# Introduction to Reliability Engineering
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**Lecture 24**
**Markov Analysis (Contd)**

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Hello everyone, welcome back to our discussion on Markov analysis. In previous class, we discussed about degraded systems, we also started discussing about the how to incorporate repair in our Markov diagram and solve it. So, today we will try to solve that problem.

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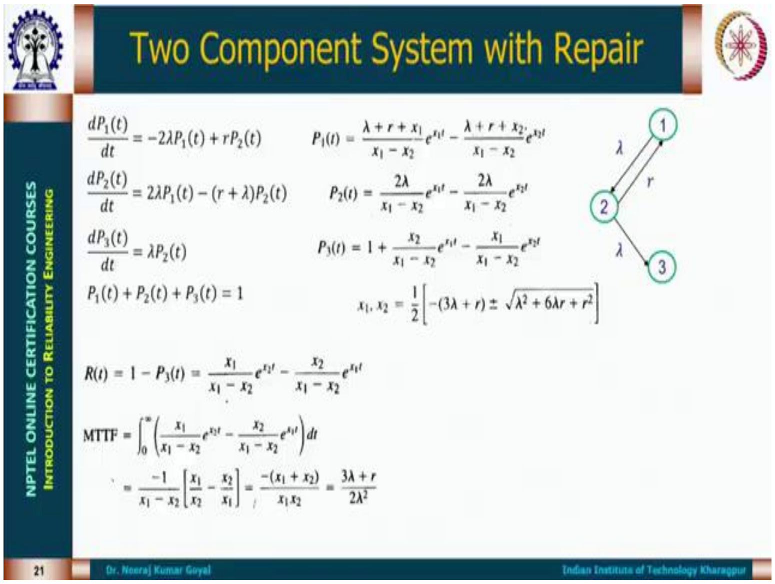


So, in previous class we discuss two component system with repair. So, here we consider that that this let me correct this and make it 2 lambda so, there is no confusion here. So, here we have 2 (sys) 2 components working or 2 systems working one of them is failed in state number 2 and here both the components have failed the when 1 failure and 1 working state is there then there is a repair possibility and once repair is successful within available time, then my system can be restored to back to the condition 1 system state 1 where both components will again start working.

So, here to solve this as we have seen or discussed earlier, we will try to use these 3 equations for P 2 and p 3 as well as p 1 p 2 p 3 is equal to 1 and using these equations, we will try to make a set of equations and try to solve it using the Laplace diagram.

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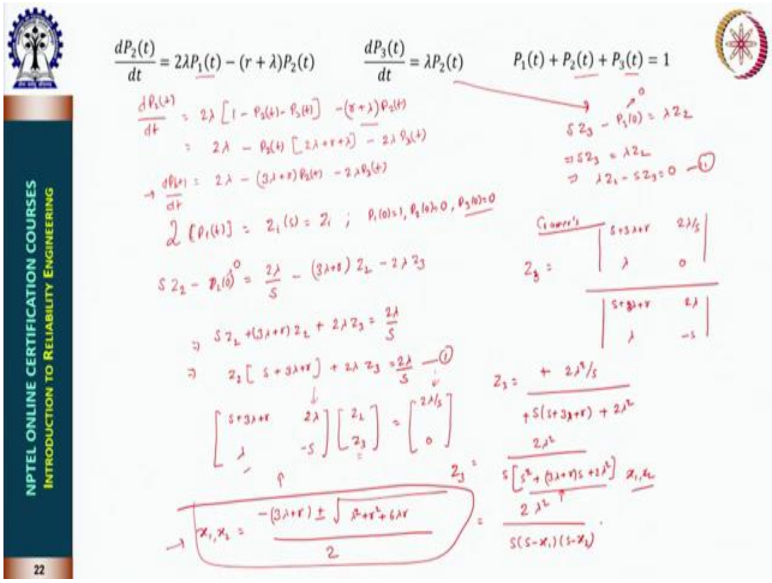


I have written these equations here for the reference so, that we can follow up to solve this. To solve this first what we what we will do we want this equation to be 2 probabilities. So, first I will convert this p 1 into P 2 and p 3 so, my both equations when they will then be belonging to P 2 and p 3 only.

So, this will be dP 2 t over dt is equal to 2 lambda into 1 minus minus P3T minus r plus lambda into P 2 t. Now, this when we solve then if you open this will become 2 lambda minus P 2 t now, P 2 T is having r plus lambda negative here and minus 2 lambda here. So, that is 2 lambda plus r plus lambda minus 2 lambda into P 3 t this if we solve further this will become 2 lambda minus here if I solve this this will become 3 lambda plus r into P 2 t minus 2 lambda P3T.

