Health Economics

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Week – 10

Lecture 49- Data Envelopment Analysis: CCR Model

Welcome, friends, to our NPTEL MOOC module on Health Economics. We have been dealing with this week with the discussion on health efficiency as we have already clarified the different concepts of efficiency. In this lecture, we will be discussing Data Envelopment Analysis. We already gave you the background of DEA in the last lecture. We are going to give you further details and their models, as well as how mathematically they are constructed, and explain them with clarity. Hence, here we will be discussing the input and output CCR model, which we have already discussed here.

And we will be explaining you the slacks in CCR as well and their conditions on efficiency scores and slack values. So, let us go back to our previous lecture in a minute and we discussed the basic DEA especially we emphasize on data driven approach, we discussed DEA assesses efficiency in two stages. Especially in the data-driven approach, we have discussed the noneconometric and non-parametric models, and the DEA especially assesses efficiency in two stages. One is through your input or output orientation and second is assigned an efficiency score that is basically called piecewise linear envelope model.

And in the third input orientation model, we emphasize on how to keep output fixed and explore the proportional reduction in input usage which is deriving the best output or output to be fixed and optimizing the use of inputs that is called input orientation model. In the output orientation, it is just the reverse; keep the input constant and explore the proportional expansion in output. And how these are discussed in detail we will clarify further. In the basic DEA model, we start with the orientation that is maybe input oriented or output oriented. Input-oriented, we have the model through the CRS or the variable returns to scale model.

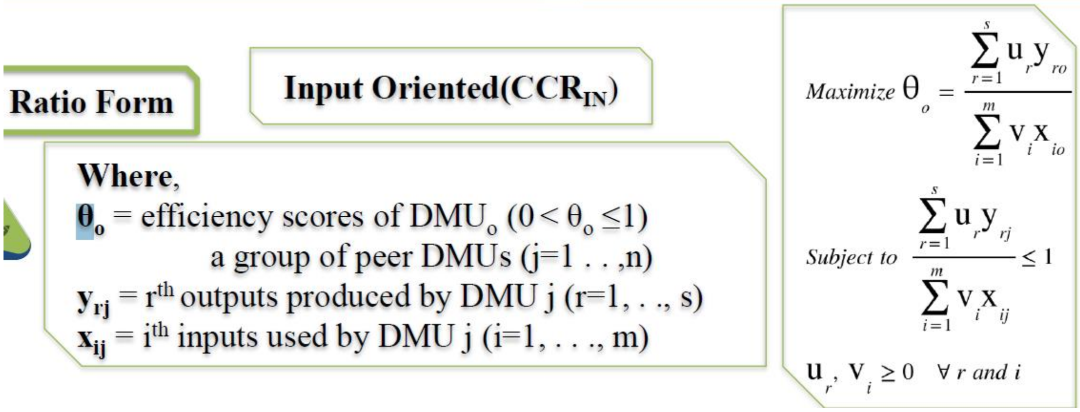
In the CRS, we have CCR models. CCR stands for the names of the authors who started it: Charnes, Cooper, and Rhodes. CCR follows CRS, which is a constant returns to scale model with their assumption. In VRS, the variable returns to scale where it was initially introduced by Banker, Charnes, and Cooper; that is why it is called BCC. In the output model, again, through the output orientation model, we can have CCR and also BCC models.

So, through the CCR, we will discuss their input, CCR inputs, or BCC input. And similarly, in other cases, in the output orientation, we will be emphasizing the CCR output and BCC output. There are other models especially emphasized called additive models, and weight-restricted models, also called multiplier models. And the last mostly also discussed in the paper on slag based models. We will also explain what Slacks are.

Let us start with that CCR model. It is the first model introduced in 1978, based on Farrell 1957. Later on, the CCR of 1978 is noted, and they assume the CRS function, Constant return to the scale production function. DEA forms a frontier using efficient organizations. So, originally assumed input orientation which we have already started. Later, they even extended the output orientation model. That is, Cooper et al. 2007 paper suggests that to have an adequate number of degrees of freedom, the size of the n should exceed the number of inputs and outputs several times. This is also considered a rule of thumb, and n must be greater than that of inputs and outputs. There are different formulations and estimations based on three forms of CCR.

