Research Methodology
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Lecture - 39
Propagation of Errors, Part 01

So, we have learned that whenever we make a measurement, we never make *one* measurement. In order to get a value of a parameter or a variable, we always make a number of measurements, and from there we try to obtain what is a reasonable value that we can state.

The idea is that, if that experiment is repeated by anybody anywhere in the world, he or she should get the same result. For that purpose, we have to state the result that I have got, as well as the error bar. And the basic concept was a that we have to make a number of measurements, and there is a minimum bound to the number, which is around 25 for the measurement of a value. While if it is a measurement of a proportion, that number is far larger. That we have seen.

The second idea was that, whenever we are making a measurement, there is a population out there and we are taking samples. Even when we are measuring something like the mass of an electron, we are making say 25 measurements: 25 readings we are taking. But we know that we could have taken an ideally infinite number of measurements. If we had done so, then the average that we get is a reliable estimate of the mass of an electron. But we cannot take that.

So, there is a theoretically infinite number of measurements: that is a population. From there we are taking a sample and we have got a value. If we had taken a different sample, that means, we had made the measurement with another 25 readings, we would got a different value. A third time we would get another different value. So, there would be a distribution of the means that we get, and the central limit theorem ensured that the distribution of the means would be a normal distribution.

Once you had that, then the idea was that the mean of the normal distribution of the means, the mid value, would be the population mean, and we have got a particular value. Then we can talk in terms of the confidence level with which we can state that the population mean will lie within a certain range of the mean that you have got. That range is given as the error bar. The standard error of the mean is stated as the error bar.

And in different areas there are different confidence levels that are demanded, and based on that we define different confidence levels. Therefore, the error bar is defined with the appropriate confidence level which is appropriate for that particular field. In some fields it is basically one standard error, in some fields it is, say, 95 percent confidence, which is 1.96 standard error and so on and so forth. This is something that we have learnt. We have also learnt that sometimes in a data set that you have got, there can be outliers.

The outliers happen because of various reasons: they may happen due to instrumental error, they may happen due to a error in measurement, they may happen due to some physical phenomenon—that is also possible. And so, whenever we have got a measured value which is apparently a bit incongruous, out of a range, then we have to define the range that is reasonable. That is done by the box and whisker plot, which is done not on the basis of the means, but on the basis of the medians.

The median, the median and the max, their median, and the min, their median, and then that defines the inter quartile range. Let me just clarify it once again because there is sometimes an create confusion about it.

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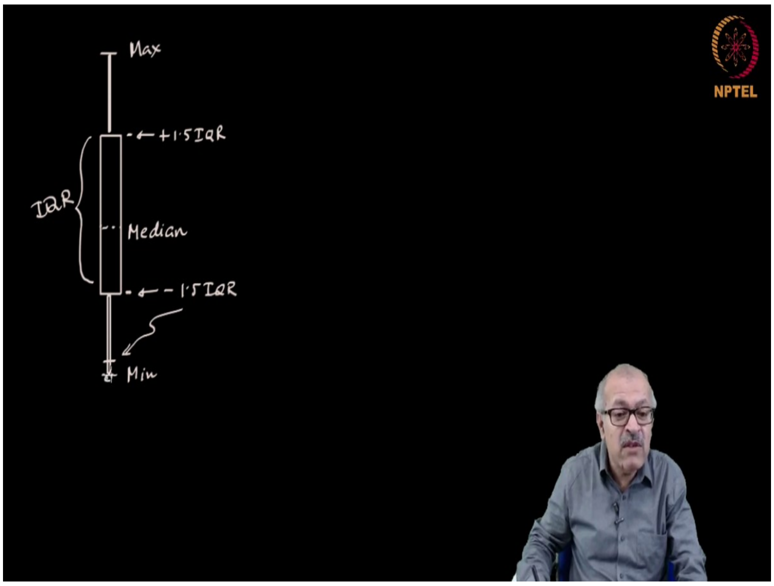

You have the box and you have whisker. The whiskers are designated like this. Here is, suppose, the median of the whole data set. Then the median of this is the max, and this is the minimum value of the data set, the median of this is the end here: the median between this median and the min is the end here.

