Fundamental Fluid Mechanics for Chemical and Biomedical Engineer

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Dimensional Analysis -Ipsen Method

In this lecture, we are going to continue the discussion of dimensional analysis. In the dimensional analysis, we talked about π Theorem or Buckingham π Theorem that, how do we choose or how do we find out the non-dimensional groups for a particular problem using the repeating variable method.

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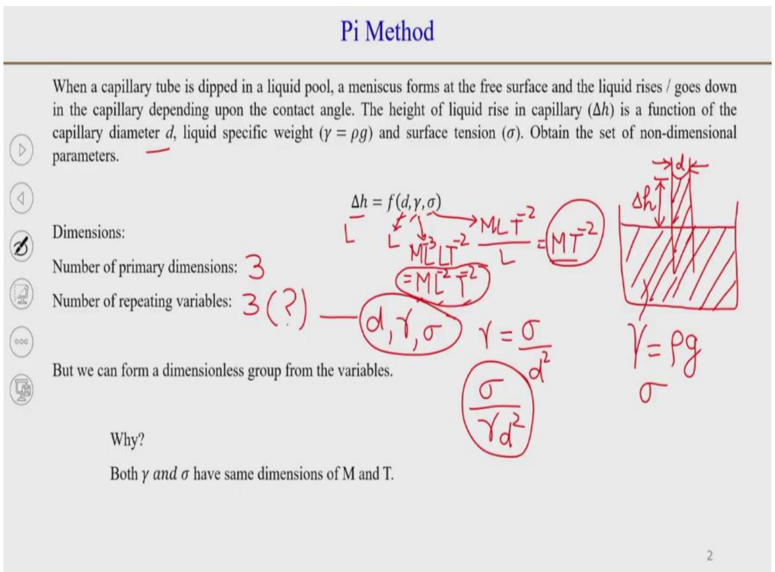

Let us look at a different problem, which talks about that when a capillary tube is dipped in a liquid pool. So, if you have pool of liquid, which is filled with some liquid, let us say water and a capillary is dipped in this pool of liquid. Now because of the contact angle, the water will rise in this capillary and the height of this water rise in the capillary is let us say Δh. The specific weight of this liquid γ, which is basically a product of density of the fluid and acceleration due to gravity. So, the liquid rise in capillary Δd depends on the diameter of the capillary, which is d, the specific weight of the capillary and surface tension.

Of course, it will also depend on the contact angle, but that will give us a non-dimensional number anyway. So, what we need to do is combine these and find a set of non-dimensional parameters. So, we have that Δ h is a function of capillary diameter d, the specific weight γ and surface tension σ. So, let us start with writing out the dimensions of these. So, the dimension of h is L, which has dimension of length, d again has the dimension of length, σ is dimension of force divided by length. So, MLT^{-2} divided by L, which is MT^{-2}. And the dimension of σ is ρg. So, ML^{-3} into LT^{-2}.

So, this gives us the dimension of γ is equal to M, L^{-2}, T^{-2}. Now, the number of primary dimensions in this case we have 3 primary dimensions, M, L and T. So, the number of primary dimensions are 3. So, if we use our previous theory, then, the number of repeating variables should be equal to 3, but is that so? So, if we look at that, if the number of repeating variables are 3, then, the independent variables that we have here is, these 3 variables can be d, γ and σ.

So, if you try to find a dimensionless group from this, but before that, one condition is that these repeating variables should not form a group among themselves. A small inspection will tell you that γ is MT^{-2}. So, the dimension of σ is MT^{-2}. So, if you look at the γ, which has dimension of MLT^{-2}, which is σ over d^2.

So, if you see the dimension of σ is MT^{-2} and the dimension of d is L so d^2 L^2. So, you can form a non-dimensional group σ over γ d^2 is a non-dimensional group in itself. So, the condition or the requirement that the repeating variables should not make a dimensionless group among themselves is not satisfied here. So, what is the problem?

Let us, if you revisit the π theorem, the problem lies here. If we look at the dimensions of σ, which is MT^{-2} and dimensions of γ, which is ML^{-2} and T^{-2}, the dimensions of M and T are same in both the cases. So, M^1, T^{-2}, M^{-1}, T^{-2}. So, they are going to be same, so that will not help.