We start with the ratio form, which measures efficiency in the ratio form. This is also called fractional form. In the input-oriented model where CCR input is written, we have already clarified where, in the ratio form, the attempt is to maximize or maximize the ratio of weighted multiple outputs to weighted multiple inputs. Basically, it is output to input, and its respective weights are taken, as well as the respective weights of output and inputs. This is subject to the ratio of other DMUs. Other DMUs are basically the particular DMUs that we have started explaining for their output and inputs, so whether some of the other DMUs must be less than 1, then only we can discuss the possible optimization level through the input-oriented model.

So, this is how it is explained. So, the attempt is to θ refer to the maximization of output to input. So, here Y stands for the output and X stands for the input, output variables and there might be R^{th} number of output variables. Even for the respective DMUs, it stands for the organizations or DMU. And the respective weights are U and V for the output and inputs, respectively, subject to other DMUs; there are some of other DMUs with their output oriented to that of the inputs or their efficiency ratio that must have been less than that of 1 because we have clarified very clearly that the optimum level reaches at 1 in terms of ratio.



Therefore, J here stands for other DMUs. So, I am just giving the direction. This is actually called other DMUs. So, the respective weights must be positive for all R^{th} levels of output and i^{th} levels of inputs. Hence, our objective function is highlighted here.

This is to maximize the efficiency ratio subject to the constraints of other DMUs. We have just clarified this is precisely called ratio form. U and V are the respective weights of the decision variables, and the optimum value of U and V is the solution for the efficiency ratio. Next, in the ratio form, while we are going through the input-oriented model through the CCR, we have just mentioned once again that this theta of the O or O stands for the organizations or the DMU O unit of the organization. This is precisely called the efficiency score.

This must be at maximum of till 1, 0 till 1. If it reaches 1, that means the efficiency score is considered to be the best. Hence, a group of other DMUs that must be accounted for is for J till the end, which must be less than 1. Y_{rj} stands for R^{th} outputs produced by the DMUs. DMU_j, so those must be till one till s. And X_{ij}, that is, i^{th} inputs of DMU_j that must be till 1 to M. U and V should be positive when a number of constraints vary to the number of DMUs. The weights are derived from the data and used to be considered to be endogenous, and from the model itself, we will identify. So, in the optimization problem, this gives the most favorable weights to DMUs, that is, the Oth unit that constraints allow weights, the best possible weight, to be identified, and no other sets of weights lead to a higher level of efficiency. This means that the input-output ratio for each DMU is maximized relative to that of other DMUs. So, it is a constant function. We have clarified that these DMUs are comparatively better than all other DMUs.

Coming to the multiplier form suggested by Charners and Cooper, they suggested transforming the ratio form. The ratio form is considered to be complex because it is considered a number of infinite solutions, and usually, since it is fractional in nature, whereas the multiplier form, it is attempts to simplify the procedure and takes a linear function, also called primal linear programming. This also follows the input-oriented format. Hence, the equation is maximizing that theta, theta stands for the efficiency score, and that is basically a linear combination of its respective weight of the output of the DMU unit and r^{th} number of output of that DMU and subject to the gap between the output to input and the additional value with their respective weights of the output or the j^{th} units of output to that of j^{th} units of inputs that must be less than 0.

$$\text{Maximize } \theta_o = \sum_{r=1}^{s} u_r y_{ro}$$

$$\text{Subject to :}$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0$$

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$u_r,\ v_i \ge 0$$

It should not be positive. The gap should not be positive. Another condition is that the respective weight of the i^{th} inputs must be equal to 1. Input and its respective weight value summation should be 1. So, here, we have to note that it is not just the constant that is of the nth unit because we have taken another constant function. So, the number of constants increases to n+1. Ratio and multiplier forms are considered to be equivalent in nature, and we have taken notations as per the previous model.

