Fundamental Fluid Mechanics for Chemical and Biomedical Engineer
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Dimensional Analysis - π Theorem

Hello. So today's lecture, we are going to talk about dimensional analysis. So, before we talk about dimensional analysis, let us look at what our base and derived quantities.

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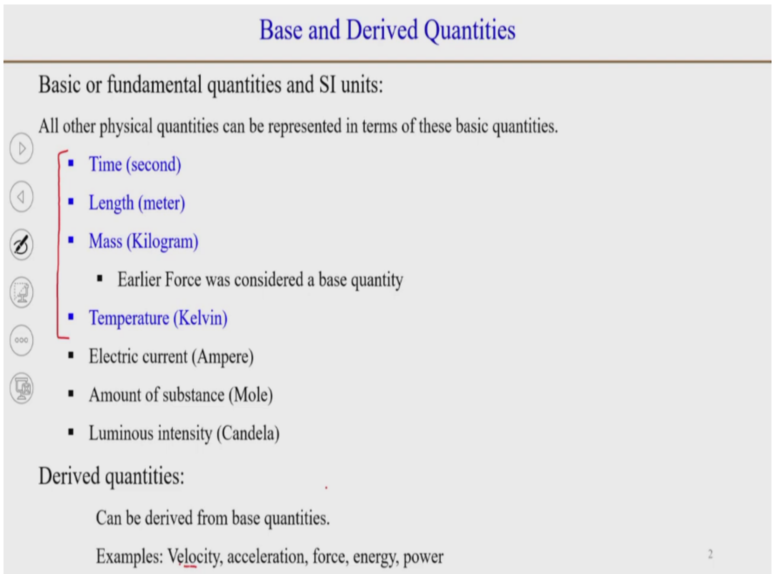

So, according SI System, in the SI System, it has been decided that there are some fundamental or base quantities, based on which or using which units of all other quantities can be derived. So, in the current system, the quantities which are considered to be base quantities is time, the unit, SI unit for it is second, length SI unit is meter, then mass SI unit is kilogram. So, before few decades back, force was considered as a base quantity, but now in place of force, it is mass that is considered a base quantity, then temperature the SI unit is Kelvin, Electric Current SI unit is ampere, amount of a substance where the SI unit is Mole and luminous intensity where SI unit is Candela.

So, in this course, in fluid mechanics, what we will be considering or the main parameters, which will be of our concern will be time, length, mass and temperature if there is heat transfer involved or if the flow is compressible. So, in such cases, we will consider in the base quantities that are of relevance in this course are time, length, mass and temperature. Now, the derived quantities as the name suggests that all the quantities that can be derived from the base quantities are called derived quantities.

And some of the examples, for example, that we will use in this course extensively, velocity, acceleration, force, energy and power. So, all these are derived quantities, whose units we can write in terms of the base quantities. For example, the unit of velocity is meters per second, so length per unit time. Similarly, acceleration LT^{-2}, force mass into acceleration, so, MLT^{-2} and so on.

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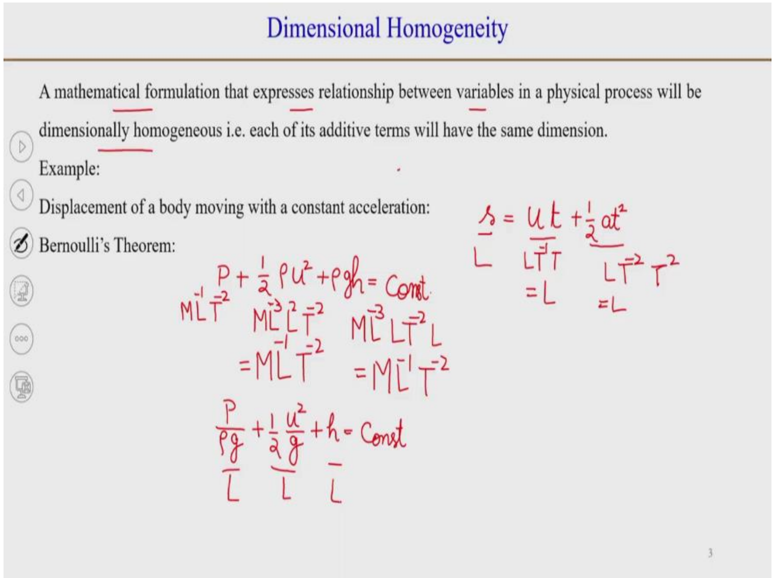

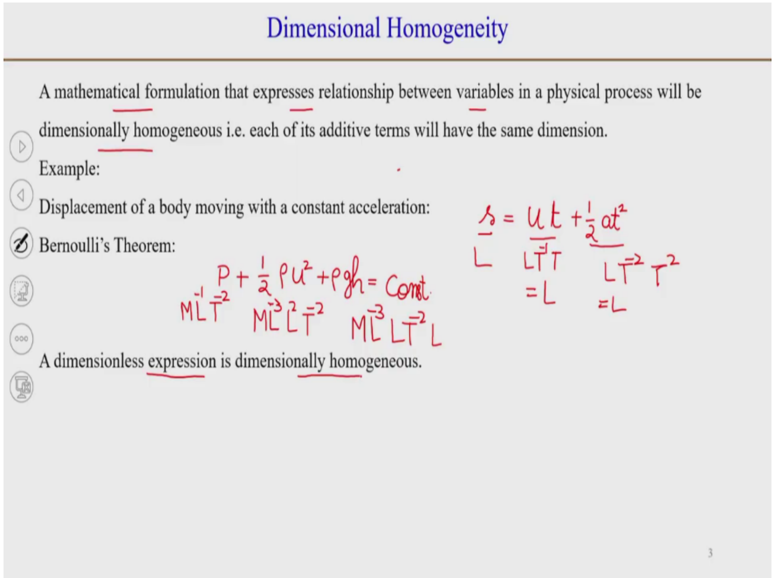

So, we have looked at the principle of dimensional homogeneity in earlier classes. So, the principle of dimensional homogeneity suggests that basically you cannot add or subtract apples with oranges. So, that means a mathematical formulation that expresses a relationship between physical variables in a physical process that should be dimensionally homogeneous. That is, the dimensions of each term that you add should be the same. Let us look at some examples so this statement becomes clearer.

