Exam Date & Time: 28-Sep-2020 (02:00 PM - 05:45 PM)

# PSG COLLEGE OF ARTS AND SCIENCE

Note: Writing 3hrs: Checking & Inserting Image : 30mins

MSc DEGREE EXAMINATION MAY 2020
(Fourth Semester)

Branch - MATHEMATICS

**CONTROL THEORY [18MAP15]**

**Marks: 75** **Duration: 210 mins.**

**SECTION - A**

**Answer all the questions.**

1) The fixed points of the operator $px=x^2$ are _________. (1)
(i) 0,1 (ii) 1,2 (iii) 0,2 (iv) 1,3

2) The nxr matrix function R(t) defined on [O,T] is a reconstruction Kernel if and only if _________ = I (nxn identity matrix). (1)
(i) $\int_0^T R(t)H(t)dt$ (ii) $\int_0^T R(t)X(t_{10})dt$
(iii) $\int_0^T R(t)H(t)_\Delta x(t_{10})dt$ (iv) None

3) The controllability Grammian matrix is M(0,T)=________. (1)
(i) $\int_0^T X(T,t)B(t)dt$ (ii) $\int_0^T X(T,t)B(t)B^*(t)X^*(T,t)dtdt$
(iii) $\int_0^T X^*(T,t)B^*(t)dt$ (iv) $\int_0^T X^*(T,t)B(t)dt$

4) If ________, then the system $\dot{x} = Ax + Bu$ is controllable. (1)
(i) rank B=n (ii) rank B<n (iii) rank B>n (iv) rank B=0

5) The system $\dot{x} = Ax$ is stable if all the eigenvalues of A have _______ real parts. (1)
(i) positive (ii) negative (iii) no (iv) none

6) Let X(t) be a fundamental matrix of $\dot{x} = A(t)x(t)$. Then the system is _______ if and only if there exists a constant k>0 with $\|x(t)\| \le k$, $t \in J$. (1)
(i) Stable (ii) Unstable
(iii) Uniformly stable (iv) Asymptotically Stable

7) The pair (H,A) is detectable if and only if the pair _____ is stabilizable. (1)

(i) $(A^*, H^*)$ (ii) $(-A^*, H^*)$ (iii) $(A^*, -H^*)$ (iv) $(-A^*, -H^*)$

8) The control problem $x(0)=x_0$, $x(T)=x_1$ for the system $\dot{x} = Ax + Bu$, $x \in R^n$, $u \in R^m$ is solvable if and only if ______.

(i) $x_1 - e^{AT}x_0 \in C(A,B)$ (ii) $x_1 - e^{AT}x_1 \in C(A,B)$ (1)

(iii) $x_1 - e^{-AT}x_0 \in C(A,B)$ (iv) $x_1 - e^{-AT}x_1 \in C(A,B)$

9) The matrix differential equation $\dot{K}(t) + K(t)A(t) + A^*(t)k(t) - K(t)S(t)K(t) + Q(t) = 0$ is called ________ equation. (1)

(i) Euler's (ii) Ricatti (iii) Poisson (iv) Lagrange's

10) If $u(t) = -R^{-1}(t)B^*(t)K(t)x(t)$, then T attains a ______.

(i) local minimum (ii) local maximum

(iii) zero (iv) none (1)

## SECTION - B

**Answer all the questions.**

11)

a) Prove that the observed linear system $\dot{x}(t) = A(t)x(t)$ and $y(t)=H(t)x(t)$ is observable on [0,T] if and only if the observability Grammian matrix $W(0,T) = \int_0^T X^*(t,0)H^*(t)H(t)X(t,0)dt$ is positive definite, where the star denotes the matrix transpose. (5)

[OR]

b) Check the system $\dot{x}(t) = Ax$ and $y=Hx$ is observable or not, where $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ and $H = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ (5)

12)

a) Show that the dynamical system described by $\dot{x}_1 = x_1 + x_2$ and $\dot{x}_2 = -2x_1 + x_2 + u$ is controllable. (5)

[OR]

b) Prove that the system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ is controllable on [0,T] if and only if for each vector $x_1 \in R^n$ there is a control $u \in L_m^2[0,T]$ which steers 0 to $x_1$ during [0,T]. (5)

13) State and prove Gronwall's inequality. (5)

a)

[OR] (5)

b)

Consider the differential equations $\frac{dx_1}{dt} = x_2 - x_1(x_1^2 - x_2^2)$ and $\frac{dx_2}{dt} = -x_1 - x_2(x_1^2 - x_2^2)$. Show that the solution of the above system is asymptotically stable.