Now, let us consider the Markov transformation sorry, Laplace transformation if we apply Laplace transformation then we are calling Laplace of any probability function Pit let us say that is zis, since s will be common for all so, I will be using for short form as zi. So, if we do this then we know we take the Laplace transform for all for this complete equation, if we do that then Laplace transform of and we know that P 1 0 is equal to 1 p 2 0 is equal to 0 initial conditions are P 3 0 is equal to 0.

We know Laplace transform of differentiation term produces the s into Laplace transform of the term that is z 2 minus z minus p 2 at 0 that is equal to Laplace transform right side so 2 lambda this is constant value if you take Laplace so, then it will become 1 upon s minus 3 lambda plus r Laplace transform of P 2 will be z 2 minus 2 lambda Laplace transform of P 3 will be z 3 P 20 is 0.

So, my equation will turn out to be s z 2 I can take this z 2 term there so this will become plus 3 lambda plus r into z 2 this also I can take left hand side this will become 2 lambda z 3 which is equal to 2 lambda upon s so, here this equation I can write as z 2 into s plus 3 lambda plus r plus 2 lambda into z 3 is equal to 2 lambda upon s.

Similarly, I can solve this equation if I take a Laplace transform for this this will become s into z 3 minus p 3 at 0 will be equal to lambda z 2 and P t equal to 0 at 0 is 0 from here, so, this becomes s z 3 is equal to lambda z 2 this I can solve take on one side so, this will become lambda z 2 minus s z 3 equal to 0.

So, I have these two equations let us see equation 1 and equation 2. Now, these two questions if I make a set of equations, then I can make this equation in the matrix form if I convert this into matrix form then for this equation it will become s plus 3 lambda plus r this is 2 lambda and from here if I take z 2 multiplied by lambda and z 3 multiplied by minus s into z 2 z 3 will be equal to this equation is 2 lambda upon s and this will be 0 this right side is 0 this is 2 lambda upon s.

So, from system of two equations I have made the matrix equivalent now, this matrix equalent I can solve using the Cramer's so, Cramer's rule says that if I want to know the z 2 here then z 2 I can evaluate first let us see, we can evaluate z 3 what happens here if we see here my reliabilities probably p 1 plus P 2 and what is p 1 plus p 2, p 1 plus P 2 is 1 minus p 3 t so, or because this is on reliability ft is equal to P 3 t.

So, my reliability is 1 minus f t that is 1 minus P 3T. So, if I want if I know the z, if I know the P3T then I can know the reliability directly I do not have to calculate p 1 T and P 2 T still, we can calculate that, but for the reliability calculation if that is our only aim then knowing z 3 is enough or knowing P 3 is enough so, for P3 calculation, we can calculate the z3.

So, z 3 as we see, z 3 will be equal to so, if I want to calculate z3 z3 is on the second second row, so, I have to replace this second column with the output column. So, this determinant of s plus 3 lambda plus r first column will remain same second column will replace by the output column.

So, that will become 2 lambda upon s 0 divided by the determinant of my metric which I have to solve that is s plus 3 lambda plus r lambda 2 lambda minus s. To solve this I can take z3 is equal to this into 0 will be 0 and this will become minus 2 lambda square divided by s divided by if I multiply this this will become s into s plus 3 lambda plus r that is also minus minus s into this and this will be minus into lambda into 2 lambda minus 2 lambda square all are minus so I can make them plus.

And if I look at the denominator part or I can see this is 2 lambda square upon s into solving the denominator will give s into s s square plus 3 lambda plus r into s and this is 2 lambda square. So, once I have this this is my z 3 now z3 if I want to solve, I have to first convert this equation and we have to get it in parts.

So, this equation I can write it as this is equal because this is s squared so, I can find out the 2 routes for this. So, let us say those routes are x 1 x 2, then in that case this z 3 will be equal to s into s minus x 1 and S minus x 2, what is x 1 x 2 here x 1 x 2 are the 2 routes. So, for the quadratic term the 2 routes can be evaluated as x 1 x 2 is minus b.

So, minus of b is here 3 lambda plus r minus of 3 lambda plus r plus minus square root of b squared b squared b square of 3 lambda plus r 3 lambda plus r if I square, this will become 9 lambda squared plus r square plus 3 to the 6 lambda into r and minus 4 AC minus 4 a is 1 C is 2 lambda squared, so, 4 into 2 will become 8, lambda squared divided by 2 a a is 1, so, this will become 2.

