22MAP419

PSG COLLEGE OF ARTS & SCIENCE
(AUTONOMOUS)

MSc DEGREE EXAMINATION MAY 2024
(Fourth Semester)

Branch – **MATHEMATICS**

OPERATOR THEORY

Time: Three Hours Maximum: 50 Marks

SECTION-A (5 Marks)

Answer **ALL** questions
ALL questions carry **EQUAL** marks (5 x 1 = 5)

1 If $T^*T = TT^*$, then T is called __________ operator .
(i) normal (ii) unitary
(iii) isometry (iv) projections

2 Let A and B be normal operators. If $AX = XB$ holds for some operator X, then ________.
(i) $A^*X = XB$ (ii) $A^*X = XB^*$
(iii) $AX = X^*B$ (iv) $AX = XB^*$

3 If $T \geq cI$ for some $c > 0$, then T is __________ .
(i) closed (ii) self-adjoint
(iii) invertible (iv) normal

4 An operator T on a Hilbert space H is said to be a paranormal operator if __________ for any unit vector $x \in H$.
(i) $\|T^2x\| = \|Tx\|^2$ (ii) $\|T^2x\| \leq \|Tx\|^2$
(iii) $\|T^2x\| \neq \|Tx\|^2$ (iv) $\|T^2x\| \geq \|Tx\|^2$

5 $A \geq B \geq 0$ ensures __________ for any $\alpha \in [0,1]$.
(i) $A^\alpha \geq B^\alpha$ (ii) $A^\alpha \leq B^\alpha$
(iii) $A^\alpha > B^\alpha$ (iv) $A^\alpha < B^\alpha$

SECTION - B (15 Marks)

Answer **ALL** Questions
ALL Questions Carry **EQUAL** Marks (5 x 3 = 15)

6 a For any linear operator T on a Hilbert space H, prove that the following statements are mutually equivalent:
(i) T is bounded.
(ii) T is continuous on the whole space H.
(iii) T is continuous on some point Xo on H.

OR

b State and prove the Generalized Schwarz inequality.

7 a Show that
(i) An operator U on a Hilbert space H is an isometry operator if and only if $U^*U = I$.
(ii) An operator U on a Hilbert space H is an unitary operator if and only if $U^*U = UU^* = I$.

OR

b Let $T = U|T|$ be the polar decomposition of an operator T. Then prove that $T = U|T|$ is quasinormal if and only if $U|T| = |T|U$.

8 a If T is an operator such that $\|I - T\| < 1$, then prove that T is invertible.

OR

b Let $\sigma(T)$ be the spectrum of an invertible operator T, then prove that $\sigma(T^{-1}) = \{\sigma(T)\}^{-1}$.

Cont...

9 a If T is a paranormal operator, then prove that the following properties hold:
(i) T^n is also paranormal for any natural number n.
(ii) T is normaloid operator, that is, $||T|| = r(T)$.
(iii) If T is an invertible paranormal operator, so is T^{-1}.

OR

b Show that an operator T is convexoid if and only if
$(\Sigma - \theta)\ Re\Sigma(e^{i\theta}T) = \Sigma(Re(e^{i\theta}T))$ for any $0 \leq \theta \leq 2\pi$, where $\Sigma(S)$ denotes $co\sigma(S)$.

10 a Analyze the statement of Young inequality, by providing its proof.

OR

b Analyze Lowner-Heinz inequality, by proving it.

SECTION -C (30 Marks)

Answer **ALL** questions
ALL questions carry **EQUAL** Marks (5 x 6 = 30)

11 a If T is an operator on a Hilbert space H over the complex scalars C, then prove that the following (i), (ii), (iii) and (iv) hold:
(i) T is normal if and only if $||Tx|| = ||T^*x||$ for all $x \in H$.
(ii) T is self-adjoint if and only if (Tx, x) is real for all $x \in H$.
(iii) T is unitary if and only if $||Tx|| = ||T^*x|| = ||x||$ for all $x \in H$.
(iv) T is hyponormal if and only if $||Tx|| \geq ||T^*x||$ for all $x \in H$.

OR

b Let P_1 and P_2 be two projections onto M_1 and M_2, respectively. Then show that
(i) $P = P_1 + P_2$ is a projection if and only if $M_1 \perp M_2$.
(ii) If $P = P_1 + P_2$ is a projection, then P is the projection onto $M_1 \oplus M_2$.

12 a Let M be a dense subspace of a normed space X. Let T be a linear operator from M to a Banach space Y. If T is bounded, then prove that there uniquely exists $\overline{T}$ which is the extension of T from X to Y, that is, $\overline{T}x = Tx$ for all $x \in M$ and $||\overline{T}|| = ||T||$.

OR

b Let T be a normal operator. Then there exists a unitary operator U such that $T = UP = PU$ and both U and P commute with V^*, V and $|A|$ of the polar decomposition $A = V|A|$ of any operator A commuting with T and T^*.

13 a State and prove Toeplitz-Hausdorff theorem.

OR

b State and prove Power inequality of $w(T)$.

14 a Prove that the following inclusion relations hold:
Self-adjoint ⊆ Normal ⊆ Quasinormal ⊆ Subnormal ⊆ Hyponormal ⊆ Paranormal ⊆ Normaloid ⊆ Spectraloid.

OR

b Prove that an operator T is convexoid if and only if $T - \lambda$ is spectraloid for all complex number λ.

15 a State and prove the Holder-McCarthy inequality.

OR

b State and prove the Furuta inequality.

Z-Z-Z END