So, this defines the box, and this range is the inter quartile range. This point plus 1.5 inter quartile range is the acceptable range of the data. And this point minus 1.5 IQR is again defines another acceptable range of the data. And suppose this point actually comes here. This value comes here. Therefore, whatever is the data here, that is an outlier. In that case, we specify that as an outlier like this.

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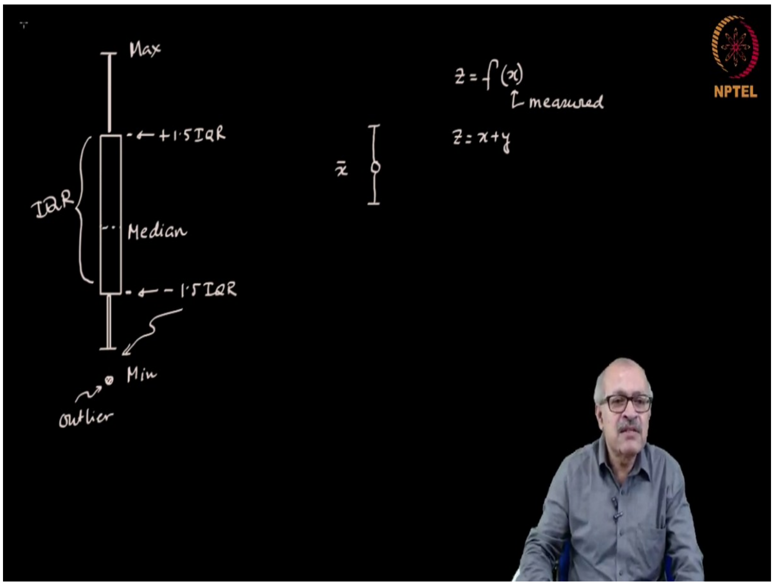

And then the minimum value becomes the lowest value of the data set that is above this value. Maybe it is here. So, then that becomes the min. This is how the whole thing is done.

Now notice, after this is done, then this data point goes out and a few data points remain. The things that we have talked about regarding the calculation of the mean, the standard deviation, the standard error of the mean, the confidence interval, all these calculations can be done with this, the remaining dataset, leaving out this outlier.

So, this is the outlier, but the outlier has to be presented in a paper as an obtained data value. You cannot, by yourself, eliminate that, because there may be information contained in the outlier. So, if you have done all the calculations with the data set and then later you identify an outlier, you might have to do the entire calculation again excluding the outlier. That will be a reasonable thing to do, but always point out that there was an outlier, that is the point.

So, with this, we have learnt how to represent data. This is the box and whisker plot. Normally in most presentations, we present it as a data point with an error bar designated like this.

Now, there are various situations in which you want to measure something, but you have access to something else which you can measure. Sometimes you can measure something and what you actually want to measure is a function of what you have actually measured.

A situation like: you want to measure some z which is a function of x, and it is the x that you have measured. There are certain situations where you have measured two values x and y and the one that you want to actually measure, maybe something like z, is equal to x plus y or x minus y or x times y.

The question then is, I have measured x as a value, and I have measured the standard error. How does it propagate as a value of z and a standard error of z? In this case I have measured a value of x; that means, a value as well as the error bar. Similarly, I have measured a value of y with an error bar. How do these propagate to decide a value of z and the error bar on z? That is a question that we will deal with today.

Let us take the first issue, where z, which I want to measure, is x plus y. x has been measured, y has been measured. I will need to define the symbols. What I will do is: x bar is the measured mean value of x. Similarly, y bar is the measured mean value of y. Now the deviations from x bar and y bar, let us denote these as delta x and delta y. So, these are the errors in x and y.

