Environmental and Resource Economics
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Dynamic Optimization and Renewable Resources Part -2

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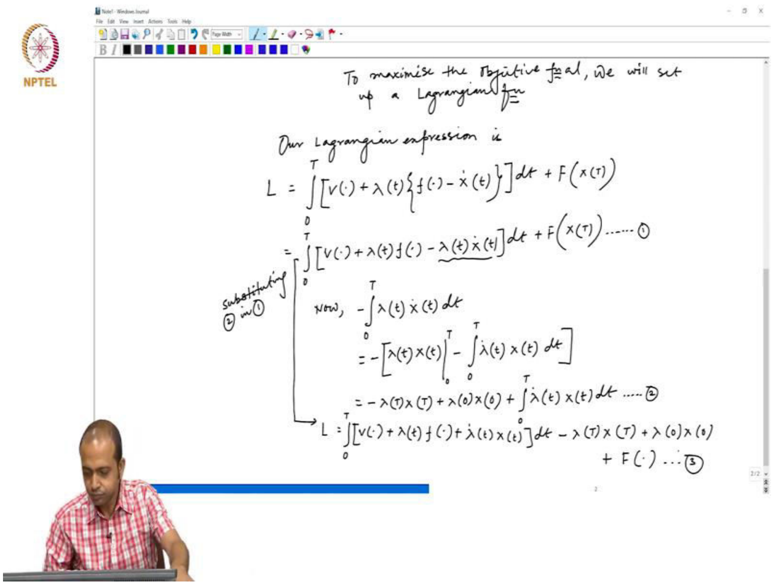

Is this clear, so basically I am substituting whatever expression I got from lambda t x dot t, here and then substituting 2 into 1 we are getting V dot lambda t this and then in place of this we are just putting this one, because this is under integration, so this part we are just putting here and other things these are all constant thing, minus lambda t xt, these are the things we are putting here.

It is very simple, so if you express this, from this entire thing the moment you put capital T and 0 these are all becomes kind of parameter things we are putting it separately and the integration is still operational on lambda dot t x t. So that is why within the bracket we are putting lambda dot t x t and then we are putting the integration, that is the thing.

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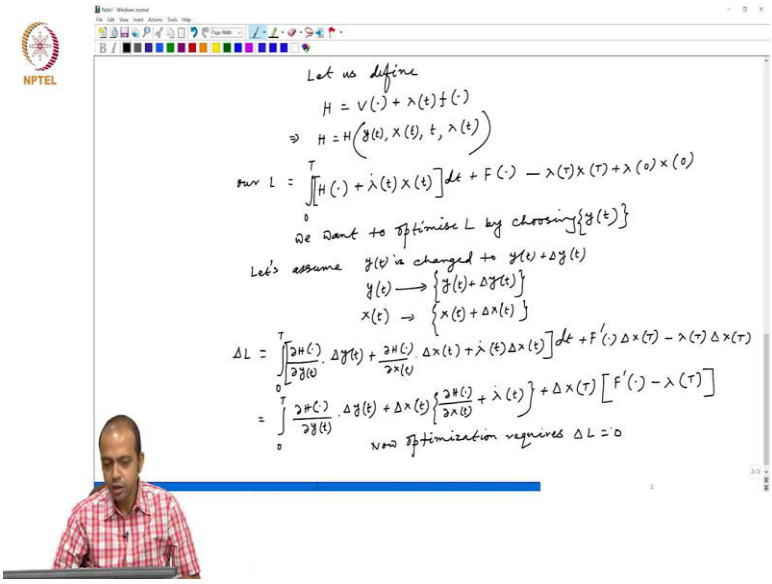

Now in the next step what we will do, let us define capital H equals to V dot plus lambda t into F dot, so if we do so see this V is actually a function of if you recall yt, xt and t, so ultimately this H will also become like this. So this implies that H is a function of yt and then xt this t and lambda t. This is how instead of V dot lambda t F dot we are just putting H, so our Lagrangian expression would become much more precise.

In that case our L is then equals to integration 0 to t instead of this V dot lambda t F dot we are we are putting H dot plus lambda dot t and then xt and entire thing is dt plus F dot. The other parameter thing minus lambda t capital T and then x capital T plus lambda 0 x 0, this we are putting as it is. What is our objective? We want to optimize L by choosing what is our control variable yt, by choosing yt that is our control variable.

