Environmental and Resource Economics
Professor Sabuj Kumar Mandal
Department of Humanities and Social Sciences
Indian Institute of Technology, Madras
Dynamic Optimization and Renewable Resources Part - 1

Welcome once again to our discussion on resource economics and we are discussing dynamic optimization. Yesterday we just got an introduction on this dynamic optimization topic and we basically discussed the important elements of a dynamic optimization and we understood what is the context is in which dynamic ptimization is required. Basically we said that when the economic agent is involved in an intertemporal choice so that means it is not a static optimization that is needed here rather the individual is involved in an optimization which spans over a period of time.

For example, let us say we have an asset or we have a resource from which we are going to derive benefits over a period of time, and in each and every period of time we need to decide what is the optimal extraction for that period of time given the demand condition and then at the end of the period our objective is to get the maximum benefit throughout, the total maximum benefit for the entire life of this resource.

So this is something we can think of lifetime utility maximization and then we said that in dynamic optimization we should have an initial state, we should have a terminal state. Initial state means let us say initial amount of initial stock of the resource and terminal state is the stock that will arrive at the end of the period. Then we have a decision variable, because every optimization requires a decision variable and here our decision variable is let us say rate of extraction of the resource or rate of investment.

Then we have several paths to achieve the terminal state and we will select that path which will either maximize our utility or minimize the cost, maximize the utility or minimize the cost and then we have objective functional instead of function and we also discussed about the difference between function and functional basically we said that functional is something wherein we are mapping an entire path to a real line.

So these are the salient features of a dynamic optimization and today what we will do, we will get a more formal discussion on this dynamic optimization technique.

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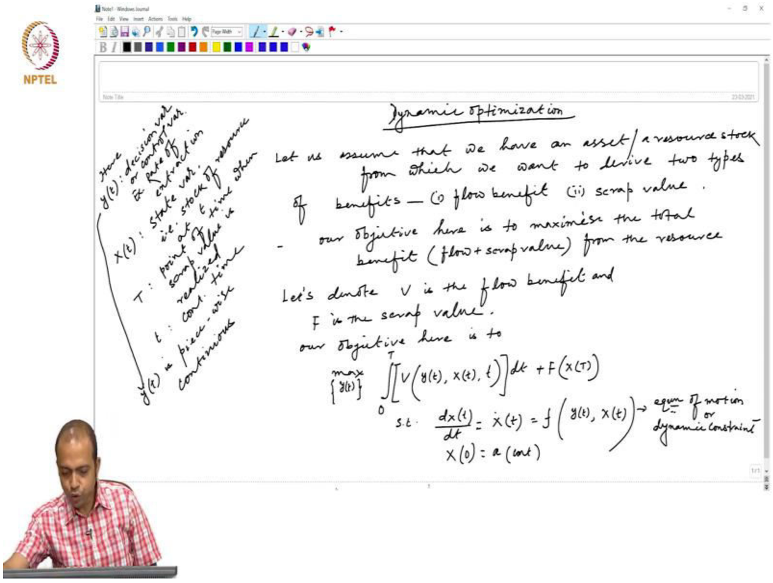

So let us start with that, this is dynamic optimization, dynamic optimization we are talking about. Let us assume that we have an asset or a resource stock from which we want to derive two types of benefits.

One is called flow benefit and another one is called the scrap value. These are the two benefits, for example, we may think of let us say we have an asset in the form of a machine, so machine has a specific time period that let us say 10 years, 15 years or 20 years and during that period we will be deriving some kind of flow benefit from the machine. At the end of the 10, 15 or 20 years when the lifetime of the period gets in, so what we will do, we will sell the machine and we will get some value that is called scrap value, so that is the idea.

So some flow benefits, some scrap value and our objective is to basically maximize the total benefit. Let us also assume that, so our objective here is to maximize the total benefit that means flow plus scrap value from the resource. That means here our objective is let us denote V is the flow benefit and F is the scrap value. So what is our objective here? Our objective here is to maximize the flow benefit.

That means flow you will get over a period of time. What you are maximizing, V and V is actually a function of let us say yt, then xt t dt, this is the flow benefit plus F of xt, this is the scrap value and scrap value is actually realized at the end of the period that means t is basically the terminal state. So here yt is the decision variable, yt you can say that this is the decision variable or control variable, for example you can think of rate of extraction is your decision variable yt and then xt is called state variable or we can think of this is basically a stock of resources t.

Stock of resources at t and capital T is point of time when scrap value is realized, t is continuous time we are talking about. So these are the notation and what is our constraint here we are trying to maximize this with respect to yt, so yt is the control variable subject to what is the constraint, constraint is dx t dt which you can think of as x dot t, it is actually a function of how the stock of the resource will change over a period of time that is actually a function of your rate of extraction and your initial stock, this is the constraint.