The most important usage of the efficiency score is through the development of the envelopment form. The envelope we have already mentioned in the earlier lecture that it envelops the scores. We will clarify this, and we will just compare the multiplier form that we have just discussed.

The objective function was to maximize, whereas the envelopment function was to minimize the objective function since it is the input-oriented model. Again, here, the constraints are to out of the m^{th} input constraints, the efficiency score times its m^{th} inputs, which is the maximum score possible in the case of inputs. So, λ times its Xij should be less than that of the maximum score. Then, in the case of output also, since we get the input score, the minimization of the input score is through the envelopment form.

$$\text{Minimize } \theta_o$$

$$\text{Subject to :}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io} \quad i = 1, 2, 3, \ldots .m \ (\text{input constraint})$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} \quad r = 1, 2, 3, \ldots, s \ (\text{output constraints})$$

$$\lambda_j \ge 0 \quad j = 1, 2, 3, \ldots\ldots, n \ (\text{non-negativity constraints})$$

In the output-oriented model, we will find just the reverse. So, here you need to note that the number of output constraints has to be till S. And then what is this lambda of the j^th^ unit? These have to be positive or non-negative constraints. I will clarify further that this **λ** is basically the raw weights assigned to the peer DMUs.

Again, since we have taken the summation with respect to weight, we are modeling through the linear program, and there are dual formulations as well. This seeks efficiency by minimizing the efficiency of a focal, that is, the DMU O unit, Oth unit subject to two sets of inequality. First, inequality emphasizes that the weighted sum of inputs of the DMUs should be less than what we have already said, less than or equal to the inputs of the focal DMU being evaluated. The sum of the other DMUs and their inputs must be less than that of this. Where in the second inequality, we identify the constraint function with respect to the output. The inequality asserts that the weighted sum of the outputs of the non-focal DMUs should be greater than, should be higher than that of the respective focal DMU.

As I already discussed, there must be some slacks. Slacks will basically identify the best and optimum combination of inputs in the output model. We will find out which one is indeed the best. Slacks exist if at least one input could be reduced further without using other input or if at least one output increases given the level of input.

With the same input, we can increase the output, or with the same output, we can reduce the input. That is basically called slacks. You can easily see that we know that this is not the optimum one so far as inputs are concerned. This is precisely why A' is considered to be the optimum one, but we can clarify further whether it is the best one. So, basically, with the lower inputs, we can still attain the optimum level of output.

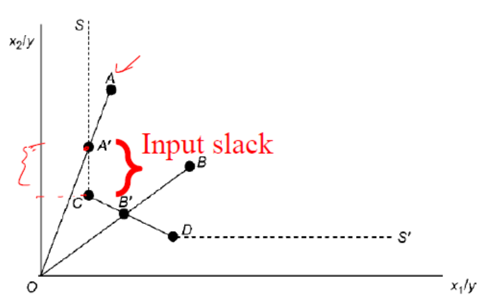


Fig: Efficiency measurement and input slacks.

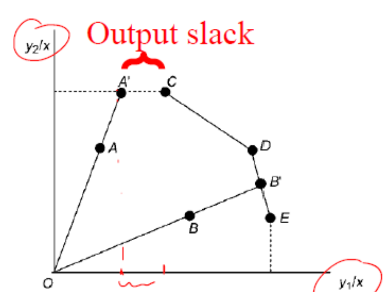


Fig: Efficiency measurement and output slacks.

Therefore, this is called input slack, and X_2 can be minimized. Similarly, you can explain the X_1 level further. And so, A'C is called input slack. Similarly, in the case of output slack, the variables should be reversed. So, we know that the output A' to C, A' is also optimum, and C is also optimum, but the best one is that we can actually minimize or maximize output with the given level of inputs.

So, this is what is called A’ to C, the amount by which output Y_1 can still be expanded. So, here, this distance corresponds to this distance, and this distance can be expanded with the same level of inputs. So, in the envelopment form, only technical efficiency scores are discussed. We have already discussed earlier what allocative efficiency and technical efficiency are. Here, in the envelopment form, we are emphasizing the technical efficiency aspects.