Consider the displacement of a body, which moves with a constant acceleration s. So, the formula that all of us would have studied at some point of time,

$$s = ut + \frac{1}{2}at^2$$

So, in this, we see that the unit for each term s, ut and 1/2 at^2, the unit of each term will turn out to be that of length. So, unit of s is of course L, unit off ut is LT^{-1} into T. So, that will again be equal to L, unit of at^2 will be A meter per $second^2$. So, LT^{-2}, then T^{-2}. So this unit will again be L. So, according to the principle of dimensional homogeneity, the dimensions of all the terms are same.

Similarly, if you look at Bernoulli's theorem, which all of us have studied in different forms. So, let us look at one form, P + 1/2 ρ U^2 + ρgh is constant. So, if you look at this statement, the unit of or the dimensions of each of these will be pressure force per unit area. So, it will be ML^{-1}, T^{-2},. Similarly, ρU^2, ρ ML^{-3} and U^2 so, L^{-2}, T^{-2},.

So, that will give us again, ML^{-1}, T^{-2}. ρgh ML^{-3}, g LT^{-2}, and h is L. So, the unit will, its dimension will be ML^{-1}, T^{-2}. So, all the terms have the same dimension. You might have seen this equation in other forms also. So, say, this is written in terms of head P over ρg + 1/2 U^2 over g + h is constant.

So, again, you will see here that the unit for all the terms is L or the dimensions for all the terms is L, the unit can be meter, centimeter or mm depending on what system you are considering. So, the idea behind principle or dimensional homogeneity is that the dimensions of all the terms that you add in a equation, they are going to be same. We should also remember that when we write an expression in dimensionless form, all the terms will have 0 dimensions. So, that equation will be dimensionally homogeneous.

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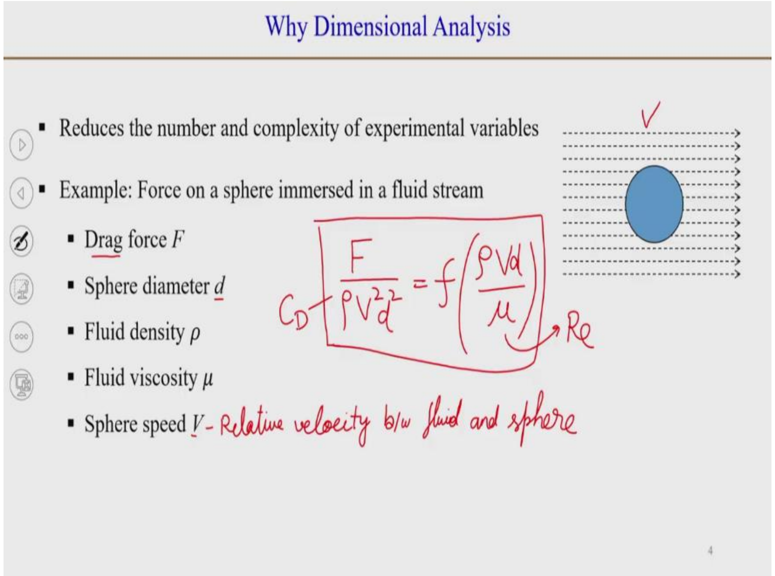

Now, let us look at why do we need to do dimensional analysis? Because it reduces the number and complexity of experimental variables. So, let us look at an example. We consider force on a sphere that is immersed in a fluid stream. So, you have a sphere and the fluid is flowing with a velocity, let us say V. So, we want to study, or we want to find out a relationship for the force on this sphere in terms of all the variables that might be affecting this flow. So, let us look for the variables or let us think about the variables that are going to affect the force on this sphere.

So of course, the first thing is we are talking about the force or more specifically that does restrict ourselves to drag force on this sphere. So, the drag force on this sphere, it depends of course on the geometrical dimensions of this sphere. So, the geometrical dimension of the sphere, we can represent it in terms of its diameter, which we say D. Then, the density of the fluid. So, fluid properties ρ and viscosity of the fluid μ.

So, this is dynamic viscosity of the fluid and the sphere speed or the relative velocity between the fluid and the sphere, so there can be different problems related to it. We can have a sphere moving in a quiescent fluid or this sphere which is stationary and the fluid surrounding it is moving, or both of them are moving. So, in all the cases, what V is the sphere speed with respect to the fluid or the relative velocity between the fluid and the sphere. So, let us just mention it here that V is the relative velocity or relative speed between fluid and this sphere.

Now, if we want to do the experiments and find out the dependence of force on these different parameters depending that how does force depends on diameter of the sphere, density of the fluid, viscosity of the fluid and speed of the sphere. Then, we might need to do a number of experiments. We might need to choose the sphere of different dimensions, let us say from 1 mm to 1 meter to find out the dependence of diameter. So, we might need to take let us say some eight 8 to 10 data points for this sphere diameter.