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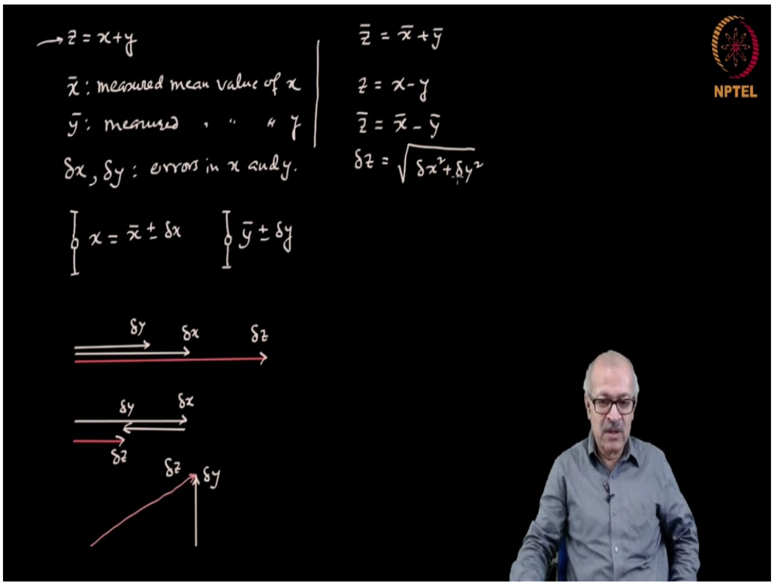

Basically what I have done is, I have measured x with an error bar. So, x is specified as x bar plus minus delta x. Similarly y is specified as y bar plus minus delta y. Then we are trying to find out how should we state z. Now, the mean value of z is easy to calculate, easy to understand. This mean value of z, the z bar, will be equal to x bar plus y bar. There is no difficulty in understanding that. But how do the errors propagate? That is the issue. ‘Errors’ means our delta x and delta y, how do they propagate to produce an error bar in z, delta z?

Now, it is easy to see that, if the errors are random errors, not systematic errors, then the errors in x and y are independent random errors. They are independently random. So, delta x can be positive or negative; delta y can be positive or negative. And they randomly occur. There is no correlation between them, and as a result of which, in some cases, when you add x and y, sometimes delta x and delta y might add, sometimes delta x and delta y might subtract from each other.

So, sometimes there will be larger error, sometimes there will be smaller error. Then what is the mean? It is something like this. Let us do that graphically. Suppose your x is this much and suppose your y is this much.

So, if the z is this, just addition of them, then it would be like this. This is where they are and they are adding up; x and y are adding up to produce z. The other possibility is, your x is here and I will better put with arrows; here I have x and from here I will draw the y so that it will be easier for you to see. I will draw it this way: y is this, so that your z is only this much.

So, in one case they were adding up. This was, say, delta x. This was delta y. And this was delta z. So, the errors added up because at the same time when x was in a positive direction, y was also in a positive direction. So they added up. In this case delta x was in positive direction, delta y was in negative direction and the delta z was naturally a subtraction of them. So, all possibilities exist: sometimes they will add up, sometimes they will subtract, and intermediate possibilities also exist.

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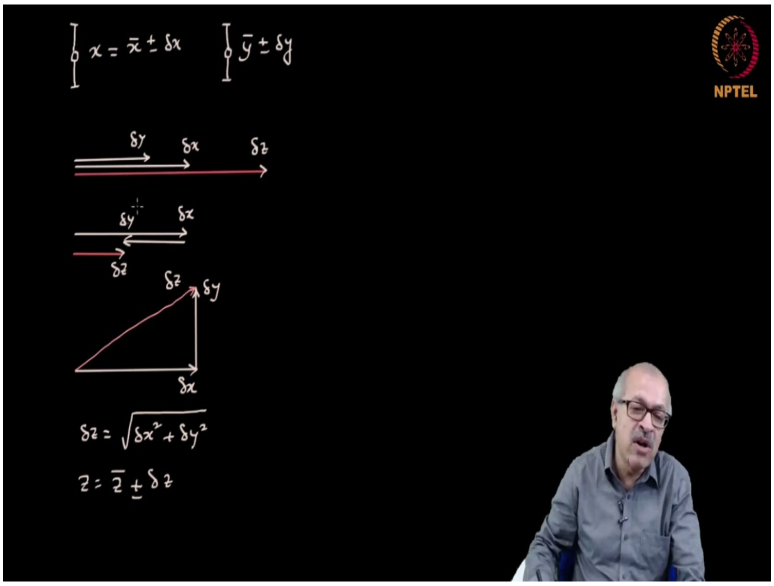