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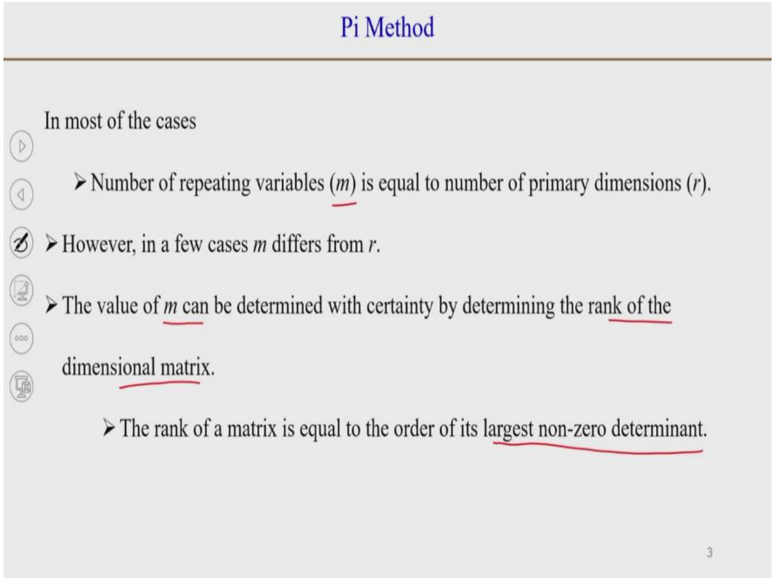

Now, the cases like these, they will arise very rarely. Most of the problems, the theory that we developed in the previous class that the number of repeating variables will be equal to the number of primary dimensions will hold good for most of the problems except a few. So, in most, most of the cases, that is true that the number of repeating variables m is equal to number of primary dimensions r.

Then, the more fundamental definition or the more fundamental principle is that the value of m, it will be the rank of dimensional matrix. So, just to remind ourselves that the rank of a matrix is equal to order of largest non-zero determinant. If you make a determinant from the dimensions of the matrix, then, the rank of such a matrix will be the value of m. So, let us revisit the problem again.

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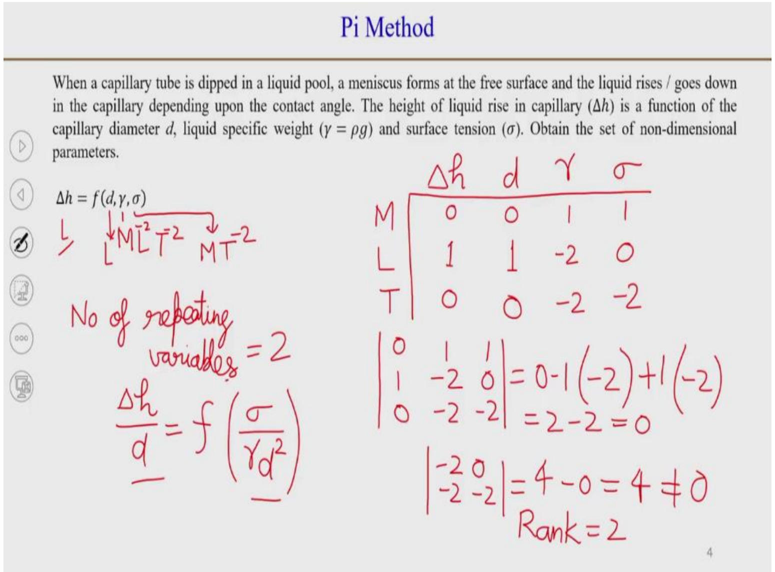

We have Δ h is equal to Fd γ, σ function of d γ σ. So, the dimension of h is L, the dimension of d is L, dimension of γ is ML^{-2} T^{-2} and the dimension of σ is MT^{-2}. So, we can make a dimensional matrix. And to make the dimensional matrix, what we need to do is we can write in a column the dimensions MLT for all the variables.

