**Environmental & Resource Economics**
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**Dynamic Optimization and Renewable Resources Part - 3**

So, welcome once again to our discussion on resource economics and we are discussing dynamic optimization and in our previous class we are trying to understand how the process of optimum control is involved in the context of dynamic optimization. We derived three conditions from our Lagrangian expression and the Lagrangian expression was derived from the objective functional where we have we said that we have a resource and we are trying to derive some kind of benefits from this resource and the benefits are of two types, flow benefit and the scrap value.

Scrap value and then from that, we derived the Lagrangian expression and optimizing that Lagrangian function we derived three conditions and those three conditions we said that these three conditions are called maximum principle and the entire process is known as dynamic optimum control, the entire process is known as optimum control then at last we were discussing one particular example what is that.

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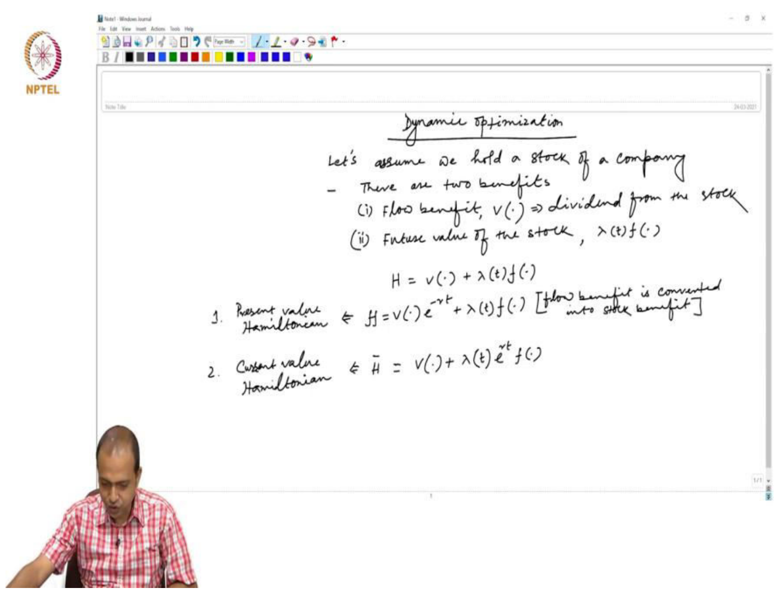


So, this is dynamic optimization. So, let us assume we hold a stock of a company so, if he hold the stock of the company then there are two benefits, what are those number one flow benefit and this flow benefit is given by that v function.

The v function and then secondly, there is a another benefit which is flow benefit denoted by v and that indicates dividend of dividend from the stock that is flow. So, if you buy and hold the stock of the company the company will pay you dividends as long as you hold the stock that is called flow benefit.

And at the end, when you sell the stock to someone else, then what you will get the other benefit which is called future value of the stock and that is denoted by lambda t into f dot and we said that the H function we define one function H equals to v plus lambda t f dot. So, this is flow benefit and this is called future value of the stock.

So, some kind of stock benefit. Now, this stock benefit is realized at one point of time, but this is a stream of benefits. So, that would be realized over a period of time. Now, suppose I want to convert this flow benefit into stock value stock in a sense this is when I say stock means at one point of time what would be the benefit.

So, if we want to convert that stream of benefit at present time, then we need to multiply this function with e to the power minus r t that means a stream has to be discounted to get the present value of the flow benefit. So, then what we can write that if we multiply this equals to v function e to the power minus rt plus lambda t into f dot.

So, I have simply converted the stream or flow benefit at one point of time by discounting that and if we do so, then this H function is known as Hamiltonian function and that is called present value Hamiltonian. So, here we have converted flow benefit into one point of time. So, here flow benefit is converted into stock benefit by stock I mean the benefit which is realized at present instead of getting the benefit over a period of time I want to get it now.

So, then obviously, you need to convert this v function into present time by discounting it where r is the discount rate. So, you have to multiply that with e to the power minus r t. So, this is one type of Hamiltonian function. Similarly, what we can do let us say I want to convert this future benefit which would be realized at one point of time into a stream like this into a flow.

So, what we need to do and then we need to multiply this by e to the power let us say I am doing this v I am keeping as it is and this I am multiplying with e to the power r t into f dot, and that function is what I am saying so, this is then I am doing this is let us say H bar I am writing and this H bar is known as current value Hamiltonian. Now do not get confused with the current and present though the English meaning of present and current both are same here these are two different functions we need to understand conceptually here I have converted the stock into a flow and then I am adding it, the idea is in this H function I have two types of benefit one is flow and another one is stock.

Now when you are adding we can convert the flow into stock or stock into flow by multiplying either e to the power minus r t or e to the power r t that is the idea. Now, let us assume that or we can write. So, this is from here in the next page or what I will write H bar equals to this I have written.