Now, in statistics it has been shown that, on an average, the delta z can be calculated as follows. delta x being drawn like this, and delta y being drawn in quadrature. Then this will be the average. So, this is our delta x, this is our delta y and this is delta z. So, since all the possibilities exist, they can add up, they can subtract, and all intermediate values exist. So, this can be this way, this can be this way, and the middle will be this, when it is at the quadrature. Then it is easy to express your delta z. delta z will then be expressed as delta x square plus delta y square root. So, this is how we express the standard error in the ultimate quantity z and we will express z as z bar plus minus delta z. This is how we will express.

$$\delta z = \sqrt{\delta x^2 + \delta y^2}$$

What if we want to calculate z as given as x minus y? In that case the average value, the mean value of z is easy to calculate. That will be x bar minus y bar, no problem. But what about the delta z? Now, if they are subtracted, then also all these possibilities exist. The errors can add up, the errors can subtract, and all intermediate possibilities exist. So, delta y can be this way, delta y can be that way. Exactly the same situation will pertain and therefore, the delta z will still be given as square root of delta x square plus delta y square.

Now, notice one important thing. As a result of the subtraction, the mean value of z can be small, but the error remains the same. Therefore, whenever there is a subtraction, whenever we want to measure something that is a subtraction of two measured values x and y, then there is a possibility that the error might swamp the result itself. The error might be big, because that is not reducing because of the subtraction. What is reducing is the mean value. The percentage error might be a very large. So, in actual measurement, we try to avoid situations like this. We try to avoid situations where we have to subtract two quantities. This is an important lesson in planning experiments.

Whenever we are planning an experiment, we always try to avoid a situation where we have to subtract two values. When we cannot, we at least make sure that the two values are widely different, so that this value would not be too small.

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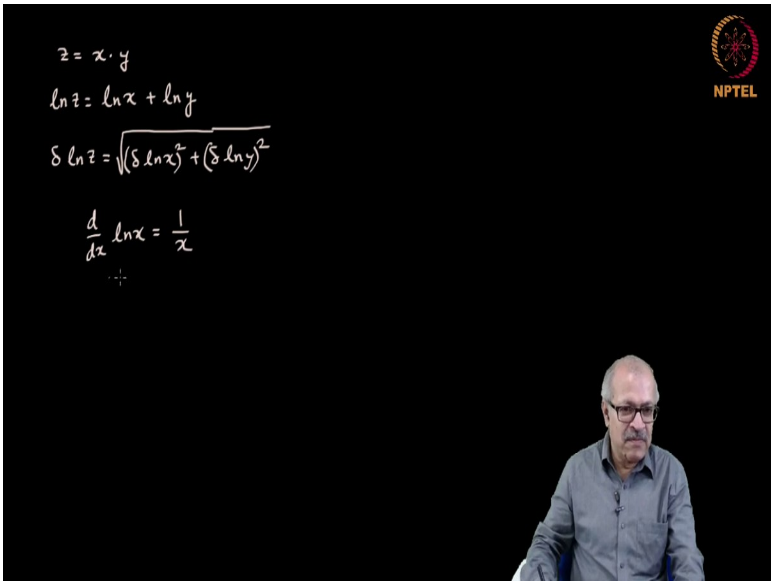

The next question is, what if z is equal to x times y? Then how would the errors propagate into an error in z? We have already learnt that if you have two quantities that add with each other: how the errors propagate. We have learnt that.