Now, this if I solve further then 9 lambda square, and this is 8 lambda is square, so, 9 minus 8 will become 1. So, here to save space, I am doing the correction here itself, not I am doing I am subtracting it from 9 and what I get here I will get only lambda squared. So, that will be lambda squared plus r squared plus 6 lambda this becomes my x 1 x 2.

So, x 1 will we can take anyone x 1 is 4 let us say positive because we take the square root positive and x 2 can be the value when we take the negative value of the square root, this gives me the 2 values. So, here these values I can solve separately because I know the lambda value and r value for a given problem and I can find out x 1 x 2 to solve this equation for z 3 I can solve this equation in z of z 3 in terms of x 1 and x 2 itself which I will solve in next slide.

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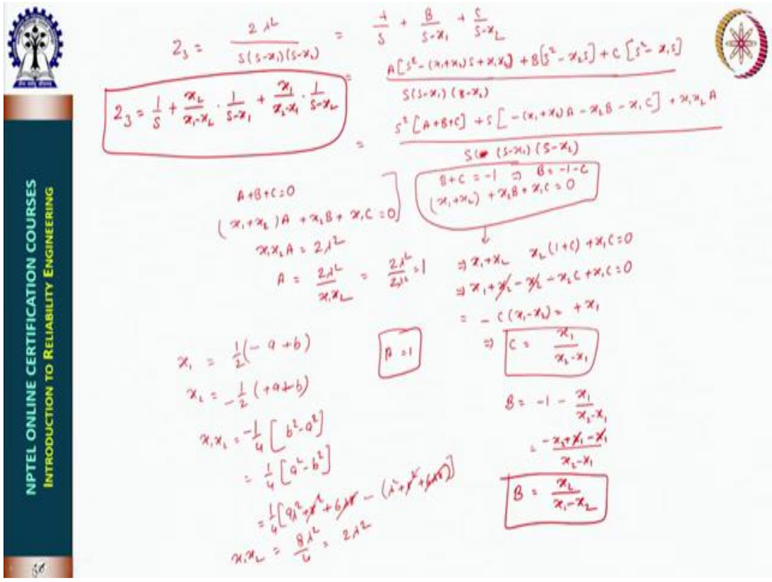

So, what I have I have z 3 is equal to 2 lambda square upon s s minus x 1 s minus x 2, 2 lambda squared upon s minus x 1 into x minus x 2 this I can solve as I can make this as it is equal to A upon S plus B upon S minus x 1 plus C upon S minus x 2. If I solve this here then as we know we can solve this by part so, this will become s into s minus x 1 into S minus x 2 and so, if LCM will be this if I divide by s that this will become s minus x 1 into S minus X 2 so, A into S minus X 1 into x minus x 2 that is s squared minus x 1 plus x 2 into s plus x 1 x 2 plus B into B into if I divided by x minus x 1 so, this will become S minus S into S minus X 2.

So, that will become S squared minus x 2 s similarly, C will get multiplied with s into x minus x 1 so, that will be S squared minus x 1 s. Now, if I compare the term denominator is equal and if I compare the terms on here then we will say because there is no s squared term here. So, for S squared terms like I will just do this calculation again x squared is a plus b plus c and s terms if I see then plus S into minus x 1 plus x 2 into a, s here is minus x 2 B and S here is minus x 1 C and remaining terms are plus x 1 x 2 a there is no term without s here. So, that is the only term divided by S into S minus S minus X 1 into x minus x 2.

So, if we compare this then a plus b plus c will be equal to 0 and all are negative. So, if I want I can write it in terms of x 1 plus x 2 into a plus x 2 into B plus x 1 into C because there is no s term here so, that will also be equal to 0 and last term is x 1 x 2 A that is equal that is the term without s.

So, that is equal to 2 lambda squared here or I can say is equal to 2 lambda square upon x 1 x 2 this x 1 x 2 if we see here x 1 is 1 with plus sign and x 2 will be with minus sign and we know so, I can say this is A and this is B, so, I can say this is a minus b one will be a plus b another will be a minus b.

So, x 1, I can write it as x 1 is equal to minus a plus b and x 2 is half of 8 and x 2 I can write as minus a minus b or if I take this and what is A and B here. So, if I take x 1 x 2 that will be equal to 1 by 4 I can take minus here so, this will become plus here 1 by 4 minus and a plus b into b minus a will give you b squared minus a squared or this will be 1 by 4 a squared minus b squared and what is my a here is s 3 lambda plus r.

