

PSG COLLEGE OF ARTS & SCIENCE
(AUTONOMOUS)

BSc DEGREE EXAMINATION DECEMBER 2024
(Fourth Semester)

Branch – MATHEMATICS

MODERN ALGEBRA

Time: Three Hours

Maximum: 50 Marks

SECTION-A (5 Marks)

Answer ALL questions

ALL questions carry EQUAL marks (5 x 1 = 5)

1. The two mappings σ and τ of S into T are said to be equal if -----for every $s \in S$.
a) $s\sigma = s\tau$ b) $s\sigma \geq s\tau$ c) $s\sigma \leq s\tau$ d) $s\sigma \neq s\tau$
2. A homomorphism ϕ from G into $\bar{G}$ is said to be an isomorphism if ϕ is one-to-one
a) Automorphism b) Onto c) Isomorphism d) Homomorphism
3. Every permutation is the product of its-----
a) m -cycle. b) Alternating group c) Permutation d) Cycles
4. The homomorphism ϕ of R into R' is an ----- if and only if $I(\phi) = (0)$.
a) Transpositions. b) Permutation c) Isomorphism d) Homomorphism
5. Let R be a -----with unit element. An element $a \in R$ is a unit in R if there exists an element $b \in R$ such that $ab = 1$.
a) Commutative ring . b) Associative ring c) Division ring d) Euclidean ring

SECTION-B (15 Marks)

Answer ALL questions

ALL questions carry EQUAL marks (5 x 3= 15)

- 6 .a) If a and b are integers, not both 0, then (a, b) exists; moreover, we can find integers m_0 and n_0 such that $(a, b) = m_0a + n_0b$.
(OR)
b) Prove that If G is a finite group and $a \in G$, then $o(a)/o(G)$.
- 7.a) Prove that If HK is subgroup of G if and only if $HK=KH$.
(OR)
b) Prove that the subgroup N of G is a normal subgroup of G if and only if every left coset of N in G is a right coset of N in G .
- 8.a) Prove that every permutation is the product of its cycles.
(OR)
b) Prove that let G be a group and ϕ an automorphism of G . If $a \in G$ is of order $o(a) > 0$, then $o(\phi(a)) = o(a)$.

Cont...

9.a) If R is a ring, then for all $a, b \in R$

1. $a0 = 0a = 0$.
2. $a(-b) = (-a)b = -(ab)$.
3. $(-a)(-b) = ab$. If in addition, R has a unit element 1, then
4. $(-1)a = -a$.
5. $(-1)(-1) = 1$.

(OR)

b) Define (i) zero-divisor (ii) integral domain (iii) field

10.a) Prove that let R be a Euclidean ring. Then every element in R is either a unit in R or can be written as the product of a finite number of prime elements of R .

(OR)

b) Prove that let R be a commutative ring with unit element whose only ideals are (0) and R itself. Then R is a field.

SECTION - C (30 Marks)

Answer ALL the questions.

ALL questions carry EQUAL marks (5 x 6 = 30)

11.a) Prove that Any positive integer $a > 1$ can be factored in a unique way as

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t} \text{ where } p_1 > p_2 > \dots > p_t \text{ are prime numbers and where each } \alpha_i > 0.$$

(OR)

b) State and prove Lagranges theorem.

12.a) State and prove fundamental theorem of homomorphism.

(OR)

b) State and prove Cauchy's theorem for Abelian groups.

13.a) State and prove Cayley's Theorem.

(OR)

b) Prove that if G is a group, then $\mathcal{A}(G)$, the set of automorphisms of G , is also a group.

14.a) Prove that If p is a prime number then J_p , the ring of integers mod p , is a field.

(OR)

b) Prove that finite integral domain is a field.

15.a) Prove that $J[i]$ is a Euclidean ring.

(OR)

b) Prove that

(i). If p is a prime number of the form $4n + 1$, then $p = a^2 + b^2$ for some integers a, b .

(ii). If p is a prime number of the form $4n + 1$, then we can solve the congruence

$$x^2 = -1 \pmod{p}.$$

Z-Z-Z

END