Similarly, we will need to do the experiments with different fluids then only you will be able to change the density. And the problem will come when you change a fluid, you will have changed density as well as changed viscosity. And moreover you might not see a large variation in density and viscosity and the parameters that you want to see. So, similar case with the velocity.

So, another way to do it, group in non-dimensional numbers. So for example, as we will see later on that they can be grouped in, so what we could do? We could group them in non-dimensional form that F over ρ V^2 d^2 is a function of ρ Vd over μ. You might recognize that F, ρ Vd over μ is Reynolds number and F over ρ V^2 d^2 is drag coefficient what is called C_D.

So, now the problem becomes simpler that if you want to find out, if you have done experiments, the easiest thing to change in this parameter will be the velocity. You can do the experiment, let us say with one fluid air or water on one diameter of the sphere change the velocity of the fluid over it and you will get the Raynold's number dependence of C_D. And in doing that, you will reduce the time and effort that is required to do the experiments. So, dimensional analysis becomes a powerful technique to do a number of things. So, as I said that it reduces the number and complexity of experimental variables.

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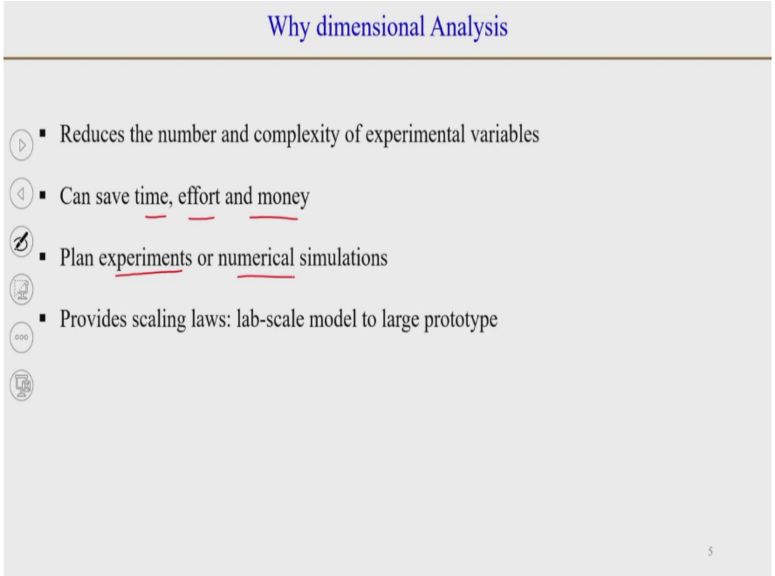

We saw that in the previous example, we reduced the variables that the drag force dependent on 4 variables. We reduced it to that we can do the experiment by varying only one dimensionless group. Now, once we do that, so of course this will save us a lot of time, effort, and money. And doing experiments, we can plan our experiments to prove a theory or numerical simulations, so that the time and effort to do experiments and simulations. And once we have figured out what are the important dimensionless groups, we can vary that dimensionless groups and find out the dependence on the parameters.

Now, another advantage of dimensional analysis is that it provides us with scaling laws. So, most of the times when we are designing an equipment, a reactor or heat exchanger, we do experiments on lab scale. And when we are going to use it, or when we are going to install it in a chemical plant, the size is going to be very different from it, maybe 10 times, or maybe even 100 times larger than what we had in lab scale.

So, what we need to do that how do we make sure that the conditions that are in the lab scale, they are similar what will be in the plant scale, because if the conditions and the two scales are not same, then, we will not be able to relate the two. So, how do we do it? How do we scale up? So, problem of scaling up is also we can address using dimensional analysis using the principle of geometric similarity, kinematic similarity, dynamic similarity and so on and so forth. So, we will look in detail what are those principles in the next lecture.

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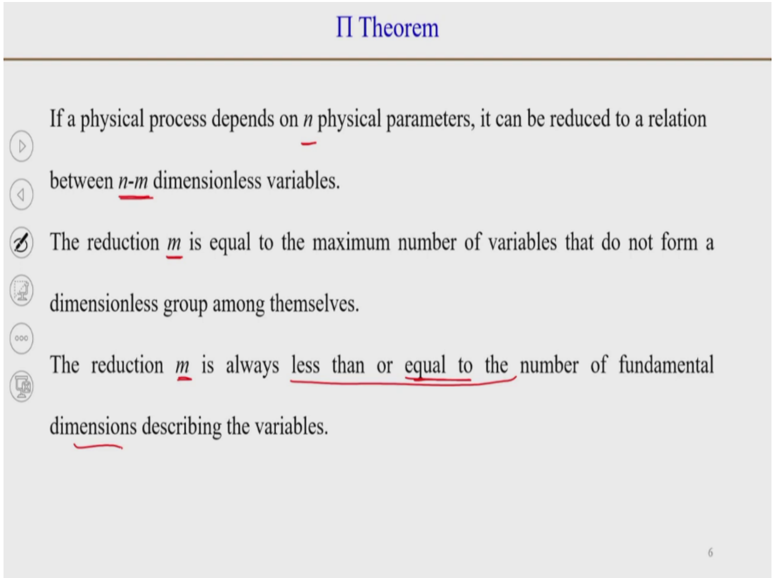

So, coming up to as we said that using dimensional analysis, we can reduce the number of variables. So, the question comes from how many dimensional variables we can reduce to how many dimensionless variables or dimensionless groups? So, this can be given as π theorem or Buckingham π theorem. So, π here represents dimensionless group. So, it suggests that if a physical process, if it depends on n physical parameters, then, the number of parameters, it can be reduced to a relationship between n-m dimensionless variables or dimensionless groups.