So let us assume yt is changed to yt plus let us say delta yt, so we can write precisely yt is now changed to yt plus delta yt, and the moment yt changed into delta yt that means your rate of extraction is changed by delta yt amount. Obviously, that would give an impact on your state variable as low, so your xt then will changed into xt plus delta xt, which is very simple, xt plus delta xt. If that is the case if we take total differentiation of the Lagrangian then delta L would become integration 0 to t this del H dot del yt into delta yt plus this x, delta xt delta plus del H dot del xt, because H is a function of y as well as x, into delta xt plus lambda dot t into delta xt into dt.

So entire thing will become in the dt plus we will have this into delta xt, because here within the scrabble F is also a function of xt the stock of the resource at the terminal period, minus into delta xt minus lambda t this x would become delta xt. So this we can rearrange and write as equals to integration 0 to t del H dot del yt into delta yt plus this lambda, delta xt we will take common, delta xt we will take common.

So delta xt if you, sorry if you take delta xt common then what will happen? This would become del H dot del xt plus lambda dot t, this is just simply taking common. From here also we can take delta xt common plus delta xt will take common, so this will become F dot minus delta sorry lambda t. Now optimization requires what is the optimizing condition? Optimization requires delta L should be equals to 0.

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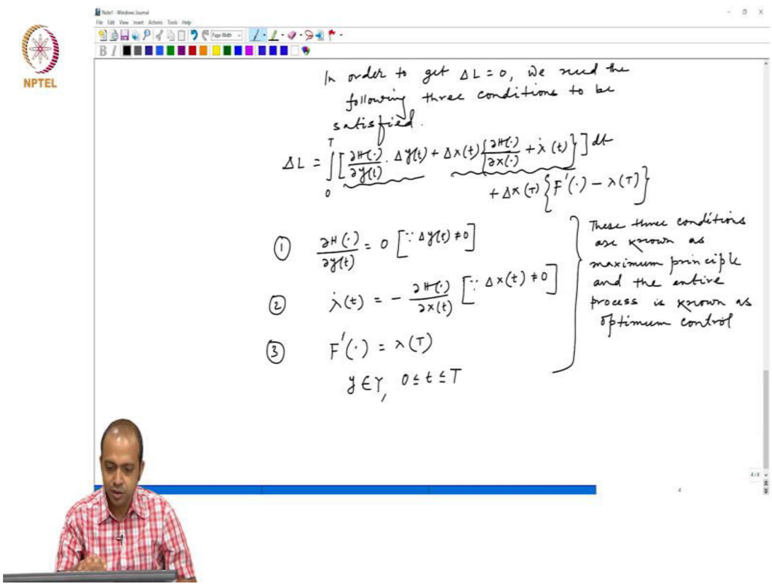

To maximise the objective fnal, we will set up a Lagrangian fn

$\lambda(t)$: marginal benefit of preserving the resource.

Our Lagrangian expression is

$$L = \int_0^T \left[V(\cdot) + \lambda(t)\{ f(\cdot) - \dot{x}(t) \} \right] dt + F(x(T))$$

$$= \int_0^T \left[V(\cdot) + \lambda(t) f(\cdot) - \lambda(t)\dot{x}(t) \right] dt + F(x(T)) \;\ldots\ldots (1)$$

Now, $-\int_0^T \lambda(t)\dot{x}(t)\,dt$

$$= -\left[\lambda(t)x(t) \Big|_0^T - \int_0^T \dot{\lambda}(t)x(t)\,dt \right]$$

$$= -\lambda(T)x(T) + \lambda(0)x(0) + \int_0^T \dot{\lambda}(t)x(t)\,dt \;\ldots\ldots (2)$$

substituting (2) in (1)

$$L = \int_0^T \left[V(\cdot) + \lambda(t) f(\cdot) + \dot{\lambda}(t)x(t) \right] dt - \lambda(T)x(T) + \lambda(0)x(0) + F(\cdot) \;\ldots (3)$$

Dynamic optimization

Let us assume that we have an asset / a resource stock from which we want to derive two types of benefits — (i) flow benefit (ii) scrap value.