So, A squared will become 9 lambda square plus r square plus 6 lambda r and what is b, b is square root of lambda squared plus r square plus 6 lambda r. So, minus lambda square plus r square plus 6 lambda r whole divided by 4 this if we see 6 lambda r will get cancelled r squared will also get cancelled, and 9 lambda squared minus lambda squared will give 8 lambda squared divided by 4 which is equal to 2 lambda squared.

So, my x 1 x 2 turns out to be 2 lambda squared. So, this a will become 2 lambda square upon 2 lambda square that will be equal to 1 my a is 1 so, now, I can use the same thing in these two equations. So, from these two equation b plus c will be equal to minus 1 and if I put a equal to 1 this will become x 1 plus x 2 plus x 2 B plus x1c is equal to 0.

Now, these two equations we can solve I can do that I can replace b here. So, b will be equal to minus 1 minus c I can place this here so, this will become x 1 plus x 2 minus x 2 because x 1 become minus x 2 into 1 plus C plus x 1 c is equal to 0.

So, this if I solve further then x 1 plus x 2 minus x 2 minus x 2 c plus x1 c is equal to 0 this is cancelled and if I take this then c into x 1 minus x 2 is equal to minus x 1 or I can say C is equal to x 1 upon x 2 minus x 1 if I multiply by minus sign on both sides and this will become x 1 x 2 minus x 1 and what is b, a is equal to 1 we already got C is equal to x 1 upon x 2 minus x 1 and b is equal to minus 1 minus c.

So, minus 1 minus x 1 upon x 2 minus x 1 so, x 2 minus x 1 if we take common then this will be minus x 2 plus x 1 minus x 1 so, minus sin is 0 so, minus sin can be taken below so, this will become x 2 upon x 2 minus x 1 minus x 2.

Now, here if we see my equation z 3 z 3 will now become a is 1 1 upon s plus b is x 2 upon x 1 minus x 2 into 1 upon s minus x 1 plus C is x 1 upon x 2 minus x 1 and into 1 upon s minus x 2. Now, this equation which we have got, I am removing some equations here because we have already solved this. So, we will have a little bit more space to understand this.

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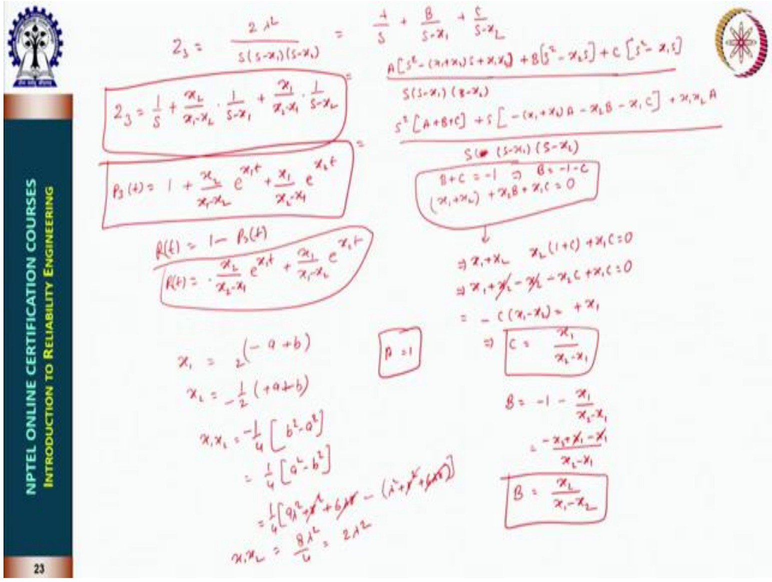


So, here, we will now take the Laplace inverse, which is quite straightforward here. And Laplace inverse as we know Laplace inverse of 1 upon s is a constant quantity. So, Laplace inverse of z 3, if you take Laplace inverse then z 3 of Laplace inverse of z3 P3T, Laplace of z 3 Laplace of p 3 was z 3, so, Laplace inverse of z3 will be P3T and Laplace of 1 is 1 upon s.

