Research Methodology
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Lecture - 55
An Example of Mathematical Modeling

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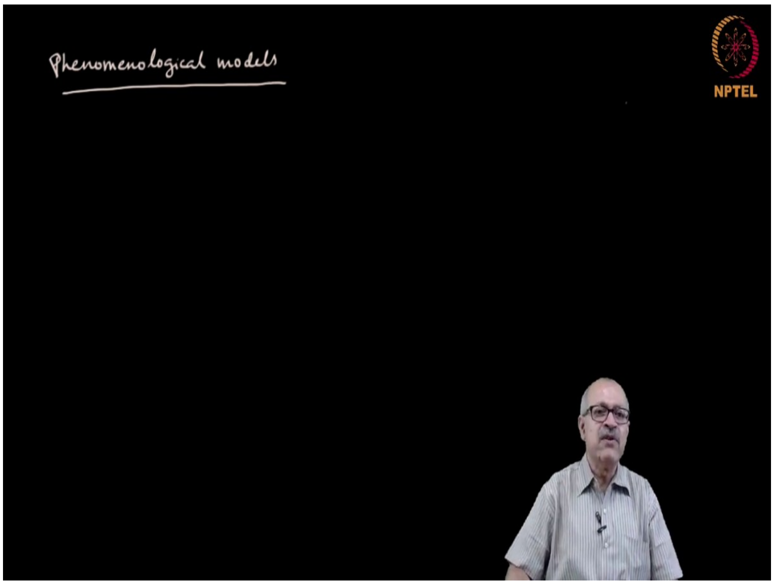

We were learning how to obtain mathematical models of various types of systems. And we have seen that there are some phenomenological models which can be obtained by a few different means, one of them being the dimensional consistency method.

And we have seen that we can also derive models starting from the first principles and then the models that we arrive at are not phenomenological models. So, there are various types of mathematical models that we obtain. Today let me give an example of a modeling procedure that proceeds from common sense.

It concerns how wild populations of different species vary with time. Field biologists have observed the number of specific individuals in the population in a species. They count them and talk about the density of that particular species. And in field biology, for a long time it has been noticed that populations oscillate. Why do they oscillate? Why do not they reach an equilibrium and stay there?

For example, if there is a predator species and a prey species—like lion and wildebeest, tiger and deer, etc. For some time the number of deer will increase and the number of tiger will decrease, and then after some time again the number of tiger will increase the number of deer will decrease and so on and so forth. But, we do not really observe both of them reaching some equilibrium, so that the number of deer that are produced by normal reproduction exactly matches the number that is eaten by tigers. Then it would reach more or less an equilibrium. But that never happens. Why?

It was a question for a long time and the riddle was solved only when we could obtain a mathematical model. So, let me just illustrate how such mathematical models are made. Imagine that the number of individual animals in the prey population is denoted by x and the number of predators is denoted by y.

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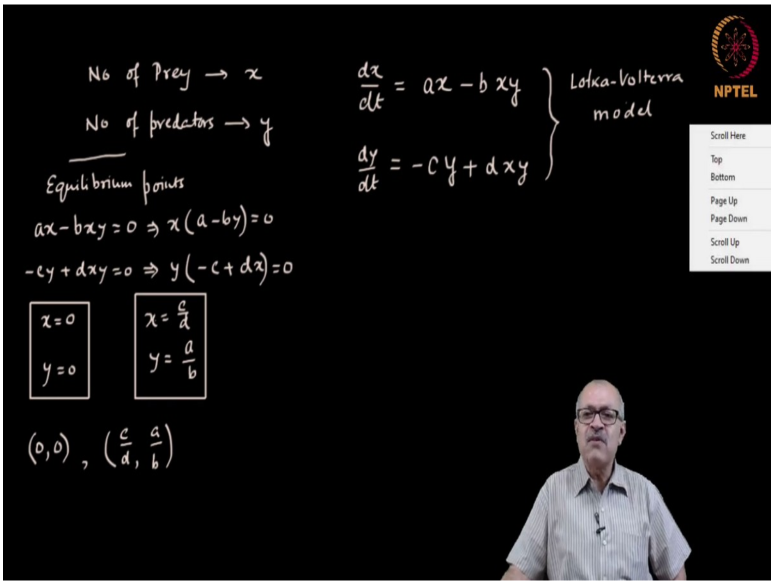

Normally x and y would be whole numbers. But, for the sake of simplicity let us assume that they are continuous variables. If the number is large then it does not really make much difference, so that when we try to find out how they change with time, we can talk in terms of derivatives. So, we assume x and y to be a continuous variables. That is a bit of approximation, but nevertheless that is an acceptable approximation so long as the number is large.