So, the reduction m, reduction in the number of variables this n-m, the m = the maximum number of variables that do not form a dimensionless group among themselves. And often, this m = or less than in most of the cases we will find that m = and in some cases, it is less than the number of fundamental dimensions, fundamental dimensions, the dimensions of base quantities, M, L, T and θ in fluid mechanics, what the problems that we will be looking into.

So, this reduction is always less than equal to the number of fundamental dimensions describing it. So, basically the π theorem or the Buckingham π theorem gives us that how many parameters from n physical parameters or n dimensional parameters, we can reduce the number of parameters to n-m dimensionless variables.

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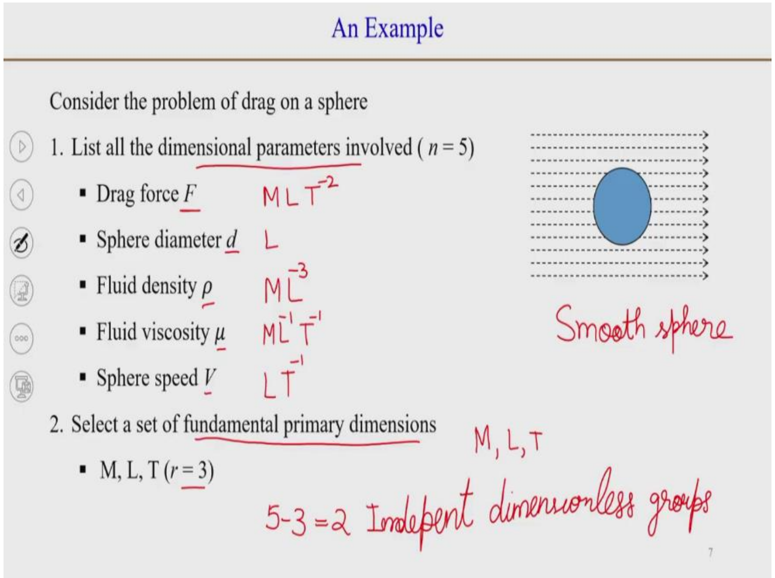

So, let us look at the same example that we discussed earlier. The steps that how do we non-dimensionalize a problem, or how do we find the dimensionless groups for a particular problem? So, if we consider the problem of drag on a sphere, now, as we, the first step is list all the dimensional parameters that are involved. So, in this case, we have listed down drag force is sphere, diameter, fluid density, fluid viscosity sphere speed. So, in all of such problems, we will be looking at one variable, which is the dependent variable, which depends on other physical parameters. So, in this case, our dependent variable force or drag force drag on the sphere.

Now, this depends on the diameter of the sphere, fluid density, viscosity, and the sphere speed. It may also depend if the sphere is not smooth, then it will also depend the roughness parameter on this sphere surface. But for this problem, we will consider the sphere to be smooth. In other cases, where it is a free surface problem that this sphere is at a free surface, it has not completely submerged, then gravity may also come into picture, but again, we consider that this sphere is emerged or submerged in the fluid.

So, the next task is or next step is that you select a set of fundamental primary dimensions. So, when we look at the dimensions for all these, so the dimension for drag force will be MLT^{-2}, this sphere diameter dimension will we L, density ML^{-3}, fluid viscosity, dynamic viscosity so kg/ms, $ML^{-1}T^{-1}$. And the sphere speed or relative speed, it will be LT^{-1}.

Now we see that the fundamental dimensions that are involved here are M, L and T. So, we list down all the fundamental or base quantities. Here we have these number of quantities 3. So, the dimensionless groups from this, we will have 5 - 3 or less. So, we will be able to construct two independent dimensionless groups.

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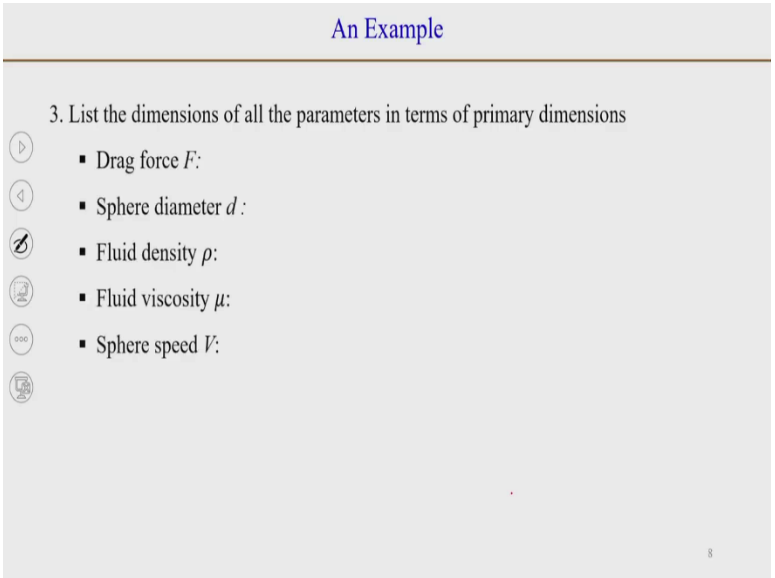