So, Δ h, d, γ and σ. So, for Δ h, what we are going to write? The values here will be the power raised on ML and T for each variable. So, for Δ h, the dimension is L. So, M^0, L^1, and T^0. For d, M^0, again, L^1 and T^0. For γ, 1 -2 and -2, it is 1, 0 and -2.

If we take the largest determinant in this case, if we take a determinant from this dimension, so this is the dimensional matrix. From this dimensional matrix, if we take the largest determinant, which will be of, if we take 0, 1, 1. 1 -2, 0, 0 -2, -2. So, the value will be 0 -1 into -2 -0 +1 into -2. So, this will come out to be 2 -2, which is equal to 0. So, this cannot be a matrix cannot be of rank 3.

So, let us check the next smaller matrix, which is -2, 0, -2, -2. And this determinant, the rank of matrix will be, if this is non-zero, then, the rank will be 2. So, let us see. 4 -0 is equal to 4, which is not 0. So, the rank of this matrix is equal to 2. So, that means the number of repeating variables is equal to 2. So, if we choose 2 repeating variables in this, the Δ h is a function of d, γ, σ. So, from that we will be able to find Δ h by d as one non-dimensional group and the other non-dimensional group as we just saw that we can form a non-dimensional group from the 3 of these.

So, σ over γ d^2 ML^{-2}. So, these will be two dimensionless groups, Δ h by d and σ by γ d^2. So, that is a limitation or a complexity in π theorem. Still, this is the most popular method, which is used for dimensional analysis.

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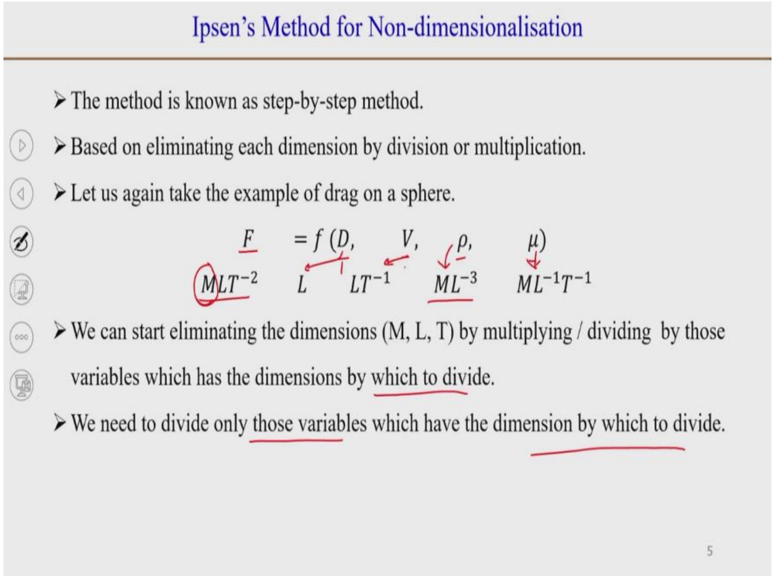

Now, there is another method, which is called step-by-step method and which is, one is likely to follow this step-by-step method if one is doing it in a sequential manner just by using say common sense. So, the Ipsen's method is called a step-by-step method and it is basically based on eliminating each dimension by division or multiplication. So, by looking at the variables, you tend to divide, you write-down the dimensions and then you tend to divide one by one by the variables so that the dimension for each, the power for each dimension M, L and T or other fundamental dimensions or primary dimensions, the power is to them become 0.

So, let us take the common example that we have been taking that the drag force on a sphere is a function of the diameter of the sphere, the velocity, relative velocity between the fluid and the sphere, the density of fluid and the viscosity of fluid. And we write down the dimensions of these. So, dimension of force is MLT^{-2}, dimension of diameter is L, dimension of velocity is LT^{-1}, density is kg per meter cube. So, ML^{-3}, the unit of viscosity is kg per meter per second. So, ML^{-1}, T^{-1}.