Now the question is, how would the number change with time? Our question concerns the character of dx/dt and dy/dt, and that is what we try to find out.

The logic would be that, in that forest if there is no predator, then how will the prey behave? It follows from common sense that the number of prey will increase and it will increase in proportion to the number that is already present.

So, if a large number is present, then it will increase more; if a small number is present, it will increase less. Quite natural. So, the dx/dt will be a function of x, a linear function of x. Let us call it ax. That is a reasonably good approximation that follows from common sense.

Now, you consider that the forest also has a predator species. If predators are present, that will reduce the number and so, there will be a negative term. It will introduce a negative term and that negative term will be proportional to the number of times the deer meet the tiger.

The number of time deers meet tigers can be approximated by a simple product of x and y. So, it will be dependent on xy. If the number of predators is small, then it will have very small number of interactions between the predators and prey. If the number of predators is small, then also there will be small number of interactions. If they are both in large number then there will be a large number of interactions. And so you can see that this is a reasonably good estimate of the number of times they meet each other in the wild environment.

So, it will proportional to xy. There has to be a proportionality constant upfront. So, in the first approximation, proceeding from common sense, that will be the equation for x.

$$\frac{dx}{dt} = ax - bxy$$

What will be the corresponding equation for dy/dt? To obtain the corresponding equation for y, we will proceed by the same kind of logic.

Suppose the forest only has the predators and no prey. Obviously the predators will die out. How will their number decrease? Their number will decrease in proportion to their existing number. So, it will be a negative quantity and it will decrease in proportion to the present number and so, let us put a proportionality constant c.

Now you put in the prey species. This forest has the prey species too. So it will have a positive effect because the existence of prey will make dy/dt a positive number. It will increase and so it will have a positive value and that also will depend on the number of times the predators meet the prey. As we have seen, it is represented by the multiplication, the product xy. So, here also it will be plus xy, but with a different proportionality constant.

$$\frac{dy}{dt} = -cy + dxy$$

So, in the first go, that will be the model of the system. Such a simple model is called a Lotka-Volterra model. Now, such models are actually quite useful in answering the question that we started with asking. We asked why do they oscillate?

Now, if we have a model like this, you can explode the behaviour of the model. In other words, how would x and y vary if this model were correct? For that purpose, the first thing that one does is to obtain what are known as the equilibrium points.

In this model, what are the equilibrium points? Equilibrium points are those where, if the number of x and the number of y are there, then they will not change, which means such values of x and y so that dx/dt will not be changing, will be 0, and dy/dt will be 0.

We are trying to figure out why we do not see equilibrium points in nature. So, we simply obtain the equilibrium points. The first equation if dx/dt is put equal to 0, you get ax minus bxy is equal to 0, and x can come common, so x times a minus by equal to 0.

The second equation gives minus cy plus dxy equal to 0. y comes common and minus c plus dx equal to 0. Now, this being 0 can come from two possibilities: x being 0 or this being 0. So, one is the possibility x equal to 0 and the other possibility is a minus by equal to 0 or y is equal to a by b. This gives either y is equal to 0 or x equal to c by d.

So, we can see, there are two equilibrium points. One is x equal to 0 and y equal to 0. The other is x equal to c by d and y equal to a by b. We can write these two equilibrium points as (0, 0) and (c/d, a/b).

So, these are the equilibrium points. Now, (0, 0) is a trivial equilibrium point which means that there is neither a predator nor a prey in the forest. Obviously, if there is no predator nor prey, no animal in the forest, we do not expect that to change.

So, this is a trivial equilibrium point and not of our interest. Of our interest is this because if it stabilized, it would have stabilized at this point. That is the only point at which it can stabilize: a possible equilibrium point.

But, why does it not stabilize in this equilibrium point? That is the next question. In order to explore that question, we notice that these equations that we have obtained are non-linear equations because there are xy terms. xy is a non-linear function and therefore, these are non-linear equations. And if you have non-linear equations then the behaviour in different parts of the space will be different.