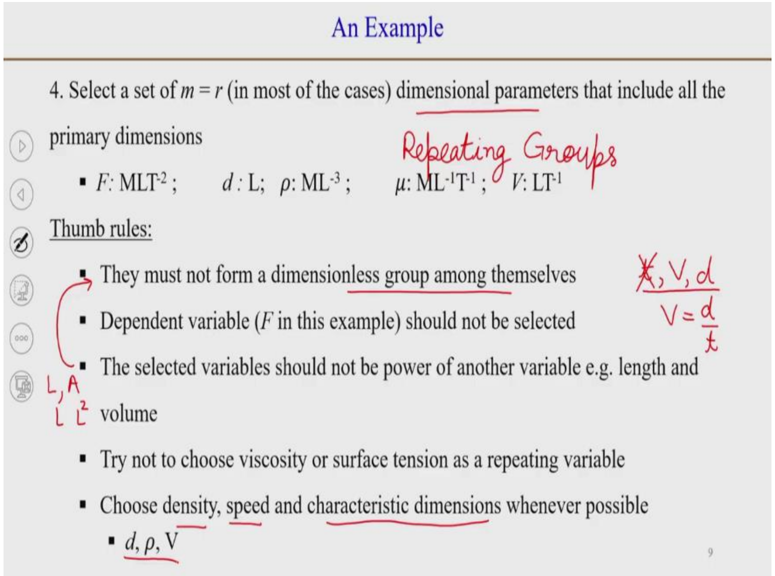

Now, the next step is that you list the dimensions of all the parameters in terms of primary dimensions. So, we have already done that. Now, what we need to do is select a set of dimensional parameters that include all the primary dimensions. So, this set of dimensional parameters is what we call, Repeating Groups. So, if you are given all the parameters that affect a particular physical quantity, then, after you have selected all those parameters, the most important part of the analysis is that, how do you find the repeating groups?

So, the first thing we need to remember is that we should choose repeating groups such that they include all the primary dimensions. So, we have 3 dimensions M, L and T. So, we should have 3 repeating groups in which at least one of the groups have each of these dimensions. So, we see here, all the dimensions written.

Now, some thumb rules to decide the repeating groups. So, they must not form a dimensionless group among themselves. So, for example, if you choose time, velocity and say length, the dimensionless groups then you can see from here that $v = \frac{d}{t}$. So, you should choose only two of these should be selected. You should not choose all three of them.

So, let us say you choose v and d as dimensionless groups. Now, the dependent variable that should not be selected, because we are trying to find the dependence of F or the dependent variable in terms of other variable. So, if it appears in all the dimensionless groups, then, our efforts will be wasted. So, we should make sure or we should remember that this dependent variable should not be a repeating variable. The selected variable should not be power of another variable. So, it is just a corollary of this that you should not have length and area, for example, because they will have L and L^2 dimensions. So, one can be represented in terms of others.

Then, we should try not to choose viscosity of surface tension as a repeating variable, because when we put it in dimensionless forms, then, we are not only looking at reducing the dimensionless groups, we are also trying to gain a physical insight into the problem. And in most of the cases, we will be looking to compare the competing physical effects or, in fluid mechanics generally it will be competing forces or competing stresses that affect a problem.

So, we will be comparing say, viscosity with surface tension or inertia with surface tension or inertia with viscosity. So, we should try that they are not appearing in all the expressions, because in that case, surface tension will be compared with all the physical effects. The other very useful rule is that if we have density, speed and characteristic dimension, whenever these parameters are there, then, we should choose these parameters as repeating groups. So, d, ρ and V in this problem are the repeating groups that we choose, so diameter, density, and velocity.

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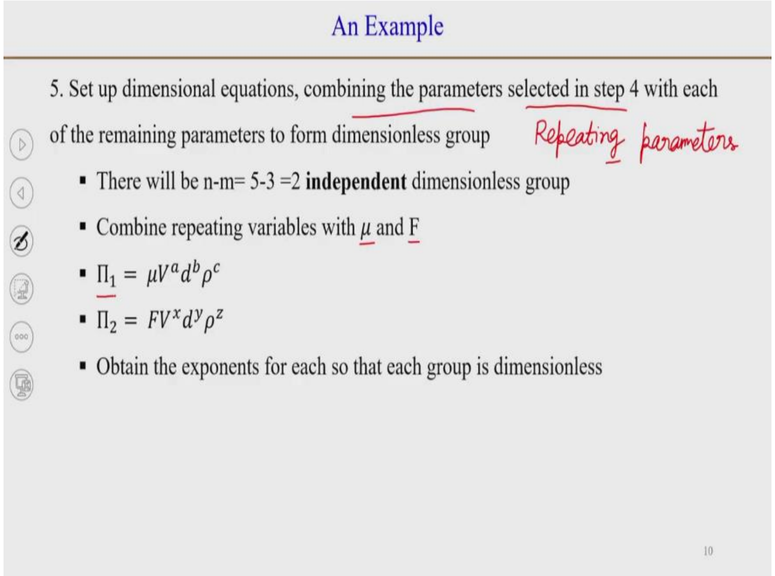

Now the next step is set-up dimensional equations, where we combine the parameters selected in step 4, so selected in step 4, these parameters we'll call repeating parameters or repeating groups, so repeating parameters. And we will combine this with each of the remaining parameters, so that dimensionless groups can be formed. So, you combine these repeating variables with μ and F, which are two remaining variables. So, we will have two dimensionless groups. Let us call these dimensionless groups as π, So π_1 = $\mu V^a d^b \rho^c$, the second dimensionless groups FV^x, d^y, ρ^z.

