20MAP18

PSG COLLEGE OF ARTS & SCIENCE
(AUTONOMOUS)

MSc DEGREE EXAMINATION MAY 2022
(Fourth Semester)

Branch – **MATHEMATICS**

OPERATOR THEORY

Time: Three Hours Maximum: 75 Marks

SECTION-A (10 Marks)
Answer **ALL** questions
ALL questions carry **EQUAL** marks (10 x 1 = 10)

1. Justify : $T^*T = 0$ iff $T=0$ for a linear operator $T:H\rightarrow H$.
   (i) true (ii) false
   (iii) not necessary (iv) not defined

2. For a linear operator $T:H\rightarrow H$, if $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x,y \in H$ then T is said to be ______ operator.
   (i) identity (ii) self adjoint
   (iii) unitary (iv) isometry

3. If $\|Ux\| = \|x\|$ & $Ux = 0$, then U is said to be ____________.
   (i) unitary (ii) partial isometry
   (iii) normal (iv) hyponormal

4. If $x \in M$, and $y \in M^{\perp}$ where $x,y \in H = M+M^{\perp}$, then $\langle x,y\rangle =$ ___________.
   (i) 0 (ii) 1
   (iii) -1 (iv) $\langle y, x\rangle$

5. For an operator T, if $(T-\lambda)$ is not invertible for $\lambda \in C$, then it is said to be
   (i) spectrum of T (ii) resolvent of T
   (iii) point spectrum of T (iv) Continuous spectrum of T

6. If T is invertible then, _____________.
   (i) $N(T)=N(T^*)$ (ii) $N(T)=N(U)$
   (iii) $N(T)\supset N(T^*)$ (iv) $N(T)\subset N(T^*)$

7. For an operator T, if $\| T^2x \| \geq \| Tx \|^2$, where $\| x \|= 1$ and $x \in H$, then T is said to be ______.
   (i) normal operator (ii) paranormal operator
   (iii) normaloid (iv) spectraloid

8. An operator T is said to be transaloid if $T-\mu$ is_______ for any $\mu \in C$.
   (i) normal (ii) paranormal
   (iii) normaloid (iv) spectraloid

9. For an operator T, if $|T|^2 \geq |T|^2$ then T is said to be ________.
   (i) absolute k-paranormal (ii) absolute 1-paranormal
   (iii) class A operator (iv) class B operator

10. For a log hyponormal operator T, T should not be invertible.
   (i) true (ii) false
   (iii) not necessary (iv) not defined

Cont...

SECTION - B (35 Marks)

Answer **ALL** questions

ALL questions carry **EQUAL** Marks (5 x 7 = 35)

11 a. Let T be an operator on H then prove that T^* is also an operator and the F ollowing properties hold (i) $\| T^* \| = \| T \|$

(ii) $(T_1 + T_2)^* = T_1^* + T_2^*$

(iii) $(\alpha T)^* = \bar{\alpha} T^*$

(iv) $(T^*)^* = T$

(v) $(ST)^* = T^* S^*$

(OR)

b. If Tis an operator on a Hilbert space H over the complex scalars C then prove that the following are mutually equivalent.

i) T is isometry

ii) $\|Tx\| = \|x\| \; \forall x \in H$

iii) $\langle Tx, Ty \rangle = \langle x, y \rangle \; \forall x, y \in H$.

12 a. Let T be an operator on H and M be a closed subspace of H then prove that the following are mutually equivalent (i) M reduces T

(ii) $M^\perp$ reduces T

(iii) M reduces T^*

(iv) M is invarient under Tand T^*

(v) TP = PT where P is a projection onto M

(OR)

b. Let T =UP be the polar decomposition of T on H then prove tat Tis normal iff U commutes with P and U is unitary on $N(T)^\perp$

13 a. If T is an operator then prove that $\sigma(T)$ is a compact subset of a complex Plane. If $\lambda \in \sigma(T)$ then $|\lambda| \leq \|T\|$.

(OR)

b. Define Normaloid and Spectraloid operators and prove that

(i) if T is self adjoint then T is normaloid.

(ii) if T is a normal operator then T is normaloid

14 a. Prove the following inclusion realtions hold,

Self adjoint $\subseteq$ Normal $\subseteq$ quasi normal $\subseteq$ subnormal $\subseteq$ hyponormal

(OR)

b. State and prove Young's inequality

15 a. Prove that (i) every log-hyponormal operator is a class A operator

(ii) every class A operator is a paranormal operator.

(OR)

b. If an operator T is absolute k –paranormal for some k > 0 then prove that T is normaloid

Cont…

SECTION -C (30 Marks)
Answer **ANY THREE** questions
ALL questions carry **EQUAL** Marks (3 x 10 = 30)

16. Define orthogonal projection and Let P_1 and P_2 be two projections into M_1 and M_2 respectively. Then prove the following
    i) $P=P_1 P_2$ is a projection $<=> P_2 P_1 = P_1 P_2$.
    ii) If $P_2 P_1 = P_1 P_2$ then $P = P_1 P_2$ is a projection into $M_1 \cap M_2$

17. Let U be the partial isometry operator on H with initial space M and final space N then prove that the following holds
    i) $UP_M = U$ & $U^*U = P_M$
    ii) N is a closed subspace of H
    iii) $U^* P_N = U^*$ & $UU^* = P_N$

18. Prove that an operator T on Hilbert space is invertible iff the following holds .
    i) There exists a positive number C, such that $\| Tx \| \geq C\| x \|$
    ii) R(T) is dense in H . i.e, $R(\overline{T}) = H$

19. State and Prove Furuta inequality

20. Let $T=U|T|$ be the polar decomposition of a p-hyponormal for p>0,then prove that i) $T_{s,t} = |T|^s U |T|^t$ is $\frac{p+\min(s,t)}{s+t}$ hyponormal for any s>0 and t>0 , such that $\max(s,t) \geq p$
    ii) $T_{s,t} = |T|^s U |T|^t$ is hyponormal for s>0 and t>0 such that $\max(s,t) \leq p$.

Z-Z-Z END