So, Laplace inverse of 1 upon S is 1 plus x 2 upon x 1 minus x 2 and we know Laplace inverse of 1 upon s minus x 1 is e to the power plus x 1 t. Similarly, plus x1 upon x 2 minus x 1 e to the power x 2 t. So, if we see here because x 1 x 2 is known to us, we already have calculated here, if you put x 1 x 2 calculated value from here in this equation, I will get the P3T and RT I can get as 1 minus P3T. So, this will be 1 minus this. So, 1 minus 1 will be 0 and this will be negative sign negative sign if I take below this will become reversal sign reversal will be there.

So, this will become x 2 upon x 2 minus x 1 e to the power x 1 t and this will be plus because this is negative so, this becomes x 2 minus x 1 this becomes x 1 minus x 2 upon x x 1 minus x 2 e to power x2 t the way we have calculated P3 t we can also calculate the p 2 T in a similar fashion by using the Cramer's rule.

If you see here my rt is x 1 upon x 1 minus x 2 x 2 T x1 upon x 1 minus x 2 e to the power x 2 t if I take this x 1 minus x 2 then this will become minus minus x 2 upon x 1 minus x 2 e to the power x 1 t minus x 2 upon x 1 minus x 2 e to the power x 1 T.

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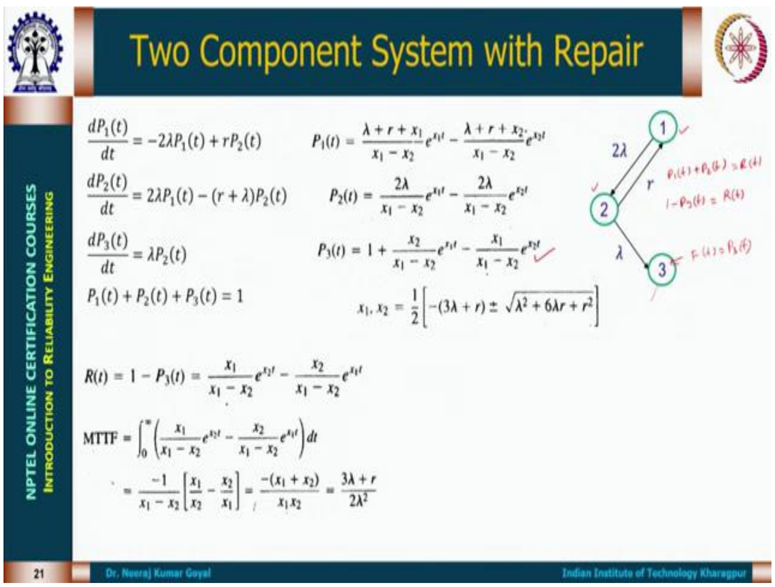


Same value which we have got there, this is P3T is also saying 1 plus x 2 upon x 1 minus x 2 e to the power x 1 t and this we have written x 1 upon x 2 minus x 1 e to the power x 2 T same value which we have got here same are reflected there. So, we are able to calculate reliability here directly.

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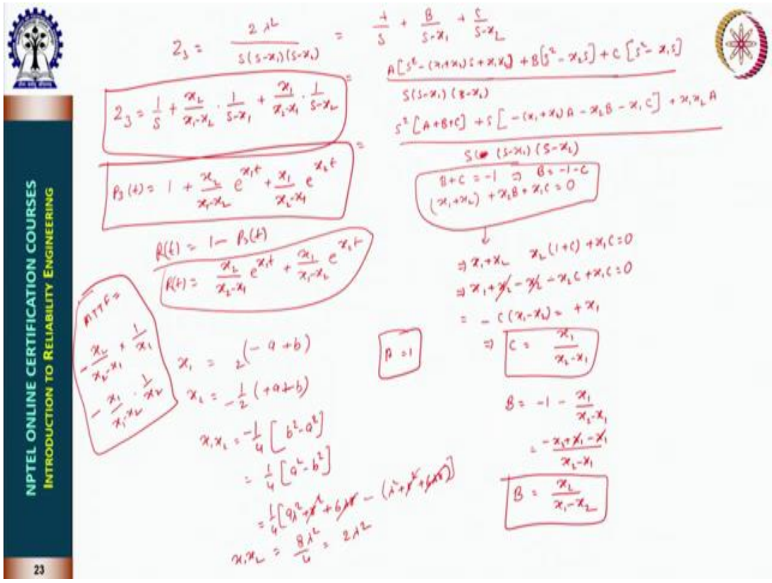


If I want to calculate MTTF, then again MTTF will be x 2 upon x 2 minus x 1 into 1 upon x 1 as we see minus lambda it comes so this will be minus sign and again minus of x 1 upon x 1 minus x 2 into 1 upon x 2.So, we can solve this and we can we will be able to get the values of MTTF also I have explained this for calculating P3T we can follow the same process and we can get the value of P2T also.

