Research Methodology
Prof. Soumitro Banerjee
Department of Physical Sciences
Indian Institute of Science Education and Research, Kolkata

Lecture - 53
Theoretical Research: Modeling Using Dimensional Analysis Part 01

In the last class, we learnt, using a rather simple example, how to go about deriving a mathematical model of a physical system from the basic principles. That means, we start from the basic principles of physics that we know and on that basis we try to derive a mathematical model of a system.

And while doing so, we have to be very clear in our minds as to what is the objective the model is supposed to serve. If it is to study the dynamical behaviour of a system then it has to be a differential equation, otherwise it will be an algebraic relationship between quantities. So, accordingly we have to proceed.

But the basic thing was that, we proceeded on the basis of our understanding about how various things in nature function. We just cast that understanding in the form of some mathematical relationships.

When we are doing so, when we are establishing some mathematical relationship between quantities, in the left hand side it can be how the quantity varies with time, or the quantity itself in case of a algebraic relationship. But in all cases the left hand side and the right hand side must have the same dimension. That is one very important thing to keep in mind while deriving any system model: the left hand side and the right hand side must have the same dimension.

Not only that, if there is an addition of two terms, then both the terms should have the same dimension: addition, subtraction, all that must have the same dimension.

Now, out of the various quantities that we handle, there are a few quantities that are fundamental, in the sense that their dimension cannot be derived from the dimension of other quantities.

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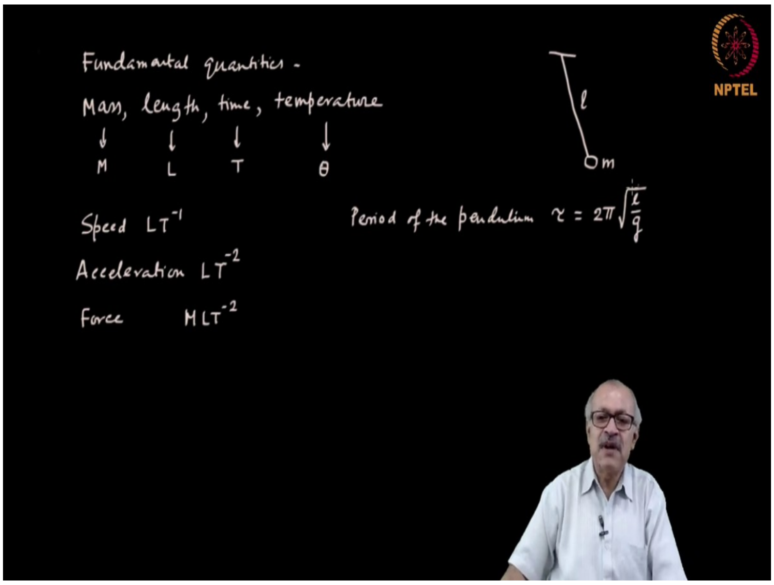

These are the fundamental quantities: the mass, length, time. These three are the basic. Also temperature, which cannot be derived from something else. Mass is given the dimension of M. We normally write in capital letters. Length is given the dimension L. Time is given the dimension T. And since time is there, temperature cannot be the same T, it is normally given as theta.

So, these are the fundamental dimensions and the dimension of every other thing is expressed in terms of these fundamental quantities. For example, the dimension of speed is length per unit time, LT^{-1}. Momentum would be mass times the speed, so MLT inverse. Acceleration would be speed per unit time, so LT to the power minus 2. Force will be mass into acceleration, MLT minus 2, and so on and so forth.

To cut a long story short, every other quantity can be expressed in terms of these 4 fundamental quantities. What I am insisting is that, whenever we write an equation, at every stage we need to keep a check whether the left hand side and the right hand side have the same dimension or not. If a quantity is added to another quantity, sometimes that quantity may be complicated with various terms coming in, we need to check whether whatever it is added to, they have the same dimension. So, these are the things that we need to keep in check. Let me illustrate that.