Then X0 this is something constant, so that means you have some initial stock. Now this constraint the first constraint dx t dt or x dot t which we said that it is a function of yt and xt this is actually called equation of motion or dynamic constraint. So this is clear, we have a resource or asset from which we are basically deriving two types of benefit, one is the flow benefit, so 0 to t we are integrating V function then we will get the total sum of flow benefit which is a function of yt that means your rate of extraction, your state variable xt and t where time is continuous.

And then at time period capital T we are basically realizing the scrap value, so the total benefit is V plus this F that we are going to maximize subject to a dynamic constraint or equation of motion which is dx t dt equals to x dot t or equals to which is a function of your rate of extraction and initial stock of capital. Here one more thing we have to keep in mind yt need not be continuous but at least piecewise continuous, yt is basically piecewise continuous. So the question is how will you optimize this? We will optimize.

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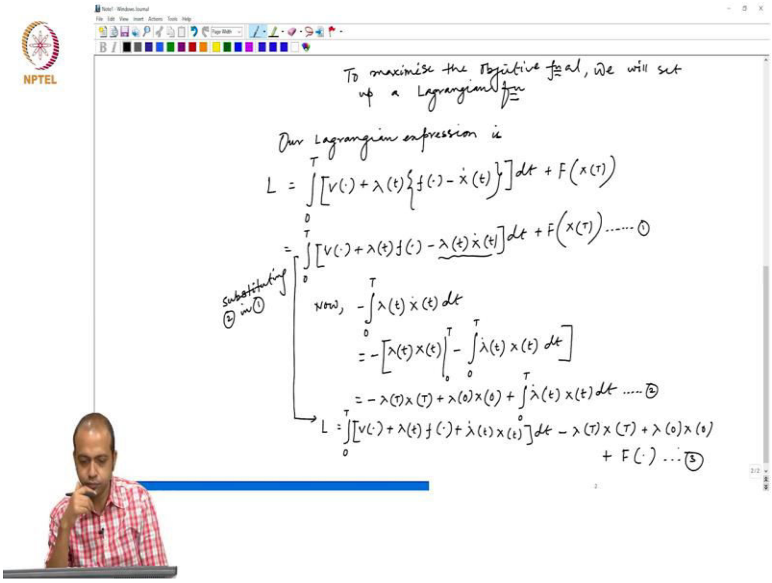

So to optimize or to maximize the objective functional we will first set off a Lagrangian function. What is the Lagrangian function? Our Lagrangian expression is basically what we will write L equals to integration 0 to t then instead of writing the entire V function what we will write, V dot plus lambda t into F dot minus X dot t and then dt plus F of xt, this part we are not changing.

That means how the Lagrangian, I have already discussed Lagrangian expression, so first you have to write your objective functional plus lambda into this, what is the interpretation of this lambda that we will again discuss but for the timing if you recall that lambda is basically a converter that will convert one thing to another thing. Here the first item V dot is some kind of benefit what is the nature of this that we will discuss.

So that lambda is actually converting this into some form, so that these two benefits can actually be added. This is how we have set up our Lagrangian multiplier. This we can write as integration 0 to t then V dot plus lambda t into F dot minus lambda t into x dot t, I am just simply multiplying this dt plus F of x and then t. Now what we will do, in the integration there are two components, there are three components actually first second and third. We will take this component separately and we will do the integration.

What we will do now, let us say this is equation 1, now minus integration 0 to t lambda t x dot t dt. This if we try to integrate look at here we have two function lambda t and x dot t, how will you integrate, can you think of? You have already studied in your plus two level, so integration 0 to t lambda t x dot t, so it is basically like integration ut vt dt. There are two function ut and vt and then what we need to apply, we need to apply the rule of integration by parts, so we need to apply integration by parts. You have to identify the first function you have to identify, the second function and then you have to do that.

What would be this would become minus of first function unchanged lambda t and then integration of the second that means x dot t and that will become x t if you integrate, first function unchanged integration of the second you have to evaluate at 0 to t minus again integration 0 to t, derivative of the first that means lambda dot t, integration of the second and then again dt, entire thing in a bracket. First function unchanged integration of the second evaluated at 0 to t minus integration of derivative of the first integration of the second and dt.

This will become minus lambda t x capital T, so minus lambda capital T then x capital T minus lambda 0 x 0 that means this would become positive plus lambda 0 x 0. This minus and this minus will become plus integration 0 to t then you will have lambda dot t x t dt. This is let us say equation 2, so what we will do now in place of this lambda t x dot t we will replace this entire expression from 2, so substituting 2 into 1, what will get?

So this will become now, this is substituting 2 in 1, what you will get, your L would become integration 0 to t then you will have V dot plus lambda t f dot, here it is minus, this minus, this will become plus, plus sorry we will take this integration only, so that means lambda t x t here this is integration, this part also integration plus what we will have lambda dot t x t dt. So V dot t lambda t f dot minus, sorry this is lambda dot t into x t dt plus, sorry minus lambda capital T x t. This part I am writing plus lambda 0 x 0 plus this value F, this is let us say equation 3.