This does not account for the slacks in the basic form of envelopment analysis. This fails to account for slacks or sometimes overestimation of technical efficiency as per the Farrell estimates for those DMU units operating with slacks. So, you can understand what is the relevance of other advanced models since slacks are not clearly accounted by the envelope form. Various ways of dealing technically with slacks, such as Bessent et al. in 1988 paper, Torgerson, Forsund, and Kittelesen in 1996, and Tofallis 2001, etc., have also been discussed. A 1993 paper by Ali and Seiford proposed technical efficiency by means of a second-stage DEA linear programming problem. This is done by taking the theta value from the first stage linear programming problem. Basically, in the second stage, we can identify whether their change is still positive, zero, or negative, and accordingly, we can identify the slacks. Running a second-stage linear programming problem and setting the input and output slacks to zero. You can understand accordingly whether it is output-based slacks or input-based slacks.

The model of slacks has two phases. Phase 1 obtains θ* for organizations that is DMU unit U or organization of O from the envelopment model. This is what it looks like. We have started with the envelopment form. Hence, our target is to minimize theta. Subject to the basic envelopment form, we already clarified its input constraint, output constraint, and non-negative constraints. So, this is the first stage. The second stage, as I already mentioned, will maximize the slack. It is related to slack. The first one is the basic envelopment form.

$$\textit{Minimize } \theta_o$$

$$\textit{Subject to}:$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io} \quad i = 1, 2, 3, \ldots m \ (\textit{input constraint})$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, 3, \ldots, s \ (\textit{output constraints})$$

$$\lambda_j \geq 0 \quad j = 1, 2, 3, \ldots, n \ (\textit{non-negativity constraints})$$

The second one is maximizing the slack. So, maximize slack with respect to Slack positive and Slack negative.

$$\text{Maximize} \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+$$

Subject to :

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta^* x_{io} \qquad i = 1, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} \qquad r = 1, \ldots, s$$

$$\lambda_j \geq 0 \qquad j = 1, \ldots, n$$

Where:
S_I^- → Input slack
S_r^+ → output slack
θ^* → DEA efficiency score resulting from the initial

So, we can also check the i^{th} unit and the r^{th} unit. I think of inputs we already mentioned, and slack might be in terms of inputs and output. So, if it is input-based, we have mentioned it as minus and output-based; it is positive towards more output with the same inputs. So, then, it is subject to it again with the same modified version of the constraints where slacks are mentioned. You can just see the changes, and the rest are almost related to the input constraints and output constraints, and accordingly, we can identify its maximization.

So, here it is given as very clearly defined minus it is written as input slack and plus it is output slacks. And θ^* is basically what we already discussed as a DEA efficiency score resulting from the initial run. First, θ will derive the initial calculation, and then, based on that, we can restrict it to its constraint function. We have to clearly mention that this superscripted minus sign is on the input, and the plus sign is on the output.

And that is related to slacks. And superscripted minus sign on input slacks that indicates reduction. Slacks basically means that the objective of the input context is to reduce. Hence, it is a negative one. Where in the case of a positive sign, the objective is to increase the output; hence, it is an augmentation of outputs. So, with the input model through the CCR, we have already started using slack again. In the initial model, we did not discuss it with slack. The minimization of efficiency score minus the sum of its slacks. So, basically, this is the efficiency score, which is already said minus what these slacks are and if it is input-oriented. The rest are almost the same and can change.

$$\text{Minimize } \theta - \sum \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right)$$

Subject to :

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta\, x_{io} \qquad i = 1, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} \qquad r = 1, \ldots, s$$

$$\lambda_j \geq 0 \qquad j = 1, \ldots, n$$

Then I will just clarify: we already discussed what is called efficient DMUs and what is called fully efficient DMU.