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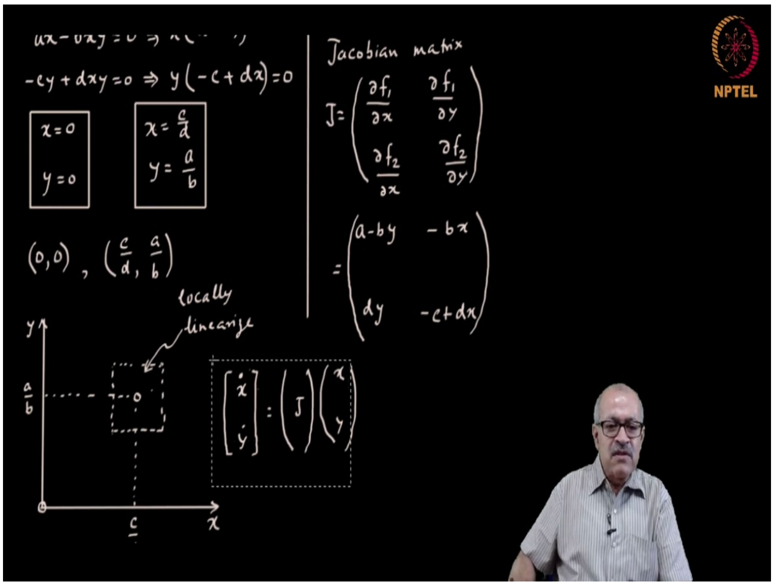

Let me plot it. You can see that this point is an equilibrium point, but is a trivial equilibrium point, not of our interest. But there is another equilibrium point somewhere here, whose x coordinate is c/d and whose y coordinate is a/b.

If it did stabilize then it would be this x this y. Then the prey population would stabilize at the value c/d and the predator population should stabilize at the value a/b. But, our question then is, why doesn’t it stabilize?

Now, in order to answer that question, we explore the neighborhood of this point. The behavior of a non-linear system can be different in different parts of this x-y space, but we can easily explore the behavior in the neighborhood of an equilibrium point. In the neighborhood of this equilibrium point we can locally linearize the behavior.

So, we can locally linearize the behavior around this equilibrium point. And if we do so then we might obtain something of interest. Why? If you start from a point something like this: that means, x value is this and y value is that, why doesn’t it converge on to the equilibrium point? How do you locally linearize this? Well, local linearizations are done by the method of Jacobian matrix.

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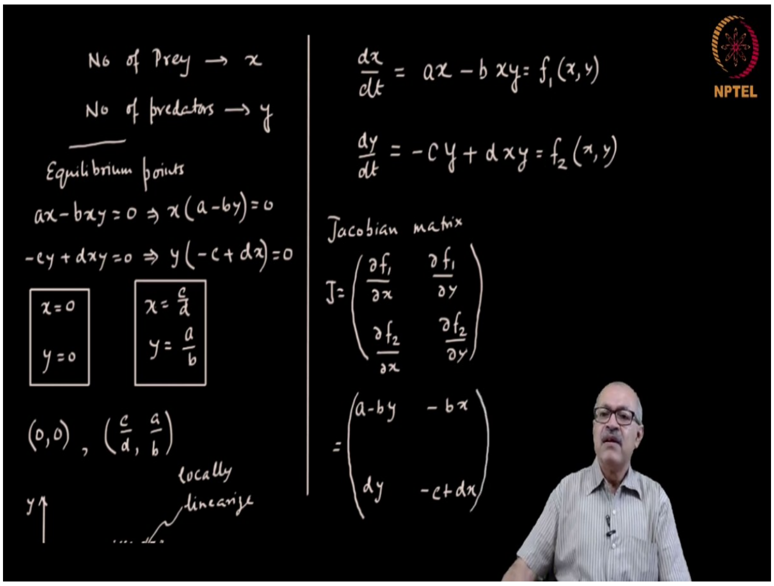

The method of Jacobian matrix: let us call it J. Let this be one function of x and y, and let this be another function of x and y. So, f_1 is ax minus bxy and f_2 is minus cx plus dxy.

Let us call them these functions. Then the Jacobian matrix is the derivative of f_1 with respect to x, derivative of f_1 with respect to y, the derivative of f_2 with respect to x and the derivative of f_2 with respect to y. Now, this matrix can be calculated for this particular set of things and you can easily see what it will be.

The first function when taken derivative with respect to a x, it becomes a minus by. The first function when differentiated with respect to y this disappears; it becomes just minus bx. This function f_2, when you take a derivative with respect to x, this disappears and this becomes just plus dy. When you take it with respect to y, it becomes minus c plus dx.

$$J = \begin{pmatrix} a - by & -bx \\ dy & -c + dx \end{pmatrix}$$

Now, you see that the Jacobian matrix will represent a local linear behaviour of the type. Let me abbreviate by putting a dot here: x dot, y dot, this vector will be this Jacobian matrix times the x y vector.

So, that is the local linearization, and the Jacobian matrix is this. Now you have to locally linearize around a particular equilibrium point. In this case we are not interested in the (0 0) equilibrium point. We are interested in this equilibrium point. So, in place of x and y, we have to put these values. So, let me put these values here.