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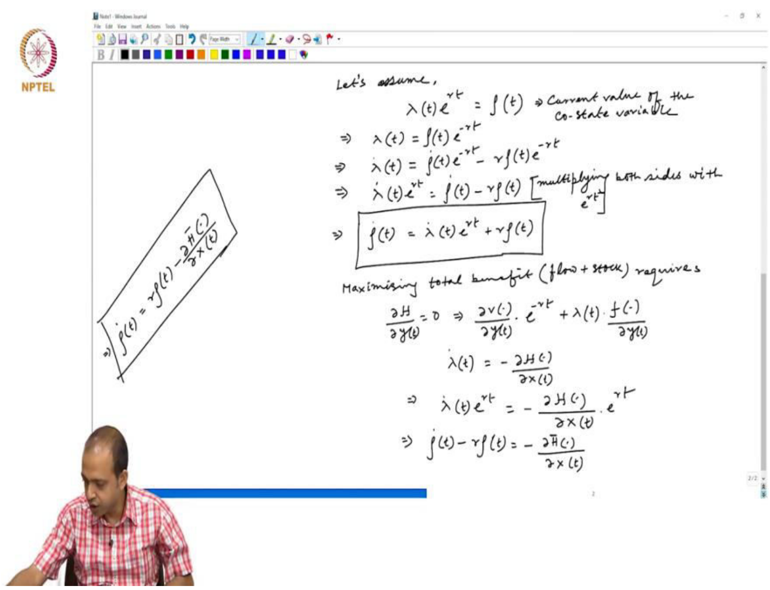

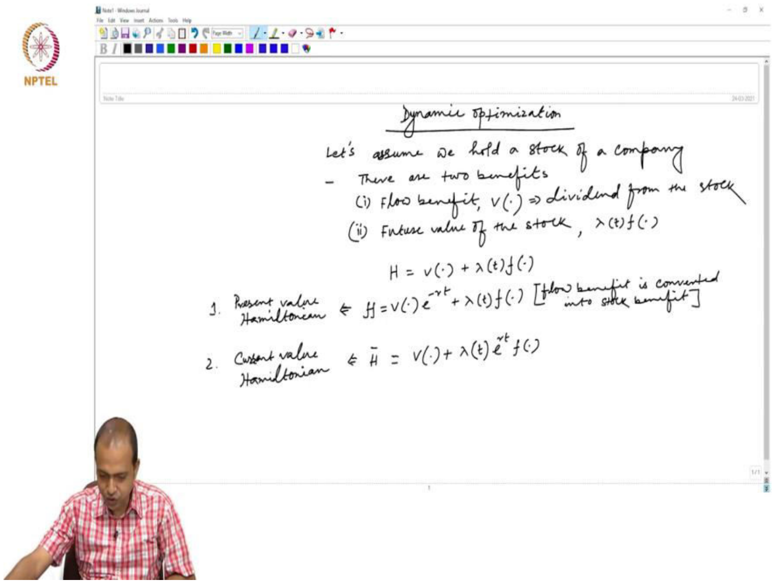


Now let us assume lambda t into e to the power r t equals to rho t. So, this rho t is known as current value of the co-state variable. Now, if you look at this rho t is how we are getting lambda t multiplied by e to the power r t. Now lambda t if you go back and see this lagrangian multiplier lambda t is we have depend the co-state variable because it was attached with the state variable x t.

So, the interpretation of lambda t is the co-state variable or we have also defined lambda t is the marginal preservation benefit if you preserve the state variable the stock then what is the benefit since lambda t was attached with the state variable x t that is the y this lambda t is known as co state variable.

Since we are multiplying this co-state variable with e to the power r t we said that rho t is the current value of the co-state variable. So, from here what we can say that that means lambda t equals to rho t into e to the minus r t. So, if you from here what I can say that lambda dot t dot means differentiating with respect to t rho dot t equals to then what I will write, sorry lambda dot t would be equals to rho dot t into e to the power minus r t minus r into rho t into e to the power minus r t.

Now, if you both side multiplied by e to the power r t then what you will get lambda t into e to the power r t equals to then the rho dot t minus r into rho t. How we are getting it multiplying both side by e to the power r t.

So, from here what we can get rho dot t equals to lambda dot t e to the power r t plus r into rho t. So, this is a condition which would be used later on. So, far why we are discussing this mathematical procedure because this results this would be useful when we talk about the optimum extraction of a non-renewable or renewable resource that means, while deriving the optimum extraction path, the small small mathematical conditions what we are deriving those would be useful.

Now, previously we have defined the H function particularly the present value Hamiltonian equals to this. So, present value Hamiltonian if we differentiate that means when you are trying to maximize the flow and stock these two benefit what we have to do we need to differentiate this Hamiltonian with respect to your decision variable which is nothing but yt very simple uncertainty equals to 0.

