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PSG COLLEGE OF ARTS & SCIENCE
(AUTONOMOUS)

MSc DEGREE EXAMINATION MAY 2025
(Fourth Semester)

Branch – **MATHEMATICS**

OPERATOR THEORY

Time: Three Hours — Maximum: 75 Marks

SECTION-A (10 Marks)
Answer **ALL** questions
ALL questions carry **EQUAL** marks (10 × 1 = 10)

Module No.	Question No.	Question	K Level	CO
1	1	Which of the following is a self adjoint operator? a) $T^* = T$ b) $T^*T = TT^*$ c) $T^*T = I$ d) $T^2 = T$	K2	CO1
	2	If A is a positive operator on a Hilbert space H, then _____ for any x,y ∈ H (a) $\|(Ax, y)\|^2 \geq (Ax, x)(Ay, y)$ (b) $\|(Ax, y)\|^2 \leq (Ax, x)(Ay, y)$ (c) $\|(Ax, y)\|^2 = (Ax, x)(Ay, y)$ d) $\|(Ax, y)\|^2 \neq (Ax, x)(Ay, y)$	K1	
2	3	An operator U on a Hilbert space H is an isometry operator if and only if _____ a) $U^* = U$ b) $U^*U = UU^*$ c) $U^*U = I$ d) $U^2 = U$	K2	CO2
	4	An operator T on a Hilbert space H is invertible, then _____ holds. a) $N(T) = N(T^*)$ b) $N(T) \subset N(T^*)$ c) $N(T) \supset N(T^*)$ d) $N(T) \cong N(T^*)$	K1	
3	5	If $T \geq cI$ for some $c > 0$, then T is ______ (a) spectrum (b) point spectrum (c) resolvent (d) invertible	K2	CO3
	6	An operator T is normaloid if _____ (a) $\|\|T\|\| = r(T)$ (b) $\|\|T\|\| \geq r(T)$ (c) $\|\|T\|\| \leq r(T)$ (d) $\|\|T\|\| \neq r(T)$	K1	
4	7	If T is a paranormal operator, then which of the following is incorrect? a) T^n is also paranormal for any natural number n b) T is normaloid operator, $\|\|T\|\| = r(T)$ c) If T is an invertible paranormal operator, so is T^{-1} d) T^{-1} is also paranormal	K2	CO4
	8	An operator T is convexoid if and only if ____ is spectraloid for all complex number λ (a) $T\lambda$ (b) $T + \lambda$ (c) $T - \lambda$ (d) $\frac{T}{\lambda}$	K1	
5	9	$A^\alpha \geq B^\alpha$ does not hold in general for any $\alpha > 1$, even if _____ a) $A \leq B \leq 0$ b) $A \geq B \geq 0$ c) $A \leq 0 \leq B$ d) $B \leq 0 \leq A$	K1	CO5
	10	If T is a positive invertible operator on a Hilbert space H and $\lambda > 1$, then _____ a) $\lambda T + (1-\lambda) \leq T^\lambda$ b) $\lambda T + (1-\lambda) \geq T^\lambda$ c) $\lambda T + (1-\lambda) < T^\lambda$ d) $\lambda T + (1-\lambda) > T^\lambda$	K2	

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SECTION - B (35 Marks)
Answer **ALL** questions
ALL questions carry **EQUAL** Marks (5 × 7 = 35)

