}+k\lambda^{k+1}\geq 0$ holds for all $\lambda>0$. (8)

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