So, now we need to select A, B, C in such a manner that these groups become dimensionless. So, we need to obtain the exponents. So, let us look at how we do it.

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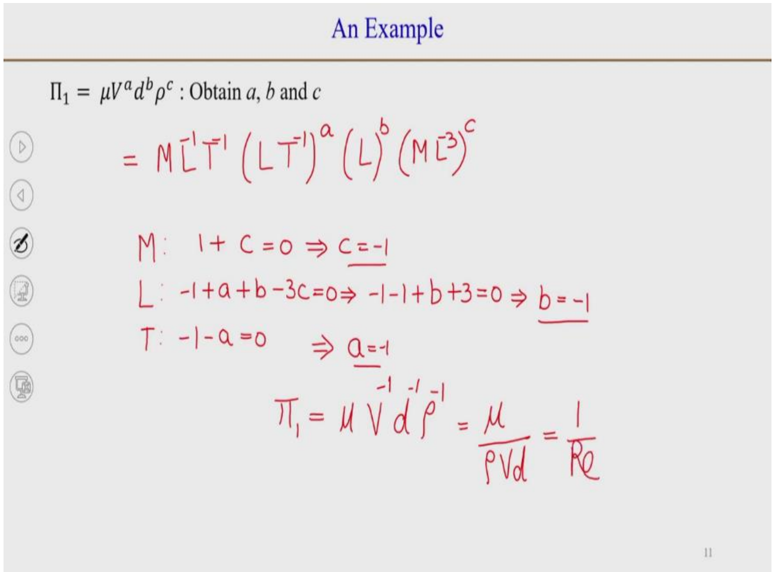

Of course, we will need to write the dimensions for each. So, for π_1, we will write μ, the dimension of μ is $ML^{-1}\,T^{-1}$, V is $(LT^{-1})^a$. D is L so $L^b$and ρ $(ML^{-3})^b$. Now, let us combine the dimensions of each of M, L and T and equate them to 0, combine the exponents of each of these. So when we combine the exponents of M, we will have

1+c = 0

and that gives us C = - 1.

Similarly, combine the exponents of L and that will give us -1 +a +b -3c = 0. We will come back to this. Let us now combine the exponents of T, so, which will give us -1 -a = 0, and we will get a = -1. And then, we substitute a and c in the second equation, so we will have -1 +b +3 = 0 or b = -3 +2. So, b = -1. So, we will have, in this case, a = -1, b = -1 and c = -1.

So, our first dimensionless group is π_1 = μ, V^{-1}, d^{-1} and ρ^{-1} or if we write this in simplified form μ over ρVd or if you recognize this, this is 1 over Raynold's number. Now, when we obtain a dimensionless group, we can obtain this in terms of Raynold's number or 1 over Raynold's numbers. From the perspective of dimensionless group, both μ over ρ V, d or ρ Vd over μ, they are dimensionless. So, we have obtained the same dimensionless group.

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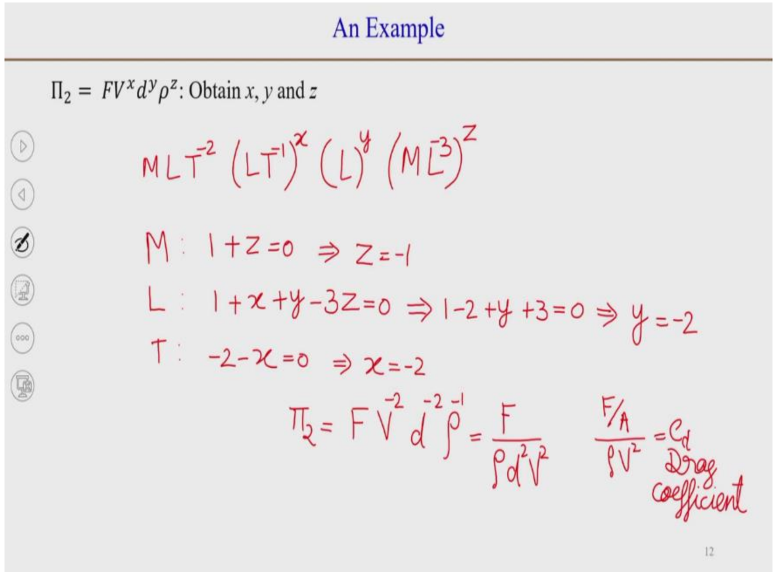

Now coming to the next dimensionless group, where we have combined force of it, the repeating variables. So, again, we will write the dimensions of F, M, L, T^{-2}, $(LT^{-1})^{x}$, $(L)^{y}$ and $(ML^{-3})^{z}$. So if we combine, we will get for M, 1 +z = 0 which gives us z = -1.

When we combine the exponents of L, we will get 1 +x +y -3 z = 0, exponents of time. So, -2 -x = 0. So, from this, we get x = -2. So, we will have a 1 -2 +y +3 equal to 0 or -1 +3, so 2 +y = 0 or y = -2. So, that means our second dimensionless group is π_2 = F, V^{-2}, d^{-2} and ρ^{-1} or if we write this in terms of fraction, F over ρ d^2, V^2. So, we have found the two dimensionless groups as we discussed before that drag coefficient. So, this is basically drag coefficient force per unit area divided by ρ V^2 and this is called drag coefficient.