Module No.	Question No.	Question	K Level	CO
1	11.a.	If T is an operator on a Hilbert space H over the complex scalars C, then demonstrate that the following holds (i) T is normal iff $\|Tx\| = \|T^*x\|$ for all $x \in H$ (ii) T is self-adjoint iff (Tx, x) is real for all $x \in H$ (iii) T is unitary iff $\|Tx\| = \|T^*x\| = \|x\|$ for all $x \in H$ (iv) T is hyponormal iff $\|Tx\| \geq \|T^*x\|$ for all $x \in H$	K3	CO1
		(OR)		
	11.b.	Let M_1 and M_2 be two closed subspaces, and let P_1 and P_2 be two projections onto M_1 and M_2 respectively. Then reveal that the following (i) and (ii) hold: (i) $M_1 \perp M_2 \Leftrightarrow P_1P_2 = 0 \Leftrightarrow P_2P_1 = 0$. (ii) $M_1 \subset M_2 \Leftrightarrow P_1P_2 = P_1 \Leftrightarrow P_2P_1 = P_1 \Leftrightarrow P_1 \leq P_2 \Leftrightarrow \|P_1x\| \leq \|P_2x\|$ for all $x \in H$.		
2	12.a.	If U is a partial isometry operator on a Hilbert space H with the initial space M and the final space N, then prove that (i) $UP_M = U$ and $U^*U = P_M$ (ii) N is a closed subspace of H Also prove that if U^* is a partial isometry with the initial space N and final space M, then $U^*P_N = U^*$ and $UU^* = P_N$	K3	CO2
		(OR)		
	12.b.	If A and B are normal operators and $AX = XB$ holds for some operator X, then show that $A^*X = XB^*$.		
3	13.a.	If T is a self-adjoint operator on a Hilbert space H, then that $T + iI$ has a bounded inverse operator. Justify this statement.	K4	CO3
		(OR)		
	13.b.	Examine the proof for the different characterizations of normaloid operators.		
4	14.a.	Formulate Young's inequality and provide its proof.	K5	CO5
		(OR)		
	14.b.	Frame the Generalized Furuta inequality and prove it.		
5	15.a.	If T is a paranormal operator and T^* is an invertible paranormal operator, then analyze the following inequalities respectively (i) $\|T\| \geq \ldots \geq \frac{\|T^{n+1}x\|}{\|T^n x\|} \geq \ldots \geq \frac{\|T^2x\|}{\|Tx\|} \geq \frac{\|Tx\|}{\|x\|}$ (ii) $\frac{\|x\|}{\|T^{-1}x\|} \geq \frac{\|T^{-1}x\|}{\|T^{-2}x\|} \geq \ldots \geq \frac{\|T^{-n+1}x\|}{\|T^{-n}x\|} \geq \ldots \geq \frac{\|1\|}{\|T^{-1}\|}$	K4	CO4
		(OR)		
	15.b.	If T is a hyponormal operator, then reveal that the following properties hold: (i) $T - \mu$ is also a hyponormal for any $\mu \in C$ (ii) T is a transaloid operator (iii) T^{-1} is also a hyponormal operator if T^{-1} exists (iv) T is a condition G_1 operator		

SECTION -C (30 Marks)

Answer **ANY THREE** questions

ALL questions carry **EQUAL** Marks (3 × 10 = 30)

Module No.	Question No.	Question	K Level	CO
1	16	If P_M is a projection onto a closed subspace M of a Hilbert space H, then P_M is an operator such that $P_M^* = P_M$ and $P_M^2 = P_M$. Conversely if P is an operator such that $P^* = P$ and $P^2 = P$, then $M = R(P)$ is a closed subspace and $P = P_M$, i.e., P is a projection onto M – Validate these sentences.	K4	CO1
2	17	Consider T as a normal operator. Then there exists a unitary operator U such that $T = UP = PU$ and both U and P commute with V^*, V and $\|A\|$ of the polar decomposition $A = V\|A\|$ of any operator A commuting with T and T^* - Validate this statement.	K4	CO2
3	18	By providing logical arguments, prove Toeplitz-Hausdorff theorem.	K4	CO3
4	19	Provide a logical proof of Holder-McCarthy inequality using well-reasoned arguments.	K5	CO5
5	20	Critize the following inclusion relations: Self-adjoint ⊆ Normal ⊆ Quasinormal ⊆ Subnormal ⊆ Hyponormal ⊆ Paranormal ⊆ Normaloid ⊆ Spectraloid	K5	CO4

Z-Z-Z END