I will try I will show you for so that you will be able to follow up better. So, that same process what we have done earlier for P3t, we will follow the same process for P2T, so that you are able to follow it.

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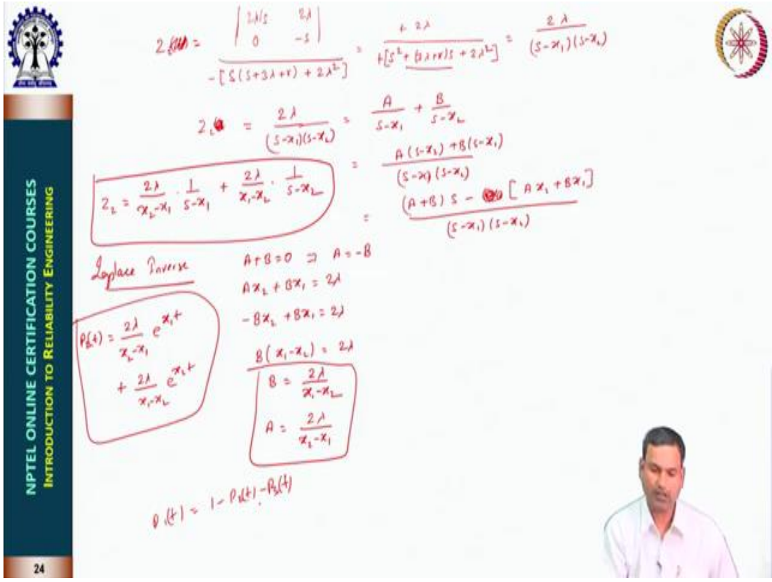

So, as an exercise, I am doing it for z2t. Now z2 t as we have seen here. If we solved from here, then z2 is the first row. So, that means first column will be replaced by the output column. So, 2 lambda upon s 0. So, (deno) determinant of 2 lambda upon s 0 and 2 lambda minus s second column will remain as it is 2 lambda minus s divided by the denominator term we already solved that denominator term term came out to be see same thing will come up s s minus like we got here.

This value was 2 lambda square upon s divided by s into this so, rather than 2 lambda squared by s this this was the determinant value for this that determinant for this was this s into s plus 3 lambda plus r plus 2 lambda squared s into s plus 3 lambda plus r plus 2 lambda square. Now, this if we solve further this will be equal to this multiply by this will be minus 2 lambda s into s will be 1 divided by and 2 lambda into 0 will be 0 and this we already saw this as s square plus 3 lambda plus r into s plus 2 lambda squared this was also negative as we got earlier this was also negative. So, this will also become positive again.

Now, we got this value now, to solve this what we can do we can again put it like 2 lambda upon s minus x 1 and x minus x 2 the x 1 x 2 will be same, because this is the same equation with what we solved earlier. So, x 1 x 2 are going to be the same value, what we calculated here we only have to solve this for getting the z 2.

So, this is not t sorry this is z 2, z 2 function of s. So, j 2 is equal to 2 lambda upon s minus x 1 into S minus X 2 now, this again I can put it into two parts a upon s minus x 1 plus b upon s minus x 2. So, if I solve this this will be s minus x 1 into S minus X 2 a into S minus X 2 plus b into S minus X 1.

So, this implies a plus b into S minus x 1 minus a into x 2 plus b into x 1 divided by s minus x 1 into S minus X 2 now, denominator is same and there is no s term so, we can make equations from here A plus B is equal to 0 and the constant term here is a x 2 plus b x 1 that is equal to 2 lambda. So, from here I can say a is equal to minus b if I put it here then this will become minus b x 2 plus b x 1 is equal to 2 lambda.

So, if I take B here common then x 1 minus x 2 is equal to 2 lambda so, b will be equal to 2 lambda upon x 1 minus x 2 and as we know what is my a a is minus b so, a will be equal to minus of this so, I can say 2 lambda minus sign I can take in denominator that will become x 2 minus x 1 so, here I have got a and b both so, my z2 z 2 I can write it as a upon x minus x 1 that is 2 lambda upon x 2 minus x 1 into 1 upon s minus x 1 plus B, B is 2 lambda upon x 1 minus x 2 into 1 upon s minus x 2 this if I solve by Laplace inverse then left hand side will give me P 2 T.