For example, if there is a pendulum oscillating, then suppose it is a pendulum of mass m suspended by a cord of length l, then we know from the fundamental theory that you have learnt in school that the time period, the period of the pendulum, let us call it tau, is 2 pi square root of l by g.

$$\tau = 2\pi\sqrt{\frac{l}{g}}$$

Let us check if this equation satisfies the dimensional consistency. That is important.

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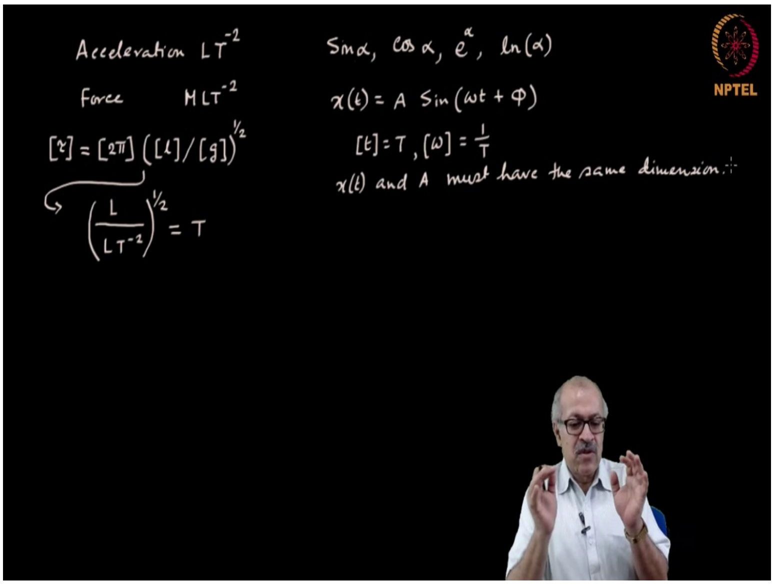

Whenever we write the dimension of something, we always put that in square bracket. So, we will have to write the dimension of tau. It should be equal to the dimension of 2 pi times the dimension of l divided by the dimension of g square root, to the power half. So, this is how we will express the dimensional equation.

Now, tau is the time period. Therefore, it has the dimension of time. Let us write down the dimension of the right hand side first and then let us see if that equals T.

The dimension of 2 pi: pi is in radians which is a ratio and therefore, it is non-dimensional. So, this will disappear. It does not have a dimension. The dimension of l is L, length. The dimension of g, acceleration due to gravity is, since it is an acceleration, LT to the power minus 2. This whole thing to the power half. L cancels off, T becomes T square to the power half. It is dimension of time. In a similar fashion, whenever we have to write an equation then at every stage we need to check whether this is dimensionally consistent or not.

Now, often in equations, we come across expressions of the form sin alpha, cos alpha, or e to the power alpha or the ln of alpha. These are transcendental expressions. What would be the dimension of these things? Both the argument of the expression as well as the result of the expression – what will be the dimension of that?

In deciding that, we have to note that all these kind of expressions take only non-dimensional quantities as argument and return non-dimensional quantities, always. Because of that, whenever we write something after a sin; that means, sin of something, then we have to make sure that whatever you are taking a sinusoidal function of, that is a non-dimensional quantity.

Similarly, e to the power something, or exponential of something: that something has to be non-dimensional quantity. Log of something: that something has to be a non-dimensional quantity and the result of taking a log is also a non-dimensional quantity. That is why, whenever we use a any of these terms, we have to always check.

Just to give a simple example, we often write x as a function of time is equal to some constant times sin omega t plus some phi. Now, if the phi is in radians, it is non-dimensional. But how about this? Notice that the dimension of t is T and the dimension of omega is 1 by T because it is cycles per second. Therefore, the product is non-dimensional. A non-dimensional quantity added to a non-dimensional quantity is a non-dimensional quantity. Therefore, you can take a sin of that.

Now, x(t) might be a length, it could be something else also. But if it is a length, then it has a dimension and therefore, A must have the same dimension. So, x(t) and A must have the same dimension.