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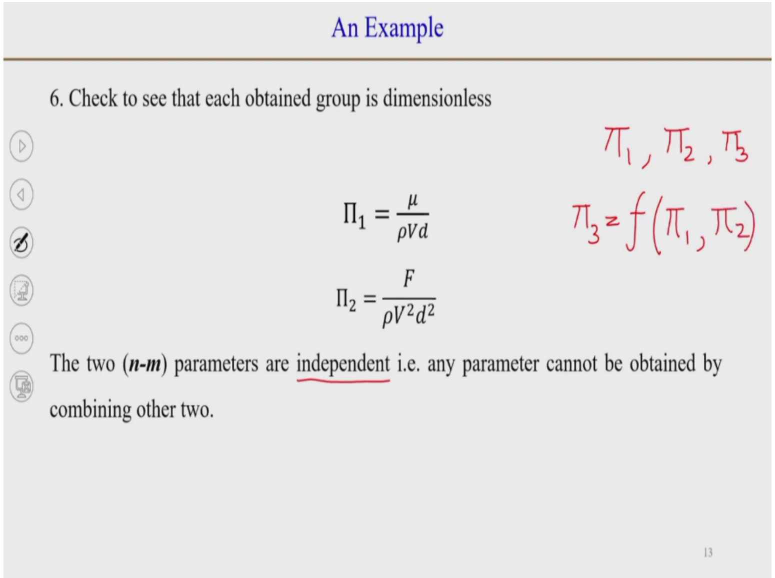

So, we have obtained two dimensionless groups. And just to make sure that these groups are dimensionless, our calculations are correct. We should make sure, we should check again that the dimensionless groups we have obtained actually has 0 dimensions in all the base, for all the base quantities. In this case, M, L and T.

And we will also see, or we can also see that these parameters, these dimensionless parameters or these dimensionless groups are independent of each other. That means you cannot find π_2 from π_1. If you have, let us say, in some problem, if you have π_1, π_2, and π_3, then it should not happen, or it will not happen if you have applied the principles correctly that π_3 is a function of π_1 and π_2. That means you cannot obtain π_3 as a function of π_1 and π_2.

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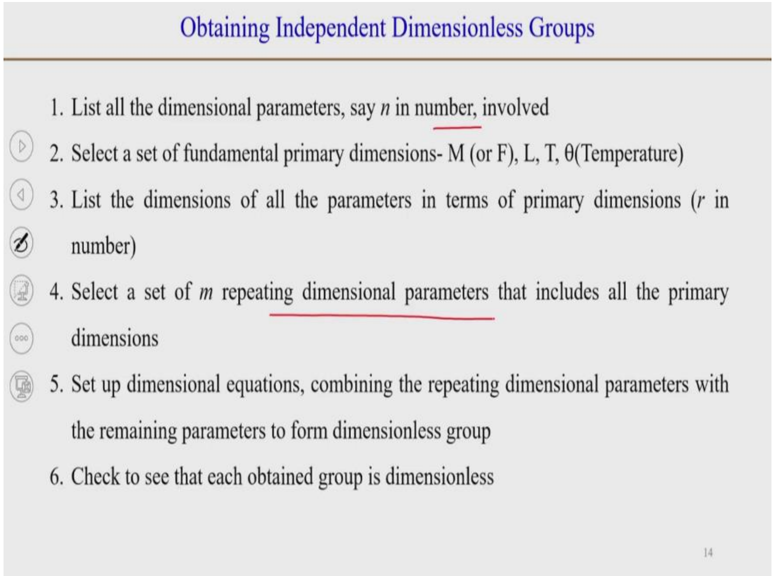

So, to summarize, we will just go through all the steps that we have done for the problem. That first thing is that we should list all the dimensional parameters that are involved in the problem. So, if we are not given, in a problem, then, we should list down all the parameters. We should think first in terms of geometry, what are the geometrical parameters that is important for a problem. For example, the diameter of a pipe, diameter of the body, over which the flow is happening, diameter of the cylinder or the diameter of a sphere.

If the cylinder is of infinite length, then, the length of the cylinder, if it is a rectangular pipe, then, the two dimensions of the rectangle, area might be important. The volume in some cases might be important, so dimensional parameters. Then, once you have checked all the geometrical parameters, then you should list down the properties of the fluid. For example, in most of the cases, you will have density and viscosity coming into picture. If there is two phase flow, for example, bubbles and droplets are involved, then, the surface tension will also be an important parameter.

If the heat transfer is coming to a picture, then, thermal conductivity, a specific heat capacity of the fluids will be important. You should also look into if the gravity is important, so, gravity is another parameter. So, try to think in detail and list all the parameters that are important for a problem. There might be some parameters that you will realize later on that they are redundant. So, they, those parameters will anyway be striked out when you just look at the problem. So, this is the first and very important step.

Once you have selected all the dimensional parameters involved in the problem, you write down their dimensions in terms of fundamental dimensions, in terms of M, L, T and θ. Then, list the dimensions of all the parameters. Once you have listed, then, you select a repeating dimensional parameter or repeating group that what are the dimensional parameters that you will be combining with all other remaining parameters to form the dimensionless groups. Once you have done that, you combine them with some, assumed exponents of the repeating groups and obtain the exponents by making these groups dimensionless. Once you make the group dimensionless and obtain it, check if your group is dimensionless.