Now, what we need to do is we can start eliminating the dimensions. So, we can start eliminating M, L and T one by one by multiplying or dividing as the case may be by those variables which has the dimensions by which to divide. So, for example, here, let us say, if we want to eliminate M, we start by eliminating M. So, we can choose density and we can divide by density to all those variables, which contain M. So, we can divide F by density, so F by ρ. These 2 variables D and V, they do not have M there. So, we do not divide them, then, ρ by ρ, so this becomes dimensionless, we do not need to consider it anymore and μ by ρ.

So, we have eliminated M from these variables and we are also one variable down. We do not need ρ anymore. Then, we can keep doing this thing. We need to remember or we need to note here that the division is required for only those variables which have the dimension by which to divide. So, basically this is one step ahead of inspection. It puts a sequence to the division by inspection of forming dimensionless groups by looking at the dimensions only or dimensions itself.

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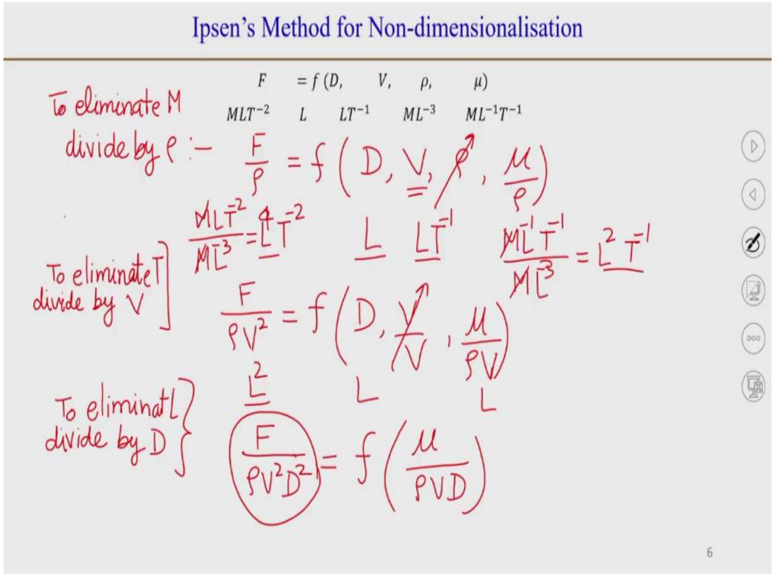

So, if you write down this problem and let us try to solve it. So, if we divide by M, so let us divide by ρ. We need to and we are doing this to eliminate M from each dimension. So, if we do it F by ρ and that will be a function of D remains same, V remains same and we will have ρ by ρ. So, we actually this variable has become dimensionless and then have, we have μ by ρ. And let us write down the dimensions again.

So, this will have dimension now, MLT^{-2} divided by ρ, which is ML^{-3} and M and M cancel out. So, what you have is L^2, T^{-2}, D is L, V is LT^{-1} and μ over ρ, so that becomes ML^{-1}, T^{-1} divided by ML^{-3}. So, M and M cancel out here and you have L^2 T^{-1}.

And if you remember, this is, μ by ρ is kinematic viscosity. Now, we have been able to successfully eliminate M from these dimensions. So, let us try another variable T. And for T, we will choose velocity V, so let us eliminate T. So, to eliminate T, let us divide by velocity V. So, we have to divide or multiply in such a manner that the dimension of time is eliminated. So, this has time^{-2}, where velocity has time^{-1}. So, we will divide by velocity squared F over ρ V^2, L remains same.

So, that will be a F over ρ V^2 will be a function of D, again V by V so this has become dimensionless and we do not need to consider it anymore. And μ that is time^{-1}. So, μ by ρ, we will divide it by V only. Now, the dimensions, if we look at the dimensions for this group will be L^2 T^{-1} and we have divided by L, V^2, this should have become L^4, the 3 +1, 4. So, now we have the dimension of this will be L^2, L^4 T^{-2} divided by L^2, so this becomes L^2, the dimension of D is L and μ by ρ will have the dimension of again L^2 T^{-1} divided by L, T^{-1}, so we will have dimension of L.

