Fundamentals of Fluid Mechanics for Chemical and Biomedical Engineers
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Lecture No. 04
Vectors: A review

Hello, in this lecture we will be talking about vectors, we in our school we would have studied about vectors. So, we will just quickly brush up some of those concepts and maybe introduce a few new concepts.

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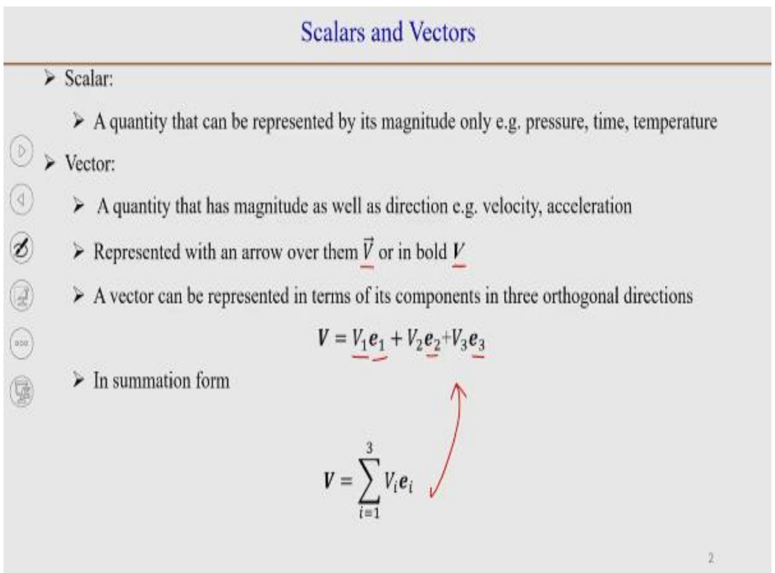

So, let us look at vectors, the first the definition of scalars. So, scalars are the physical quantities to represent them if we just need the magnitude of the quantity, we call them scalars and we have learnt about it long, long ago. So, scalars some examples can be time for example, we do not need to define a direction to represent time. Pressure, pressure is again a scalar quantity, temperature is a scalar quantity, we do not need a direction to define temperature.

Pressure somebody might say that we need to say that the pressure acts on a surface normal to it, but by the very definition of it pressure always acts normal to a surface. The formula for pressure is force per unit area, so whatever area that we are considering or we are looking at pressure, the direction of pressure will be normal to that surface. So, pressure is against a scalar quantity

Now, coming to vector. So, of course, vector is a quantity that has magnitude as well as direction. So, some of the common examples for vectors is displacement, displacement in a particular direction or the derivative of displacement. So, velocity and derivative of velocity, acceleration, force again to the represent force we need the magnitude of force and in which direction does the force act. So, they are all examples of vectors.

Now, in fluid mechanics because we are generally concerned with the velocity of the fluid, the force is acting on a fluid or the force is acting on a body because of the fluid. So, we need to analyze vector quantities often and that is the reason that we are looking at vectors here. So, vectors can be generally represented with an arrow over them or when we type it, we can also type vectors as a bold, so, it is convenient because we do not need to put an arrow every time we type the vector.

So, in this course, predominantly I will try to use vectors as bold letters, but in some cases, you might see a mix up and you might see that vectors have an arrow over them. So, vectors because we can, we are representing its direction and in space we can decompose it into three normal components. So, in general what we will be looking at in this course in the Cartesian coordinate system, so, x, y and z directions we can have the components of vectors.

We can also have the components of vector in cylindrical coordinate system where we have r direction which is radial direction, angular direction which is θ direction and the axial direction which you can say z direction or x direction, so, r θ x or r θ z coordinates are cylindrical coordinates. And then for some problems it is convenient to use a spherical coordinate system, where we have one angular direction, one radial direction which is r and two angular directions θ and φ.

So, in this lecture we will be talking about Cartesian coordinate system however, as we go along and when we need to use cylindrical coordinate system, we will discuss cylindrical coordinate system. In this course, we will not talk about the spherical coordinate system.

Now, if we take the unit vectors along the 3 directions as e_1, e_2, e_3 so, the three orthogonal directions they can be x, y, z or r, θ, z, then we can write the velocity components as v is equal to v_1 e_1, v_2 e_2, plus v_1 e_1 plus v_2 e_2 plus v_3 e_3 where v_1 is the component along direction 1 or along the direction e_1, v_2 is the component along direction 2, e_2 and v_3 is the component along the direction 3.

Now, this can also be written in the summation form, so this looks a bit intimidating, where we write

$$V = \sum_{i=1}^{3} V_i \, e_i$$

where i can go from 1 to 3. So, if we expand it, we will get this we substitute i is equal to 1. So, first term will be V_1 e_1 that is what we have here, the second term i is equal to 2 V_2 e_2 and then i is equal to 3 V_3e_3 in certain cases, it is easier to represent this in terms of summations. So, we need to be familiar with the notations in summations. However, in this course, I will try to use as much as possible the expanded form of the vectors.