And if I solve this first term will come as it is 2 lambda upon x 2 minus x 1 and this value will give me e to the power x 1 T and second value is 2 lambda upon x 1 minus x 2 e to the power x 2 t. So, this is how x 1 x 2 is known to me already and lambda is also known to me already. So, I can get the P 2 as the function of time.

So, as you see here we are able to calculate P 2 and P 1 I can calculate as 1 minus P 2 T minus P 3t, I do not have to calculate, again, using the same equations for the P1t, that I can directly calculate as 1 minus P 2 t minus P3t.

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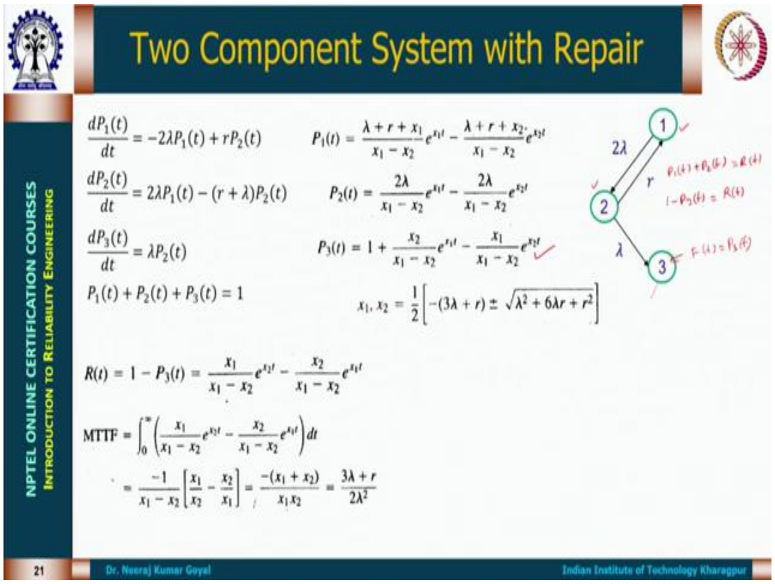


This same thing is coming here so, here by using these equations we are able to solve and we will be able to get the probability values and relative values also we are able to calculate MTTF also we are able to calculate and x 1 x 2 is also here.

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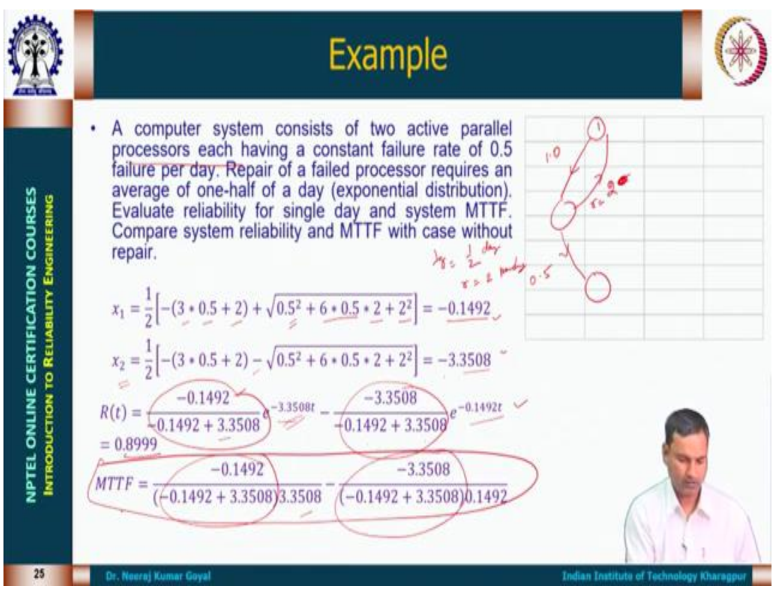


So, let us see one example here that a computer system consists of 2 active parallel processor each having a constant failure rate of 2.5 failures per day. So, that means, if I make this diagram then I had 3 states here.

$$x_1 = \frac{1}{2}\left[-(3*0.5+2)+\sqrt{0.5^2+6*0.5*2+2^2}\right] = -0.1492$$

$$x_2 = \frac{1}{2}\left[-(3*0.5+2)-\sqrt{0.5^2+6*0.5*2+2^2}\right] = -3.3508$$

$$R(t) = \frac{-0.1492}{-0.1492+3.3508}e^{-3.3508t} - \frac{-3.3508}{-0.1492+3.3508}e^{-0.1492t}$$

$$= 0.8999$$

$$\text{MTTF} = \frac{-0.1492}{(-0.1492+3.3508)3.3508} - \frac{-3.3508}{(-0.1492+3.3508)0.1492}$$

Now, here 0.5 is for the 1, 1, 1 part 1 processor. So, for 2 processors this will become 1 point 0 failure per day 1 failure per day and for 1 processor it will be 0.5 failure per day and there is a repair repair failure process requires an average 1 half of a day that means, it will be 0.5 so repair rate is half of the day half per day half of the day sorry repair rate is means repair time is 1 upon r mean time to repair is 1 by 2 half day.