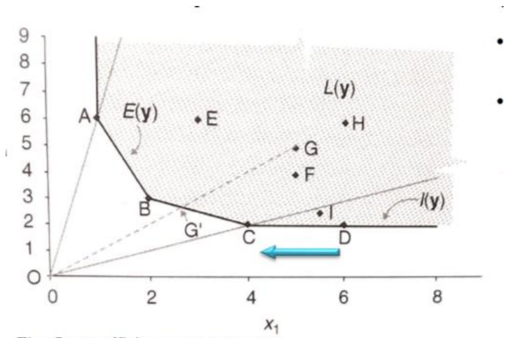


Fig.: Input efficiency measurement

Again, we can see C and D. If you just compare this in the input-oriented model, our input orientation clearly defines C as a fully optimized or fully not optimized efficient score. So, DMU D on the isoquant is called efficient with an efficiency score of 1, but not Pareto efficient since we can also reduce the X1 content to reach the one level of efficiency score. It is not Pareto efficient because we can reduce input X1 without using X2. The efficiency of 1 or 100 percent for a DMU does not necessarily mean it is Pareto efficient, but the converse is true. Fully efficient and weakly efficient DMUs We have clarified that the performance of DMUs can be assessed either as fully efficient or weakly efficient.

Conditions on efficiency scores and slack values. So, what are the conditions to reach at that fully efficient level and weakly efficient cases? So, here, the condition is that the fully efficient it has to be 1 or even, in another case, that is also with theta star also be 1, and with input slacks, that has to be 0, and even output slacks, it has to be 0. In weakly one, the other is the same, whereas in the case of slacks, at least one input should be non-zero, which is the constant we have already discussed in our model. And in another one, in the output case, at least 1 or r^{th} output should be non-zero. In input-oriented CCR models, there are levels of efficient targets for inputs and outputs. So, some targets that have to be discussed are efficient targets for the inputs and outputs.

So, as we already said, the target is from one level to another level. In the case of inputs, the target is to reduce from the efficient till the slacks the negative content should have been reduced. Therefore, a minus is given. However, in the case of output, it has to be added; more positive slacks should be added to the output. Therefore, a target symbol is given. Every time we have discussed the r^{th} level of output or the level of input. So, we have discussed the CCR input in detail.

In short, in the CCR output-based oriented model, we use ϕ. ϕ, in that case, is basically maximized. Usually, we say it in the DEA formulation or the envelope formulation. We discussed the minimize function; now we are discussing the maximize function since it is output-oriented.

$$\begin{aligned}&Maximize\ \ \phi-\varepsilon\left(\sum_{i=1}^{m}s_i^-+\sum_{r=1}^{s}s_r^+\right)\\&\sum_{j=1}^{n}\lambda_jx_{ij}+s_i^-=x_{io}\quad i=1,\ldots.m\\&\sum_{j=1}^{n}\lambda_jy_{rj}-s_r^+=\phi y_{ro}\quad i=1,\ldots.s\\&\lambda_j\geq 0\quad j=1,\ldots.n\end{aligned}$$

Rest are precisely the same. You can easily follow. However, we have also discussed this: the efficiency emphasis is removed from input and placed into output. That is basically the second constraint in that case. So, you can just check an efficient target calculation for an output-oriented CCR model.

$$\begin{aligned}&\text{Inputs: }\widehat{x}_{io}=x_{io}-s_i^{-*}\quad i=1,\ldots.m\\&\text{Outputs: }\widehat{y}_{ro}=\phi^*y_{ro}+s_i^{+*}\quad r=1,\ldots.s.\end{aligned}$$

Again, in the case of target calculation, you can just check. Here, we have also identified, and this is how it is very clearly discussed. So these are all so far as DEA and its extension are concerned. We will also be giving you further directions related to its applications in our next lectures. So these are important readings for your preparations. I think you will have a number of queries, and we will be happy to address them in our query section. Thank you. So before I thank you, I think I should clarify what is there in our next lecture. We will be discussing the BCC model because we have only discussed the first one so far.

This is the one CCR. So, in the next one, we are discussing the BCC model and another econometric measurement of efficiency. So, I suggest you follow our next lecture for further clarification. Thank you.