20MAP18

PSG COLLEGE OF ARTS & SCIENCE
(AUTONOMOUS)

MSc DEGREE EXAMINATION MAY 2023
(Fourth Semester)

Branch – **MATHEMATICS**

OPERATOR THEORY

Time: Three Hours　　　　Maximum: 50 Marks

SECTION-A (5 Marks)

Answer **ALL** questions
ALL questions carry **EQUAL** marks　　(5 x 1 = 5)

1 The identity operator I is defined by
(i) $Ix = x,\ for\ all\ x \in H$　(ii) $Ix = x,\ for\ all\ x \in G$
(iii) $Iy = y,\ for\ all\ y \in H$　(iv) $Iy = y,\ for\ all\ y \in G$

2 An operator U on a Hilbert space H is said to be an isometry operator if
(i) $\|Ux\| = \|x\|$ for any $x \in G$　(ii) $\|Uy\| = \|y\|$ for any $y \in G$
(iii) $\|Uy\| = \|y\|$ for any $y \in H$　(iv) $\|Ux\| = \|x\|$ for any $x \in H$

3 Which notation is said to be the compression spectrum of T
(i) $\alpha(T)$　(ii) $\Gamma(T)$
(iii) $\chi(T)$　(iv) $\omega(T)$

4 $\overline{W}(T) = co\sigma(T)$, an operator T is said to be
(i) Transaloid operator　(ii) Condition operator
(iii) Convexoid operator　(iv) Normal operator

5 An operator T belongs to class A if
(i) $|T^2| \le |T|^2$　(ii) $|T^2| \ge |T|^2$
(iii) $|T^2| = |T|^2$　(iv) $|T^2| \pm |T|^2$

SECTION - B (15 Marks)

Answer **ALL** Questions
ALL Questions Carry **EQUAL** Marks　　(5 x 3 = 15)

6 a. If T be an operator then prove that $\|T^*T\| = \|TT^*\| = \|T\|^2$

OR

b. If an operator P is a projection, then prove that $\|x\|^2 = \|px\|^2 + \|(1-P)x\|^2$.

7 a. If an operator U on a Hilbert space H is a unitary operator if and only if $UU^*=U^*U=I$.

OR

b. Let T = UP be the polar decomposition of an operator T. Then T is normal if and only if U commutes with P and U is unitary on $N(T)^{\perp}$.

8 a. If an operator T is normal, then prove that $R_\sigma(T) = \phi$.

OR

b. If T is a normal operator, then prove that T is normaloid, i.e., $\|T\|=r(T)$.

9 a. Let X be a positive invertible operator and Y be an invertible operator. For any real number λ, $(YXY^*)^\lambda = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*$

OR

b. If an operator T is convexoid if and only if $T-\lambda$ is spectraloid for all complex number λ.

Cont...

10 a. Prove that every invertible p-hyponormal operator is a log-hyponormal operator.

OR

b. Prove that every class A operator is a paranormal operator.

SECTION -C (30 Marks)

Answer **ALL** questions

ALL questions carry **EQUAL** Marks (5 x 6 = 30)

11 a. Let P_1 and P_2 be two projections onto M_1 and M_2, respectively. Then
(i) $P = P_1 + P_2$ is a projection if and only if $M_1 \perp M_2$
(ii) If $P = P_1 + P_2$ is a projection, then P is the projection onto $M_1 \oplus M_2$

OR

b. Let T be an operator on a Hilbert space H. Then T* is also an operator on H, and the following properties hold:
(i) $\|T^*\| = \|T\|$
(ii) $(T_1 + T_2)^* = T_1^* + T_2^*$
(iii) $(\alpha T)^* = \bar{\alpha} T^*$ *for any* $\alpha \in C$
(iv) $(T^*)^* = T$
(v) $(ST)^* = T^* S^*$

12 a. Let S and T be bounded linear operators on a Hilbert space H. If T*T = S* S, then there exists a partial isometry operator U such that the initial space $M = \overline{R(T)}$ and the final space $N = \overline{R(S)}$, and S=UT

OR

b. State and prove Fuglede-Putnam theorem.

13 a. State and prove Power inequality of w(T).

OR

b. State and prove Spectral mapping theorem.

14 a. If an operator T is convexoid if and only if $\|(T-\mu)^{-1}\| \leq \frac{1}{d(\mu, co\sigma(T))}$; *for all* $\mu \notin co\sigma(T)$

OR

b. State and prove Lowner-Heinz inequality.

15 a. Let $A \geq 0$ and T = U|T| be the polar decomposition of an operator T. Then for each $\alpha > 0$ and $\beta > 0$, the following statements hold:
(i) $U^*U(|T|^\beta A|T|^\beta)^\alpha = (|T|^\beta A|T|^\beta)^\alpha$
(ii) $UU^*(|T^*|^\beta A|T^*|^\beta)^\alpha = (|T^*|^\beta A|T^*|^\beta)^\alpha$
(iii) $(U|T|^\beta A|T|^\beta U^*)^\alpha = U(|T|^\beta A|T|^\beta)^\alpha U^*$
(iv) $(U^*|T^*|^\beta A|T^*|^\beta U)^\alpha = U^*(|T^*|^\beta A|T^*|^\beta)^\alpha U$

OR

b. If an operator T is absolute-k-paranormal for some k > 0, then T is normaloid.

Z-Z-Z END