Now, we need to eliminate only L, so we can divide by D. So, to eliminate L, let us divide by D. So, we will have F over ρ, V^2 D^2 that is equal to function of D by D, so it will become dimensionless and we will not consider it anymore and μ by ρ V D because L has just 1 dimension or 1 power to it. So, we will divide by D. So, we are able to find 2 dimensionless groups, F over ρ, V^2 D^2, which is drag coefficient and inverse of Reynolds number μ over ρ V D. So, without finding any repeating variable, we have been able to identify the dimensionless groups.

So, the number of restrictions that we had, that we need to look at the repeating variables, we need to find out the, if the repeating variables, are they making a dimensionless group or is the rank of the dimensional matrix is equal to the highest rank and m is equal to r or not. So, all those restrictions are not there in this Ipsen's Method.

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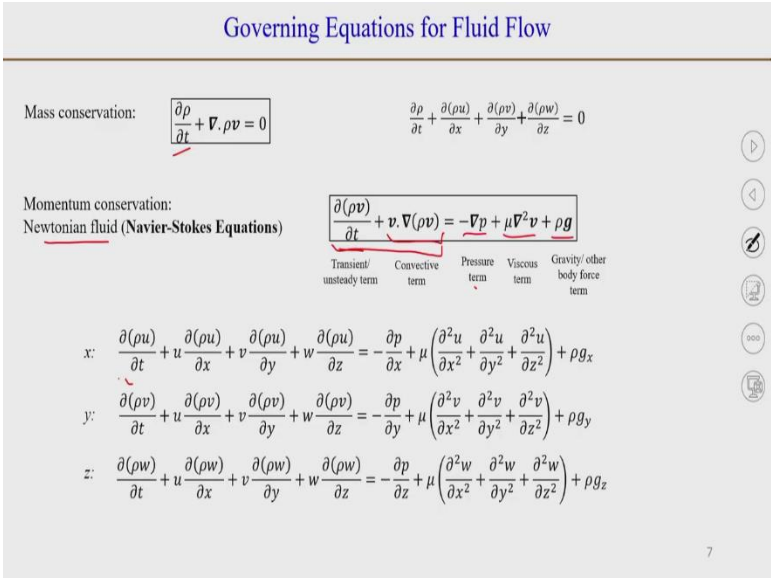

We have looked at for non-dimensionalizing the groups, we can just write down the dimensions and try to find out as you become more experienced and try to do see the dimensionless groups, you can do it by inspection only, you can do using π theorem or you can do using Ipsen's Method. What we are going to do now is look at the fundamental governing equations for fluid flow for an isothermal flow and try to non-dimensionalize those differential equations, which are called Navier-Stokes equations.

So, we, the goal is that we familiarize ourselves at the start of course itself with Navier-Stokes equations. And at the same time, we non-dimensionalize them and see that the dimensionless groups appear naturally when we non-dimensionalize these questions. So, if we write down the mass conservation equation, which is a ∂ρ by ∂t +Δ dot ρ v. So, ∂ρ by ∂t is partial derivative of ρ with respect to time.

If we write down this in Cartesian coordinate, then, as we saw when we defined, ∂dot v in the lecture on vectors, we can write down ∂ρ by ∂t +∂ by ∂x of ρ u +∂ by ∂y of ρ v +∂ by ∂z of ρ w. The momentum conservation equation, we will consider here the case for a Newtonian fluid. So, the constitute to equation, the relationship between the stress and strain is linear and that has been used in this equation.

So, the this is the Navier-Stokes equation or the momentum conservation equation for a Newtonian fluid. So it, the first term is the transient or unsteady term which is time dependent term, ∂ by ∂ of t of ρ v. The second term represents the convection of the fluid or the convection of the momentum because of the bulk flow. The first term on the right hand side is pressure term, next term is the viscous term and the last term is the term due to gravity or other body forces.

So, if you look at or as we read later on with this, the term on the left-hand side basically represents the acceleration multiplied by mass per unit volume. So, if you have a ρ, which is mass per unit volume multiplied by the acceleration, and these are different forces that are acting so a pressure term, viscous term and gravity term, which are all again in terms of force per unit volume the units are so, pressure, viscous are the surface forces and gravity or any other body force that might be acting, they will be taken into account in the form ρ g. So, these are the governing equations. If we write down the momentum conservation equation in the Cartesian coordinate system, we can expand this.