14) a) Prove that the pair (A+BK, B) is controllable if and only if the pair (A,B) is controllable. (5)

[OR]

b) Prove that C(A,B) is the invariant subspace of the matrix A. (5)

15) a) If x(t) and p(t) are the solutions of the canonical equations $\begin{bmatrix} \overset{\circ}{x}(t) \\ \overset{\circ}{p}(t) \end{bmatrix} = \begin{bmatrix} A(t) & -s(t) \\ -Q(t) & -A^*(t) \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$ and if p(t)=K(t)x(t) for all $t \in [0,T]$ and all x(t), then prove that K(t) must satisfy the equation $\overset{\circ}{K}(t)A(t) + A^*(t)K(t) - K(t)S(t)K(t) + Q(t) = 0$, where $S(t)=B(t)\,R^{-1}(t)\,B^*(t)$. (5)

[OR]

b) If K(t) is the solution of the Ricatti equation $\overset{\circ}{K}(t) + K(t)A(t) + A^*(t)K(t) - K(t)S(t)K(t) + Q(t) = 0$ and if K(T)=F, then K(t) is symmetric for all $t \in [0,T]$, that is, $K(t)=K^*(t)$. (5)

## SECTION - C

**Answer all the questions.**

16) a) Let A be a nxn matrix that is continuous on a closed bounded interval J and let $f \in L_n^2(J)$. Given $t_0 \in J$ and $x_0 \in R^n$. Prove that there exists a unique solution x(t) of $\dot{x}(t) = A(t)x(t) + f(t)$ on the interval J with $x(t_0)=x_0$. (8)

[OR]

b) Prove that the constant coefficient system $\overset{\circ}{x} = Ax$ and y=Hx is observable on an arbitrary interval [0,T] is and only if for some k, $0<k\leq n$, the rank of he observability matrix rank $\begin{bmatrix} H \\ HA \\ \vdots \\ HA^{K-1} \end{bmatrix} = n$. (8)

17) a) Determine the control function for the controlled harmonic oscillator $\ddot{x} + x = u$ which steers from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$. (8)

[OR]

b) Assume that the continuous function f satisfies the condition $\lim_{|(x,u)|\to\infty} \frac{|f(t,x,u)|}{|(x,u)|} = 0$ uniformly for $t \in I$. If $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ is completely controllable, then prove that the system $\dot{x}(t) = A(x)x(B) + B(t)u(t) + f(t,x,u)$ is completely controllable. (8)

18) a) Let x(t,s) be the fundamental matrix of $\dot{x}(t) = A(t)x(t)$. Prove that this system is uniformly stable if and only if there exist a constant k>0 such that $\|x(t,s)\| \leq k,\ 0 \leq s \leq t < \infty$. (8)

[OR]

b) If all the characteristic roots of A have negative real parts and B(t) satisfies $\lim_{t \to \infty} \| B(t) \| = 0$, then prove that all the solutions of the system $\dot{x}(t) = A(t)x(t) + B(t)x(t)$ tends to zero as $t \to \infty$. (8)

19) a) Suppose there are mxn matrices $k_1$ and $k_2$ such that $(A+BK_1)$ and $-(A+BK_2)$ are stability matrices. Then prove that the system $\dot{x} = Ax + Bu$ is controllable. (8)

[OR]

b) Prove that the control problem $x(0)=x_0$, $x(T)=x_1$ for the system $\dot{x} = Ax + Bu$ is solvable if and only if $x_1 - e^{AT}x_0 \in C(A,B)$. (8)

20) a) Consider the controllable system $\dot{x}_1(t) = x_2(t)$ and $\dot{x}_2(t) = u(t)$ with the cost functional $J = \frac{1}{2}\int_0^\infty \left[x_1^2(t) + 2bx_1(t)x_2(t) + ax_2^2(t) + u^2(t)\right]dt$, where we assume that $a-b^2>0$. Find the optimal control. (8)

[OR]

b) For the continuous nonlinear system $\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t, x(t))$ with quadratic performance criteria $J = \frac{1}{2}x^*(T)Fx(t) + \frac{1}{2}\int_0^T \left[x^*(t)Q(t)x(t) + u^*(t)R(t)u(t)\right]dt$.

Prove that the optimal control exists if $\|f(t,x)-f(t,y)\| \leq a\|x-y\|$ where 'a' is a positive constant and is given by $u(x(t),t) = -R^{-1}(t)B^*(t)K(t)X(t) - R^{-1}(t)B^*(t)h(t,x)$ where K(t) satisfies the Ricatti equation $\dot{K}(t) + K(t)A(t) + A^*(t)K(t) - K(t)S(t)K(t) + Q(t) = 0$ and $\dot{h}(t,x) = -\left[A^*(t) - K(t)B(t)R^{-1}(t)B^*(t)\right] h(t,x) - K(t)f(t,x(t))]$, $h(T,x)=0$. (8)

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