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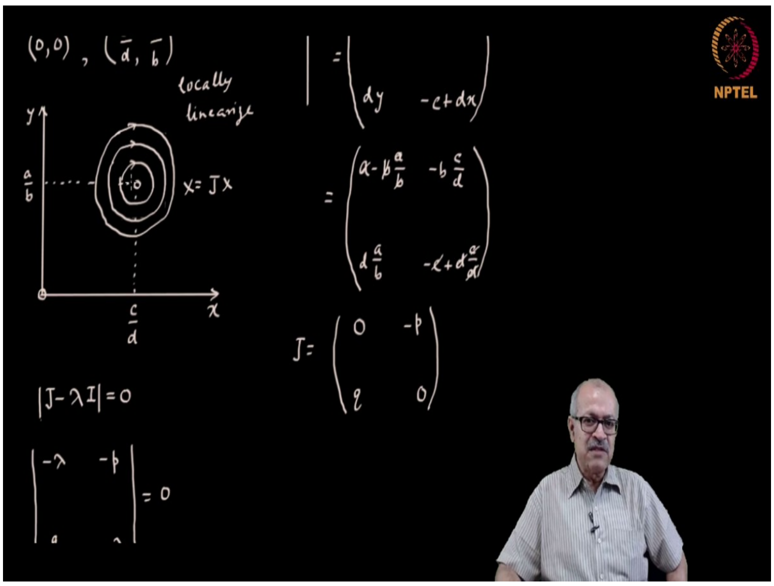

$$J_1 = \begin{pmatrix} 0 & -bc/d \\ ad/b & 0 \end{pmatrix}$$

If you put the values x=c/d and y=a/b, the Jacobian matrix becomes very simple. These two terms are 0 and 0, and these two terms have values. But, instead of writing them directly, because then we have to write a lot, let me abbreviate this as minus p and this as say q. So, you just remember that minus p is nothing but minus bc/d and q is ad/b. So, now, the equation becomes x dot is equal to this Jacobian matrix times x.

You know that the solution of the differential equation in this region which is given by x dot is equal to say the Jacobian matrix times x. This is the equation. The solution of this equation depends on the eigenvalues of the matrix. This is something that you must have learnt in the first year. So, I am not going into the theory of why this should be so, but let us obtain the eigenvalues of this matrix. You should write J minus lambda I. Lambda is the eigenvalue; I is the identity matrix. This determinant should be equal to 0. Solution of these are the eigenvalues; you know that.

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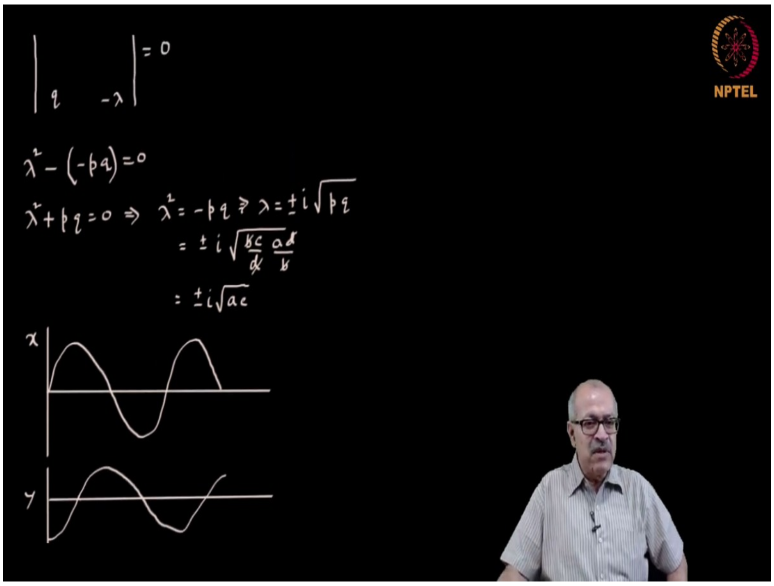

So, the determinant of (0 minus lambda I) has one here, minus pq and this is 0 minus lambda. This is lambda square minus and this product is minus pq equal to 0. So, lambda square plus pq equal to 0, or lambda square is equal to minus pq. This gives lambda is equal to plus minus i square root of pq.

This is plus minus i, the imaginary number, square root of this. Now we can substitute: p was bc/d and q was ad/b. b and d cancel off, and we get plus minus i root over a c.

So, notice that the eigenvalues have come to be purely imaginary numbers. This is important. The eigenvalues being purely imaginary says that the oscillation in x should be a sinusoidal quantity. The oscillation in y should be another sinusoidal quantity. They would be 90 degree out of phase. So, this is x, this is y.