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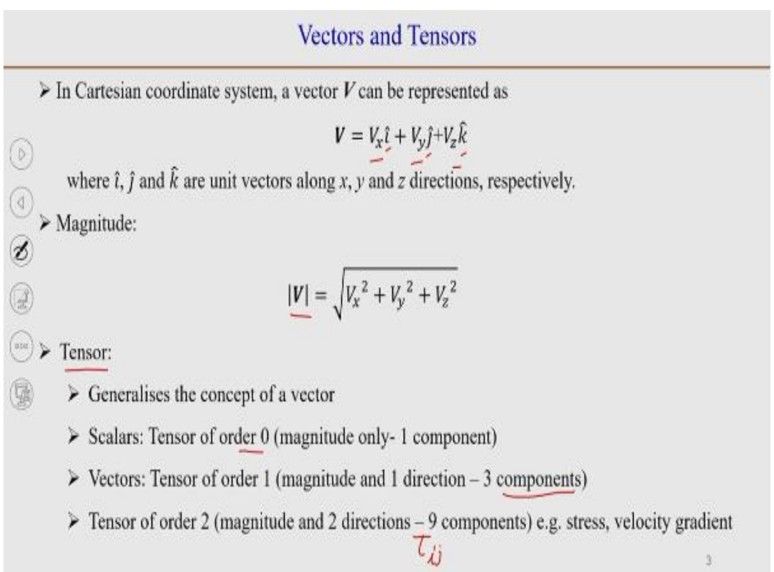

So, if we take Cartesian coordinate system, so, Cartesian coordinate system means in x, y and z directions, so, 1 is x, 2 is y and 3 is z. So, this vector can be presented as V_xi V_yj plus

V_zk, we are familiar with these notations, i, j and k, they are unit vectors along directions x, y and z, in some books you might find these directions being represented as e_x, e_y, e_z. So, when talking about Cartesian coordinate system, we will use the notations i, j and k for unit vectors along x, y and z directions.

So, the magnitude of vector, magnitude of vector V which have component V_x, V_y and V_z along x, y and z directions, we can obtain its magnitude by summing up the squares of the three components and taking its square root. So, the magnitude which is represented by

$$|V| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

Now, there is another quantity which you might have heard of, is called tensor. So, tensor is a generalization of that if we take a quantity or if we take a system in which all scalars, vectors as well as other quantities, can be taken into where we need to have more than one direction to define a particular quantity. For example, when we talk about stresses, as we will discuss in our later classes, that to define a stress, we need the direction or the surface on which the stress is acting and the direction in which they stressed is acting, so, we need to define, to define the stress components or to define the stresses, we need to tell two directions, the direction normal to the surface on which the force acts and the direction along which the force acts.

So, tensor is defined a generalized quantity, generalize in concept of vector where we might need to have more than one information. So, you can relate it with say, array what we use in programming multi-dimensional arrays. Now, scalars, so we can say the scalar is a tensor of order 0, where we have only magnitude and no direction. So, it has only one component to represent scalar we need just one component. But for vectors we need, it has one direction so, we have 3 components for vectors. So, it has magnitude plus one direction.

Now, a second order tensor and that is why we are introducing the term tensor, because some of the quantities which we deal in fluid mechanics for example, the stress, momentum flux, velocity gradient they are second order tensors. So, to represent them, we not only need the magnitude, but we need two directions to represent them, because if you have two directions, so, x, y and z component to represent direction of one thing and x, y, z component represent the direction of another.

So, for example, when we talk about stress, we are talking about the direction normal to the surface on which the stress is acting or on which the force is acting and the direction of force itself. So, we have three, in three way or we can have three components of the area vector and we can have three components of the force. So, we will have 3 multiplied by 3, 9 components of a stress.

Similarly, when we talk about velocity gradient, velocity will have 3 components V_x, V_y, V_z and it can have variations along the x direction, each component can have various and along the x direction, along y direction, along z direction. So, V_x will be where we have component $\frac{\partial V_x}{\partial x}$, $\frac{\partial V_x}{\partial y}$, $\frac{\partial V_x}{\partial z}$. So, similarly, we have 3 components, 3 multiplied by 3, 9 components, so, this is tensor of order 2 and to represent tensor we need to define two directions.

So, for example, when we write about tensor, it will be the stress tensor that will be τ_{ij} in general term, so, there will be two directions τ_i and τ_j, but when we talk about it in detail, then we will say which of these i represent the direction normal to the surface and which of the symbols represent the direction of the force acting on it.