- our objective here is to maximise the total benefit (flow + scrap value) from the resource

Let's denote V is the flow benefit and F is the scrap value.

our objective here is to

$$\max_{\{y(t)\}} \int_0^T \left[V(y(t), x(t), t) \right] dt + F(x(T))$$

$$\text{s.t.} \quad \frac{dx(t)}{dt} = \dot{x}(t) = f(y(t), x(t)) \rightarrow \text{eqn of motion or dynamic constraint}$$

$$x(0) = a \text{ (cont)}$$

Here

$y(t)$: decision var or control var. i.e. rate of extraction

$x(t)$: state var. i.e. stock of resource at t

T: point of time when scrap value is realized

t: cont. time

$y(t)$ is piece-wise continuous

Let us define

$$H = V(\cdot) + \lambda(t) f(\cdot)$$

$$\Rightarrow H = H(y(t), x(t), t, \lambda(t))$$

our $L = \int_0^T \left[H(\cdot) + \dot{\lambda}(t)x(t) \right] dt + F(\cdot) - \lambda(T)x(T) + \lambda(0)x(0)$

we want to optimise L by choosing $\{y(t)\}$

Let's assume $y(t)$ is changed to $y(t) + \Delta y(t)$

$$y(t) \longrightarrow \{ y(t) + \Delta y(t) \}$$

$$x(t) \rightarrow \{ x(t) + \Delta x(t) \}$$

$$\Delta L = \int_0^T \left[\frac{\partial H(\cdot)}{\partial y(t)} \cdot \Delta y(t) + \frac{\partial H(\cdot)}{\partial x(t)} \cdot \Delta x(t) + \dot{\lambda}(t)\Delta x(t) \right] dt + F'(\cdot)\Delta x(T) - \lambda(T)\Delta x(T)$$

$$= \int_0^T \frac{\partial H(\cdot)}{\partial y(t)} \cdot \Delta y(t) + \Delta x(t) \left\{ \frac{\partial H(\cdot)}{\partial x(t)} + \dot{\lambda}(t) \right\} + \Delta x(T) \left[F'(\cdot) - \lambda(T) \right]$$

Now optimization requires $\Delta L = 0$

And if we put delta L equals to 0 in order to get delta L equals to 0, basically we need to get three conditions. We need the following three conditions to be satisfied, following three conditions should be satisfied. So I will write the last expression once again, delta L equals to integration 0 to t then del H dot del yt into delta yt plus delta xt, we have taken common. So this would become del H dot del x dot plus lambda dot t into dt plus delta xt again common then in within the bracket we got F prime dot minus lambda t.

What are the conditions we require? First condition, if we want to get the entire expression delta L equals to 0 then del H dot del yt should be 0, del H dot del yt, this should be 0 and then then lambda dot t, so lambda dot t should be equals to minus of this because this cannot become 0, this is basically change in your stock that that cannot become 0, the moment to extract something stock will also change.

So from the first component you can think of this is the first component, from first component this should become 0, because delta yt is not 0, since delta yt not equals to 0. From the second component what we are writing that lambda dot t should be equals to negative of del H dot del xt, since delta xt not equals to 0 and from the third condition what we required that F dot t equals to lambda capital T, f dot equals to lambda T. So these three conditions should be satisfied in order to optimize our Lagrangian function.

Now these three conditions taken together for which here we should mention for all y belongs to capital Y, and this is the condition, so this is our decision variable that belongs to a set of y. So these three principles or conditions are known as maximum principle and the entire process what we have discussed up to now is known as optimum control. As of now it may appear like we are discussing only technical things but these things are the primer of dynamic optimization that we will when we discuss in our later part you will see the actual meaning of these conditions.

Now when we have said the Lagrangian I said that we have to discussed the economic interpretation of this lambda t. How will you interpret this lambda that is something we need to discuss. Now here what we can see in this entire function this V is something some flow benefit and here what is this, it is some kind of change that means F is a function of, if you look at the condition previously this is basically change in your stock and when change in stock is this so that means this is basically x dot t minus F you can think of some kind of change in stock.

So basically some amount of stock, first component is flow benefit second component is stock. We cannot simply add stock with benefit that is very simple, if you recall we have already discussed earlier. So this lambda t is a converter that will convert this type of stock into benefit. That means lambda t we can think of marginal benefit of the change in stock or indirectly you can say that marginal benefit of keeping the resource intact instead of using the resource today if you keep it what is the benefit that you are going to get.