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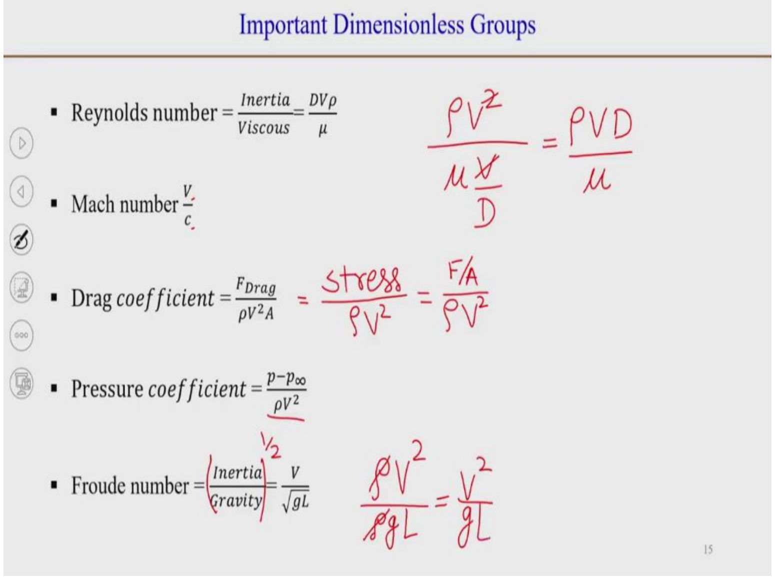

Now, I have listed down some of the important dimensionless groups. The list of course is not adjustive and only some important groups I have listed down here. So, Reynolds number, which we have seen again and again. And we will see it throughout the course, which is the ratio of inertia and viscosity. So, inertia can be represented in terms of the ρ, V^2 and viscous stress μ V over L or here what we have is d is the dimension, so we will use the same parameter. So μ V over d and V will cancel out. So, we will get ρ Vd over μ as Reynolds number.

Then, Mach number we have already, you would have heard about Mach number, which is the ratio of two speeds, the speed of the fluid as well as c is the speed of sound or it might be the speed of the aeroplane that is flying in air and divided by the speed of sound. So, this v is basically the relative speed between the fluid and the body, and c is the speed of sound. So, that is called Mach number, the ratio of two speeds.

Drag coefficient is again a ratio of viscous stress or pressure, so it is the ratio of stress. It can be viscous stress, it can be normal stress pressure to ρ V^2. So, the stress is basically force over area, so we will have force over area divided by ρ V^2. Pressure coefficient, in some cases, you might want to write pressure in dimensionless form. So, use the same principle that when you are non-dimensionalizing pressure in most of the cases, where inertia is important, then, you will non-dimensionalize pressure P -P infinity divided by ρ V^2. If it is low Reynolds number flow, then, you might want to non-dimensionalize pressure with the viscous stress. But very often the term pressure coefficient is used in aerodynamics.

Froude number, it is the ratio of inertia and gravity. So, in this form, it is^2 root of inertia and gravity. So, if we write down inertia as ρ V^2 and gravity, so that will be ρ g and L and ρ will cancel out, so it will come out to be V^2 over gL. And this is given in a^2 root form. So, this Froude number is V over root gL.

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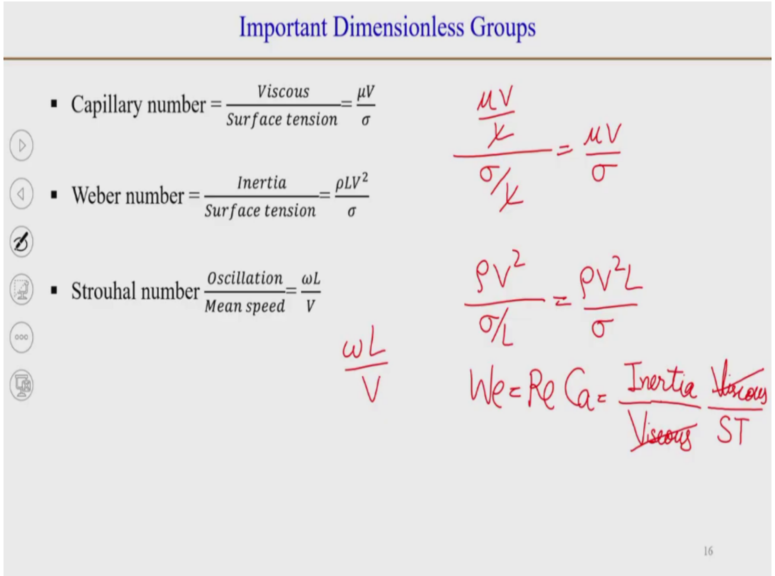

Now, capillary number, capillary number is the ratio of viscous and surface tension effects. So, viscous effect, again, viscous stress μ V over L and surface tension σ over L. So, L will cancel out and you will have μ V over σ. So, as we saw that it compares viscous and surface Tension effects in the capillary number. So, when the viscous effects are important in around bubbles and droplet, we should be looking at capillary number.

Another term, which compares surface tension is Weber number, which relates it with inertia. So, inertia ρ V^2 and surface tension σ over L. So, we will have ρ V^2 L over σ. We can also see that Weber number can be written in terms of Re into capillary number. So, Re is inertia or viscous forces and capillary number is viscous divided by surface tension effect or capillary effect. So, viscous, viscous will cancel out and you will have inertia or surface tension, which is Weber number.

Another important number is Strouhal number, which looks at when there is oscillations in the flow, which compares the oscillatory speed, or the frequency Omega with the mean speed. So, ω L over V, so oscillatory speed and mean speed, the ratio of that. So, we will stop here and continue in the next lecture, where we will talk about the scale-up. Thank you.