So far so good. It is interesting to note that in many cases even a simple dimensional analysis gives us a lot of clues regarding the relationship between quantities and allows us to derive mathematical models. A simple dimensional analysis can do that. All that we need to do is exactly the same thing that we initially did for deriving a mathematical model: that is, first identify the phenomenon that we are trying to model and then try to figure out what does it depend on, and that dependence we write. In this case we write the dimension of all these quantities on which it is dependent.

As we have done earlier, we figure out which factors are relatively unimportant, which factors are more important, and put those factors in the dependence equation and then, on that basis, we proceed. Let us give an example.

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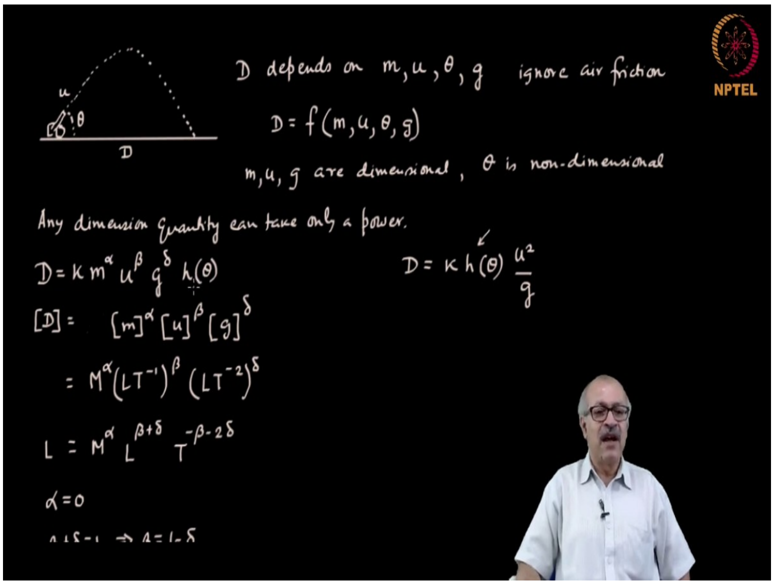

For example, let us take a problem. Suppose I have a cannon which shoots cannon balls at some angle. The ball will rise and then would fall. We know that the path is a parabolic path.

But suppose you do not know this. Suppose you do not know Newton's laws, and you are trying to figure out the distance D: how far it will actually travel. What will you do? You will first try to figure out what can that D depend on.

Firstly, your common sense will tell that, it will depend on the mass of the cannon ball. From common sense you figure out that it will depend on the velocity with which it is ejected from the cannon. So, it depends on the initial velocity u. Of course, it will depend on the angle, say, theta. At that angle it has been ejected. Your common sense tells that it will also should depend on theta. You might stop at that. But your common sense might also tell you that this distance will be different if the mass is the same, u is the same, theta is the same, it will be different, dependent on whether the whole thing is being done on the surface of the Earth, on the surface of the moon, or on the surface of the planet Mars. It means that it should depend on the acceleration due to gravity. So, you then add that also and then maybe stop.

You may also figure that the air friction might have a role to play, but in the initial part of the exercise of deriving a model, you say that, that effect is relatively unimportant, relatively small, and therefore, in the initial phase we will ignore that. So, ignore air friction. You have to state these assumptions.

So, you write your D as a function of these things: m, u, theta, g and you do not know what kind of dependence is there.

$$D = f(m, u, \theta, g)$$

Now, the line of argument will be as follows. m is a dimensional quantity, it has a dimension M. u is a dimensional quantity, speed, and therefore, LT inverse. g is a dimensional quantity, acceleration due to gravity, and therefore, the dimension is LT to the power minus 2. Theta, however, expressed in radians is non-dimensional. So, we understand that m, u, g are dimensional and theta is non-dimensional.

The moment you have identified these as dimensional, you know that these cannot appear in a sinusoidal term, e to the power x term, ln term, these cannot appear in that kind of a term. But theta can. Theta can be the argument of a transcendental function, but these cannot.

Then, what kind of functional forms can these take? The only functional form these can take is the power. Any dimensional quantity can take only a power. A power means m to the power something, u to the power something, t to the power something: these are possible. But sin, cos, log, exponential, all these are not possible. And therefore, we write D as m to the power some number alpha, u to the power beta, g to the power another thing, say, delta.