So, this present value Hamiltonian is nothing but representing two types of benefits that we can get from a resource or from an asset. One is flow another one is stock benefit. And here we have just made some changes. Some modification to bring both the benefits into same platform that is all, either flow into stock or stock into flow.

So, here it is flow into stock. If we derive then we will get the present value Hamiltonian and if we try to maximize then maximizing total benefit. So, flow plus stock requires, what is required del H del y t equals to 0 that is all. Now if you differentiate the th function what you will get you will get del v dot del y t into e to the power minus r t plus lambda t into del f dot del y t.

Because y t was there in v function y t was there in this f function also. So, differentiating Hamiltonian with respect to it means you have to differentiate this and we also know that lambda t is equals to minus delta H dot delta x t. So, this we have derived earlier one of the conditions from our maximum principles.

So, this is what we got so, that means from here if we differentiate this lambda dot t equals to minus del H dot del x t. So, this is we got earlier and then multiplying these with e to the power r t will give you minus delta del x t into e to the power r t and this lambda dot t into e to the power r t is nothing but from here what we can get rho dot t minus r into rho t.

So, this we can write rho dot t minus r into rho t equals to then when you multiply this with respect to multiply with e to the power r t then you will get the current value Hamiltonian which is nothing but minus del H bar del x t that is what we got. So, from here or you can write rho dot t equals to r into rho t minus del H bar del x t.

So, this condition also we have to keep in mind this will also be useful. So, this condition you have to sometimes you have to go back this how lambda dot t equals to del H dot del x t that we have defined earlier. Now using this condition so, what we will do? We will now try to understand how these conditions are useful in the context of a non-renewable resource extraction.

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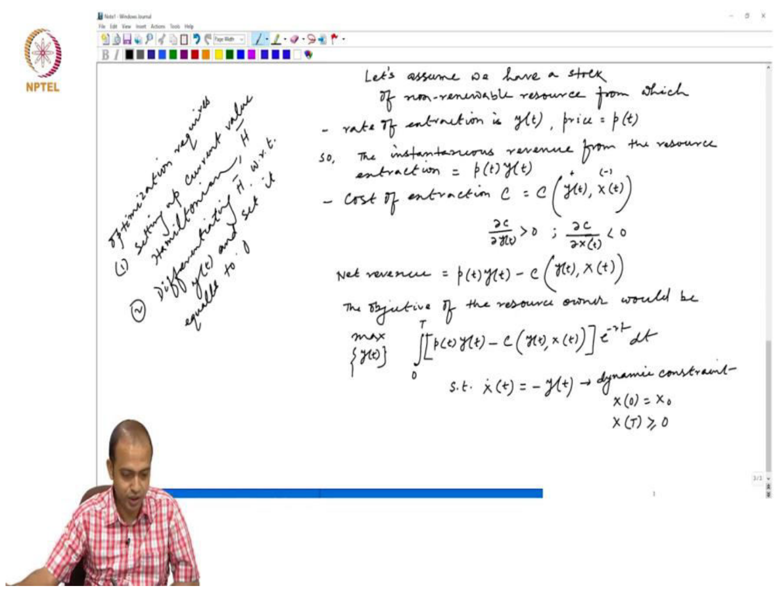


Let us now assume we have a stock of non-renewable resource from which let us say rate of extraction is y t and price equals to p t price per unit of that. So, this implies so, the instantaneous revenue from the resource extraction is equals to p t into y t very simple. Let us also assume cost of extraction C is a function of it should be a function of your rate of extraction and this.

So, obviously cost of extraction is a function of the rate at which you are extracting and the stock what you are having. So, it is positive and it is negative if you have more stock cost of extraction is less if you have rate of extraction is more than cost is more. So, that is how we assume that del c del y t is positive and del c del x t is negative.

So, net revenue or net benefit equals to then p t into y t minus C of y t to x t this is the net revenue and this is something the resource owner is trying to maximize over a period of time. So, that means the objective of the resource owner would be maximize this entire benefit 0 to T then pt into yt minus c into C function e to the power minus r t dt.

That is what the resource owner is trying to do. So, that means maximize this and what is the control variable once again the yt and what is the constant here, the constraint is x dot t that means change in stock is negatively related with rate of extraction this is called the dynamic constraint we talked about earlier dynamic constraint and x 0 equals to let us say some stock and x T is greater than equals to 0.

So, that means we have some positive scrap value. So, this is the objective of the resource owner. So, what the resource owner has to do the optimization requires, so what will do to optimize this objective functional we the first step is to set up the Hamiltonian function, the current value Hamiltonian, and then what will do we will try to differentiate the current value Hamiltonian with respect to yt and set it equal to 0.

So, that means, optimization requires two things setting up current value Hamiltonian and then number two differentiating current value Hamiltonian which is let us say H c or H bar and second step is differentiating H bar with respect to y t and set it equals to 0, these are the two steps we need to follow. So from this current value Hamiltonian would be simply the revenue minus the opportunity cost of extracting the resource today.