And if you look at it in that xy space, you can clearly see the behaviour in this plane, it will be going round and depending on the initial condition it will be just going round. That explains why you always observe oscillations. It is because this character of the model. When we obtained the mathematical model, that enabled us to understand why there should be oscillation.

It is because the eigenvalues come out to be purely imaginary. Had there been a real term here then it would be either incoming or outgoing spiral behaviour depending on whether the real term is negative or positive. But, in this case it has come to be a purely imaginary number. It also gives the insight that the eigenvalues are dependent only on this term and this term a and c.

How does the prey population increase if there were no predator population or how does the predator population decrease if there is no prey population? These factors actually determine the eigen values. So, the rate at which it will oscillate, the frequency of that oscillation will depend on these two parameters.

Now, you might ask: actually it was a very simplistic model, not really a realistic model. We assumed things. And I have not really stated the assumptions, but you can easily see what the assumptions are. For example, we have made the assumption that in the absence of the predator population the prey population will increase.

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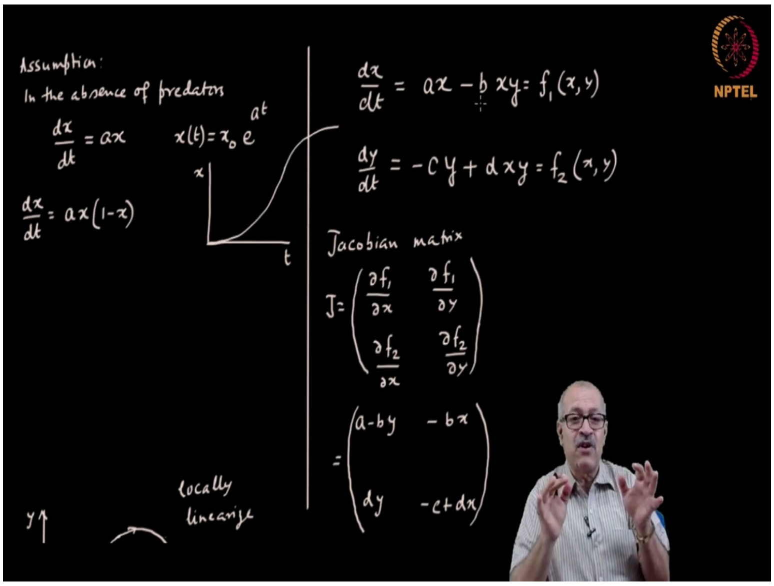

We have assumed that in the absence of predators our dx/dt will be only ax. Now, after having obtained this model, the theorists will tell the observational biologists that go forth and check. My model says that it will oscillate and it will oscillate with a frequency such and such, the frequency will be related to the eigenvalues: root over a/c. Check whether you are getting that or not. Or is the actual behaviour deviating significantly from this elliptical orbits that the model predicts.

So, again the observational biologists will check in real life situations and will come back and tell where the inaccuracies were. And then the theorist will again have to go back to the equation and figure out where we had made some assumption that might be invalid.

For example one assumption was this. Now, if this is the equation, its solution is x of t is the initial x, say x naught, e to the power at. So, it is actually a graph that continuously increases exponentially. This is the x, this is the time; this is the graph of x, which increases exponentially.

But, in a real life forest, if there is no tiger, can the number of deer increase to infinite values? No, that cannot happen because there is limitation of food in that forest and so, after some time it must taper off, something like this. That can be incorporated if we say dx/dt is, ax is there, but there should be another term. If we add and multiply another term 1 minus x, then when x is large then 1 minus x will be small and naturally the rate of increase will fall. As a result it will give a like a graph like this.

So, the next logical improvement should be to write, in place of ax, something like this. Similarly you can imagine that there can be various improvements. In the predation term we had put just a multiplication term. But, that might not be a very good representation of the actual decay of the prey population due to predation, or the actual increase of the predator population due to the availability of food. So, the functional forms here might be more complicated and people have proposed proper functional forms to take into account various possibilities.

For example, in case of tigers there has to be a period when the mother has to train the tiger cub to hunt. Otherwise they cannot hunt. Hunting is a acquired character. It is not something that is innate of a tiger. So, one has to learn how to hunt and that can be put into the model. You can also improve upon this. What I am trying to point out is that the way to produce a model is first based on the first principles. Produce a model, find out what its predictions would be, and then check those predictions against physical reality. Wherever the predictions differ from the actual physical reality, there you have to improve upon that model incrementally, trying to approximate physical reality as much as possible. So, that is the process of model building.