Now, theta. Theta can take any arbitrary function. We do not know that function. So, we have to write that as some function of theta. We do not know what this function is, but we cannot simply state that it can only take a 'to the power' kind of functional form. It can take any functional form because it is non-dimensional. And in addition, there can be some proportionality constant upfront. So, that becomes our initial model.

$$D = k\, m^{\alpha} u^{\beta} g^{\delta} h(\theta)$$

Now, let us put the dimensions. Then it becomes the dimensional equation. The dimension of D, will be equal to dimension of k times the dimension of m to the power alpha times the dimension of u to the power beta times the dimension of g to the power delta. Now, h of theta is not a dimensional quantity and therefore, there should be no dimension here.

$$\begin{aligned} [D] &= [k\, m^{\alpha} u^{\beta} g^{\delta}] \\ &= [k]\, [m]^{\alpha} [u]^{\beta} [g]^{\delta} \end{aligned}$$

You might say that I have chosen the k to be a non-dimensional quantity also because it is not necessary that k should have a dimension. Therefore, we can ignore that. So, this becomes basically the dimensional equation.

Now, if I now write the dimensions, m's dimension is capital M to the power alpha, u is LT inverse whole to the power beta, and this is LT minus 2 whole to the power delta. And if you now expand it you get M to the power alpha, L to the power beta times L to the power delta. So, L to the power beta plus delta. Here T to the power minus 1 and T to the power minus 2 or this will be T to the power minus beta and T to the power minus 2 delta therefore, T to the power minus beta minus 2 delta.

$$\begin{aligned} L &= M^{\alpha}\left(LT^{-1}\right)^{\beta}\left(LT^{-2}\right)^{\delta} \\ &= M^{\alpha}L^{\beta+\delta}T^{-\beta-2\delta} \end{aligned}$$

And in the left hand side the dimension of D is L. Now, notice the left hand side and the right hand side have to have the same dimension. That immediately tells us that M to the power alpha has to disappear because there is no M in the left hand side. So, the only possible value of alpha has to be equal to 0.

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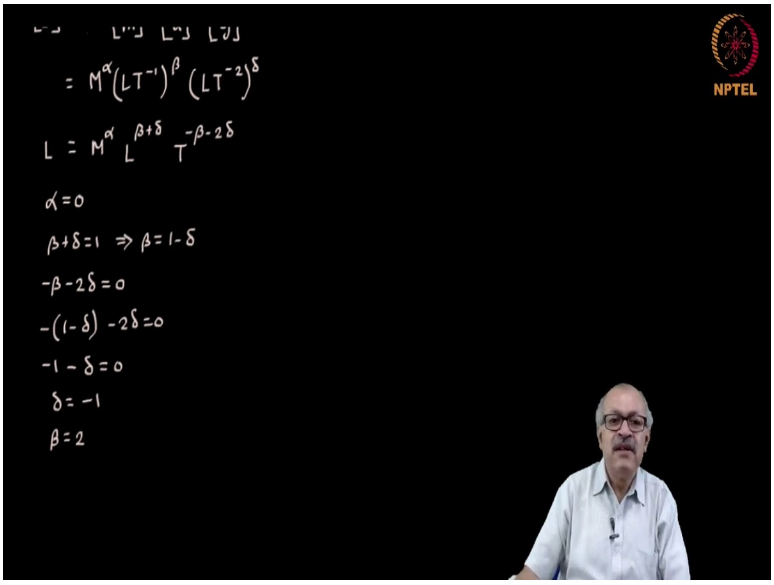

Now, here the power of L is 1, the power of L here is beta plus delta therefore, beta plus delta is equal to 1. From here, since there is no T in the left hand side, therefore, minus beta, minus 2 delta should be equal to 0.

$$\alpha = 0, \quad \beta + \delta = 1, \quad -\beta - 2\delta = 0$$

And you can solve these two equations. This gives beta is equal to 1 minus delta and then you put it here. It becomes minus 1 minus delta minus 2 delta equal to 0. Therefore, minus 1 plus delta minus 2 delta. So this is minus delta equal to 0. So, delta is equal to minus 1. Put that here: delta is equal to minus 1, it becomes 2. So beta is equal to 2.