Here it is not in that form. It is not in a addition form. It is in a multiplication form. But with the easy algebraic procedure we can get it into a summation form, simply by taking a logarithm. ln z equal to ln x plus ln y. It is in a summation form. If it is a summation form, then the error in ln x and the error in ln y would propagate in the way we have done already. That is easy. So, we can easily see that, by the same logic, the delta error in the ln z will be square root of the delta error in the ln x square plus the delta ln y square.

$$\delta \ln z = \sqrt{(\delta \ln x)^2 + (\delta \ln y)^2}$$

It might be better to put a bracket to make things clearer. So, this is clear, but how do we know? We have measured x. How do you know what is the error in ln x? That question we have to deal with now. And that is not difficult to deal with because we know that d/dx of lnx is equal to 1 by x.

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

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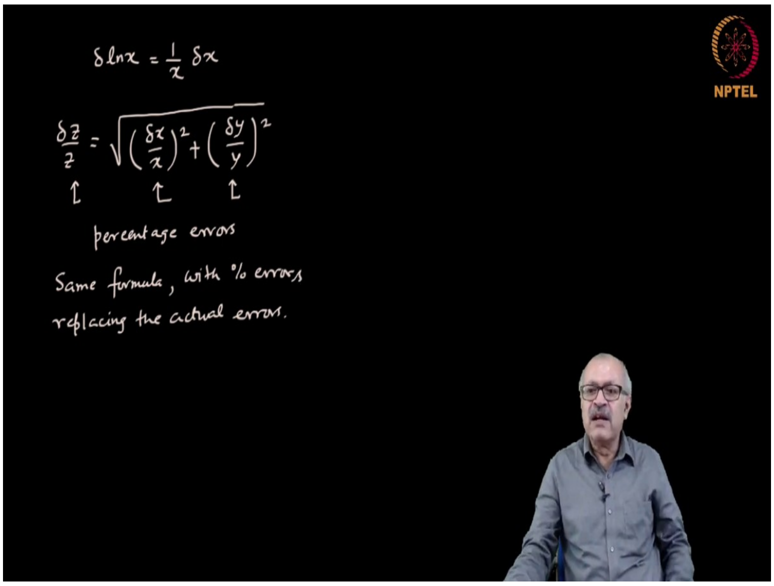

Now this means, if we now replace the d/dx by the incremental quantities, small change kind of quantities, then the delta of ln x becomes 1 by x times this delta x. Right?

$$\delta \ln x = \frac{\delta x}{x}$$

This is easy. Then we can substitute it here and we can express the delta. delta ln z will be delta z by z. So, delta z by z that will be equal to, I substitute here, it will be delta x by x square plus delta y by y square right? Easy to see.

$$\frac{\delta z}{z} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$$

Now, if this is true, we can interpret each of these. These are actually the percentage errors. This is the error as a fraction of the value. So, these are actually the fractional errors, or if you multiply that by 100, these are the percentage errors. So, you see, the way if I measure x and y and I want to measure the error in z which is x plus y, the way we did it, if we want to measure the error in x times y we would do exactly the same thing. The only thing is that, in this case, instead of the error itself, we will take the percentage error. So, in this case percentage error replaces: same formula with percentage error replacing the actual errors. So, this is how we do it for products, where we have to take a multiplication. So, the same formula, same way, there is no difficulty. There in case where we have to divide two quantities. Again the same issue will be there.

If you have to divide two quantities, then there will be minus here and the same things will occur again. The point that I had made in case of subtraction, the same issue will be there. The error in the resulting term might swamp the actual value. And so we normally try to avoid a situation like that and if we do have a situation like that, we make sure that the two ln values are somewhat different so that the ln z is not too small.