So, what we are going to do is look at the, we take a conservation equation, momentum conservation equation let us say for x and simplify it for a 2-dimensional steady flow and try to nondimensionalize it.

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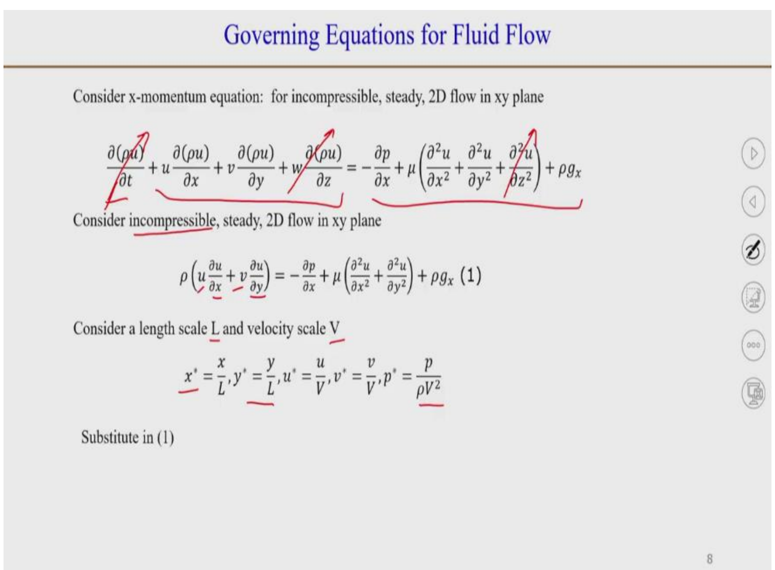

So, if we consider the x momentum equation for incompressible steady 2D flow, so let us write down the x momentum equation first. The same thing, we have unsteady term and that 3 terms represent the convective acceleration, and then the forces. Now, if we do it for a steady flow, so the first term, because it is time dependent, so, that will become 0. And because it is 2-dimensional flow so w will be 0 and ∂^2 by ∂z 2 will also be 0.

So, our equation will simplify and we consider flow to be incompressible and density is constant. So, we can take ρ as constant, and this comes out. So, ρ u ∂u by ∂x +v ∂v by ∂u by ∂y is equal to -∂p over ∂x, +μ ∂^2u by ∂x2 +∂^2u by ∂y2 +ρ g_x.

Now we need to nondimensionalize this. So, let us consider the scales for each of these variables. So, we have length variables, x and y. So these variables, we will nondimensionalize by length scale of the problem, let us say L. The velocities in the flow, u and v we will nondimensionalize velocity scale V and the pressure will be nondimensionalize by a dynamic pressure say ρ V^2, which again will have unit of pressure.

So, the dimensionless variables we will represent with an asterisk. So, x^* is equal to x over L, y^* is y over L, similarly u^* is equal to u by velocity scale V, v^* is equal to v over capital V and the dimensionless pressure p^* is equal to p divided by ρ V^2. Now, what we need to do is substitute these here, the values of x. So, in place of x, we will write x^* L, in place of y, y^* L and so on and then collect the scales together.

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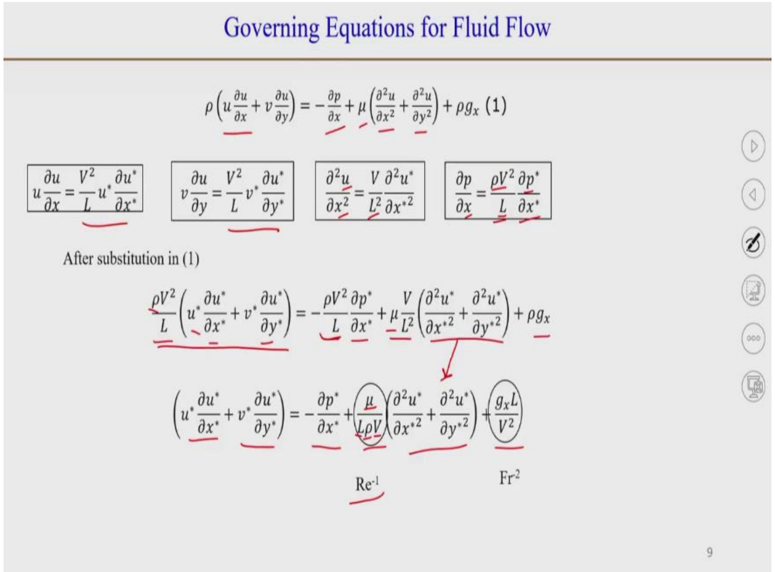