Or you can think of this is something opportunity cost of using the resource today, opportunity cost of using the resource today, and since this lambda t is basically attached with the state variable, lambda t is attached with the state variable sometimes it is also called as co-state variable. So that means from our logic of Lagrangian function we can get the interpretation of lagamda t okay.

We can say that marginal benefit of preserving the resource, this is F dot minus x dot t this is coming from the dynamic constraint, which basically indicates how the resource is changing over a period of time. Obviously, we cannot add resource change with some kind of benefit then what is that? So lambda t is converting the change if we do not change then what is the benefit that is given by lambda t. So obviously lambda t is then per unit marginal benefit of preserving the resource if the resource is not changing if you keep it intact then what is the benefit if it is changing then what is the cost that is basically indicated by this lambda t.

Now in our function in the function H what you see that H is basically here when we define the H function where we have defined our H, here if you look at H, so basically H is a summation of two benefits, now this is basically a flow benefit and lambda t F dot this is some kind of stock benefit, flow and stock. This sometimes becomes difficult to add for example, let us say we can think of let us say you have a stock or share of a company so this V would become then let us say dividend, the dividend value each and every period you are getting dividend and this lambda t F dot is the future value of the stock.

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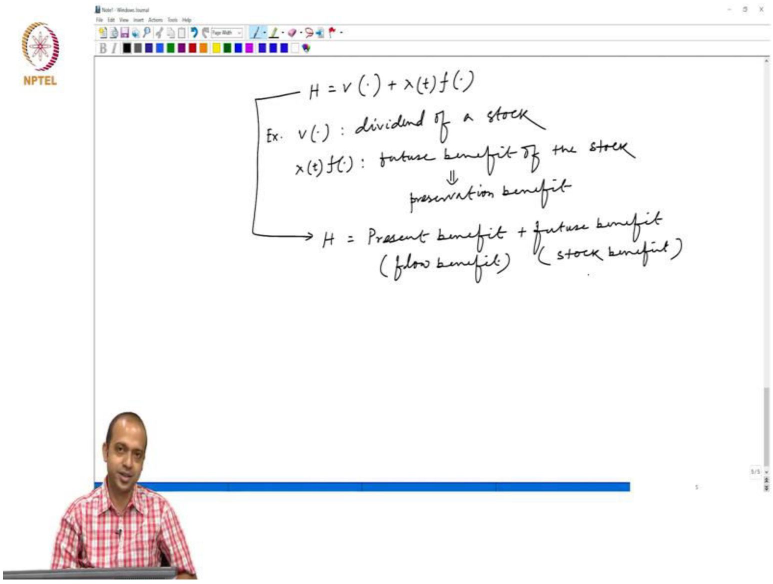

So that means what you can say that from these, from the H function, from the H function, H is basically a summation of V plus lambda t into F dot. As an example what we can say that example let us say V is, V dot is the dividend of a stock and lambda t F dot is basically future value of the stock, future benefit of the stock. That means if I do not sell the stock today and keep it for next month, so next month I will get some value for this stock.

That means that is called future benefit, so while keeping the stock with me I am enjoying dividend as flow benefit and at the end there is some kind of benefit which is called future benefit, so that means future benefit is denoted by lambda t F dot, which you can think of preservation benefit, instead of using the resource today, preservation benefit and that is how we interpreted lambda t, because this stock value, future value you cannot add with the benefit.

That is why lambda t is basically converting that future value of stock into some kind of benefit that is why it is called marginal benefit of preservation, so this is preservation benefit. That means in total what we can write that each is equals to summation of present benefit which in terms is basically the flow benefit and this is future benefit or preservation benefit. That means within this H function something which is present something which is future, now this you can think of future benefit as a stock value, stock benefit.

So what we can do actually, we can either convert this flow benefit into stock or future stock benefit into flow, so that this addition within this H function becomes easier, is this clear. So, from this example it is very clear that within this H function there is some flow benefit and there is some future stock benefit that we are trying to add. Since adding flow with stock is little difficult, what we can actually do, we can convert this future benefit into present or present flow benefit into some kind of stock so that the addition becomes easier and the moment we convert future stock into flow or flow into stock that will give us additional insights. How? That will discuss in our next class. Thank you.