So, r will be equal to 1 by 1 by 1 by 2. So, that will become 2 per day in half day repair can complete so, in 1 day you can do 2 repairs. So, this will be 2, r will be equal to 2 and lambda is equal to 0.5. So, if I use the same thing in this equation, I can solve this. So, my x1 values half of minus 3 into lambda plus r so, 3 into lambda is 0.5 r is 2 and square root of so, x 1 I am taking as the positive quantity x 2 I am taking as the negative quantity and this is lambda square plus 6 lambda into r plus r squared. If I solve this this value comes out to be minus 0.1492 x 2 is also same value but sign is negative here.

So, this will be minus 3.3508 because, once we have got x1 and x 2 both then our rt as we calculated earlier our rt was coming out to be x 2 upon x 2 minus x 1 e to the power x 1 T and x1 upon x 1 minus x 2 e 2 x 2 T. So, same thing will happen E to the power x so, if I take x 1 minus x 2 then this term will come first and this will become minus. So, here the formula use this x 1 minus x 2 this is x1 this is minus x 2.

So, x, x 1 upon x 1 minus x 2 e to the power minus x 2 minus x 1 upon x 1 minus x 2 e to the power minus x 1 this formula we have used same formula what we developed earlier and using this we can get the reliability as 0.8999 and MTTF is again as we discussed earlier, MTTF is also whatever value we get here that will be same here divided by 3.3508. And similarly, whatever value this we got here same as here divided by 0.1492 and this gives me the MTTF.

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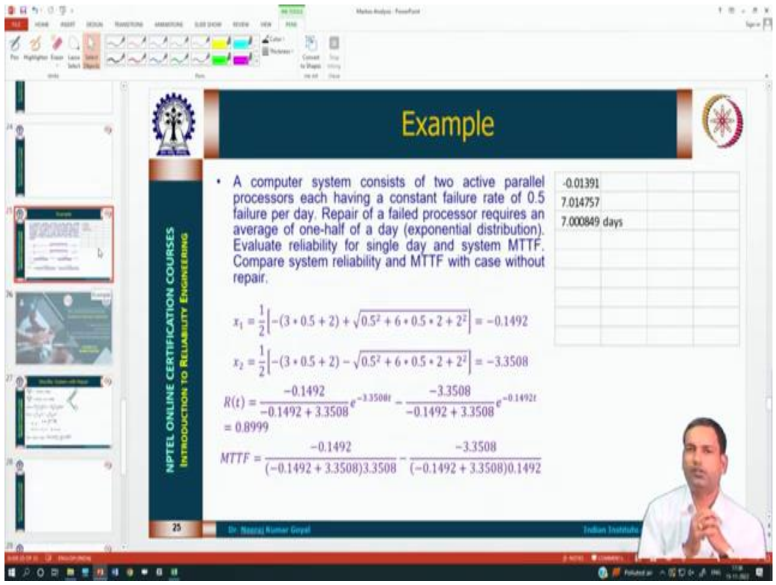


So, using same we can if you want we can solve this in Excel how much value of MTTF is coming. So, I will equal to minus of 0.1492 divided by minus of 0.1492 plus of 3.3508 multiplied with 3.3508. So, this is my first term and second term is equal to minus minus will become plus.

So, that is 3.3508 divided by minus 0.1492 plus 3.3508 multiply with 0.1492. And MTTF will be this plus this. So, my MTTF is coming out to be around 7 days here all values were in per day. So, my MTTF comes out to be approximately 7 days. So, here as you have seen, we are able to calculate reliability and MTTF when repair is considered repair whenever we consider repair as we see that we have the loop because in our equation becomes a little bit complex to solve, but they are solvable.

So, maybe little and if the problem becomes more complex, then we may be able to use other methods like approximation numerical methods to solve the differential equations the same, so we will stop here today we will discuss with one more system in next class. Thank you.