So, when we substitute it, what we get, when after substitution? So, let us look at the first term, u ∂u by ∂x. So, this u ∂u by ∂x term, when we write u will be u^* into V. So, you have 2 u in the numerator. So, that will be v^* or V^2 and 1 x in the denominator. So, V^2 by L into u^* ∂u^* by ∂x^*. Similarly, the second term, V ∂u by ∂y, that will be equal to V^2 by L into v^* ∂u^* by ∂y^*. The terms here in the viscous terms, so ∂^2 u by ∂x 2 will be, you have one u here, so v and x^2, so L^2, V by L^2 ∂^2 u^* divided by ∂x^{*2}.

Then, if we look at ∂p by ∂x, so p will be ρ V^2 multiplied by p^* and x you have so one L will come here. So, ∂p by ∂x will be equal to ρ V^2 by L into ∂P^* by ∂x^*. So, let us substitute these terms here. What we will see from these 2 terms? We can collect V^2 by L outside the brackets. So, we will have the left-hand term as ρ V^2 by L and within bracket u^*, ∂u^* by ∂x^* +v^* ∂u^* by ∂y^*.

Now, coming to the right-hand side, the first term as we saw ∂p by ∂x. So, we can substitute that will be equal to -ρ V^2 by L into ∂p^* by ∂x^*. The viscous term so μ is already outside the bracket. And these two terms for ∂u2 by ∂x2 and similarly for ∂^2 u by ∂y2. We will have V by L^2 multiplied by within bracket $\partial^2 u^*$ by ∂x^{*2} +$\partial^2 u^*$ divided by ∂y^{*2} +ρ g_x.

So, let us arrange the equation a bit. So, we can multiply it by L by ρ V^2. So, you will have left-hand side u^* ∂u^* by ∂x^* +v^* ∂u^* over ∂y^*, ρ V^2 by L when you divide by ρ^2 by L, this term will cancel out. So, you will have or you will be left with -∂p^* by ∂x^*. When you divide this mu, V by

L squire divided by ρV^2 by L, so you will be left with one L in the denominator, μ will remain in the numerator.

And in the denominator, you will have V^2 and one V at the top. So, in the denominator, you will have one V and one ρ because of the ρ present here. So, this ρ will come in the denominator. So, you will have μ over L ρ V and within bracket the same term will come here. And the last term ρg_x divided by $\rho^2 \rho V^2$ by L, so L goes up ρ, ρ will cancel out and you will have g_x L by V^2.

So, you can see that this is μ over L ρ V is inverse of Reynold's number and g_xL by V^2 or V^2 by gL is Froude number of Froude number is to the power -2. So, you have, by non-dimensionalizing the equations, you can find out the non-dimensional numbers. When you are talking about other non-dimensional numbers, for example, if you consider the transient term, you can get another non-dimensional number. If you nondimensionalize the boundary conditions, then, you can get another nondimensionalize number. So, the non-dimensional numbers can also be found out when you nondimensionalize the governing equations as well as boundary conditions.

So, now we know quite a few methods, which we can use to dimensionalize a nondimensionalize a problem. We can look at the variables and try to see by looking at their dimensions itself, we can try to form up some dimensionless groups. When we come more experienced and we know about more dimensionless numbers, we can try to relate the relations between the variables with the non-dimensional groups, we can use Buckingham π theorem using repeating variables. We can use Ipsen's Method or when we know the problem itself, we should try to see and find out what are the relevant dimensionless number for a particular problem from the, by nondimensionalizing the governing equations. Thank you. So, we will stop here.