$$\alpha = 0, \quad \beta = 2, \quad \delta = -1$$

Therefore, we have actually obtained the equation. We can now write D is equal to a constant, which is a non-dimensional constant, times m to the power alpha, alpha is 0 and therefore, this disappears; u to the power beta, beta is 2, so u square; g to the power delta, delta is minus 1, so by g.

$$D = h(\theta)\frac{u^2}{g}$$

So, that then, will be the relationship that you have obtained. It is clear that we do not yet know the functional relationship. But one very important noticeable thing is that, using simply dimensional analysis, we have come to the conclusion that the distance to which the cannon ball will go does not depend on the mass of the cannon ball. That was not following from common sense. When we did not know the Newton's equation, we did not know that it will not depend on the mass of the cannon ball. But simple dimension analysis allows us to come to that conclusion.

Now, once we know this particular term, once we know this particular term and you have g as a something constant on the surface of the Earth, then you know that D is proportional to u square and we do not know this functional form. So, how would you proceed? We have to perform very directed experiments in order to derive this functional form. How do we do that?

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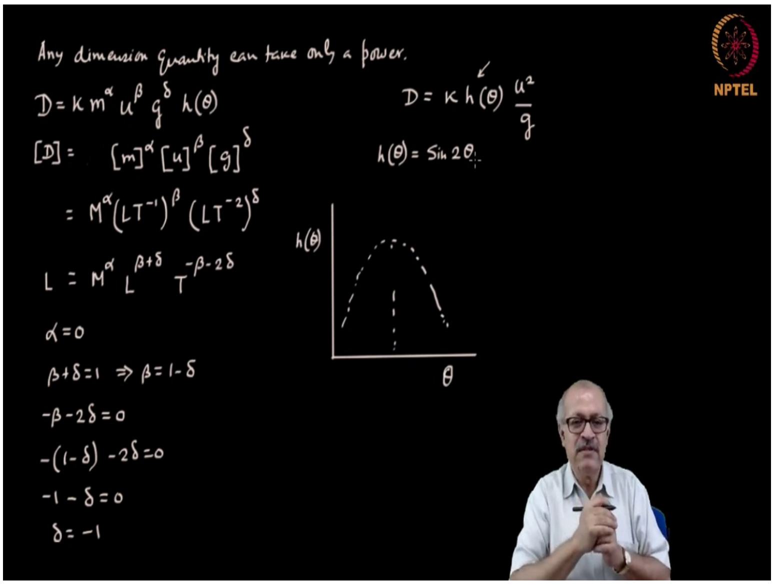

We now know from Newton's theory that h of theta is actually sin 2 theta and at theta equal to 45 degree, it maximizes. For lower values of theta and for higher values of theta, D will be smaller.

Now, when we were doing the curve fitting, we actually said that from the data we can either fit into a linear curve or a power law or an exponential curve, but beyond that we said that it would be difficult. So, in this case, either from theory you have to guess the kind of functional form, and then you can test. Because it will be a curve like this, with no prior idea what kind of functional form it will fit you never know.

That is why, since you are trying to find out some kind of a functional form, you have to plot the data in this form, theta here and h of theta here. For each theta you do the experiment. At each theta, keeping u constant, because this is the dependence you are trying to figure out, keep u constant, u square by g becomes a constant. The distances are measured and presently you assume k to be just a proportionality. So do not bother about it.

So, D times g by u square is something that is calculated, the distance is measured and that should be equal to the h theta. So, this is what is plotted. And if you plot it, you should get something like this, and here is 45 degrees.

Now, you cannot fit a curve like this into either the power law or an exponential or a linear curve. So, in order to get the functional form you have to guess what it could be. A curve like this could fit a parabola, could fit a sinusoid, and if you have some idea that it might be a sinusoid, then you fit a sinusoid, and if it is likely to fit a parabola then you fit a parabola. In this case, it would fit a sinusoidal graph.