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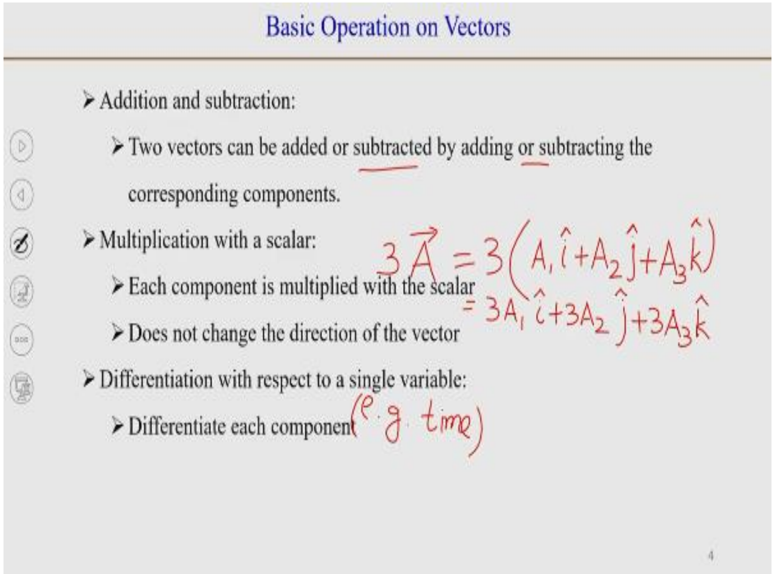

So, then some usual stuff that we are already familiar with that, when we operate on vectors, so the normal operations which we have a subtraction and addition, it is very common and by now, if you would have doing, we have been doing mathematics, this would have been, these concepts would have been built up in our subconscious mind. So, to just to repeat it, that two vectors can be added or subtracted by adding or subtracting their corresponding components.

So, if you have vector say

$$\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$$

$$\vec{B} = B_1\hat{\imath} + B_2\hat{\jmath} + B_3\hat{k}$$

$$\vec{A} + \vec{B} = (A_1 + B_1)\hat{\imath} + (A_2 + B_2)\hat{\jmath} + (A_3 + B_3)\hat{k}$$

Similarly, the same thing can be used for subtraction. Now, when we multiply a vector with a scalar So, if you have A, let us say a vector A and this is multiplied by a scalar 3, so

$$3\vec{A} = 3(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k})$$

Now, when you multiply with the scalar you can just write

$$3\vec{A} = 3A_1\hat{\imath} + 3A_2\hat{\jmath} + 3A_3\hat{k}$$

Now, that is what we will have here, that each component is multiplied with the scalar. So, 3 is the scalar here of course, when you multiply the scalar with a vector, the direction of the vector is not changed. But a scalar can have a positive or negative sign and as a result, the direction of the vector might be reversed, if you multiplied it with a scalar, which has negative magnitude.

Now, if you want to differentiate a vector with respect to a single variable, for example, with respect to say time, there is single variable time, so when you differentiate a vector, you can differentiate each component of the vector and you will get the differential or the derivative of the vector.

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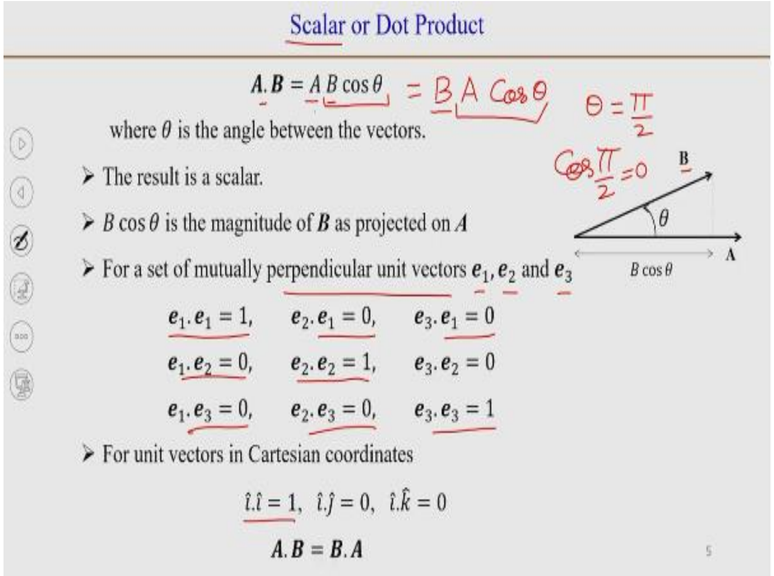

So, the next operation is product. So, when we talk about product, we can have actually three different operations on vectors. So, we can have that when you multiply two vectors, you get a scalar quantity, when you multiply two vectors, you get a vector quantity and when you multiply two vectors, you get a tensor of second order. So, when you multiply two vectors and you get a scalar that is called dot product.

So, $A.B$ where A and B are two vectors, $A.B$ is their dot product, so dot represent and that is the reason we call it dot product and it is also called the scalar product, because the quantity that we get as a result of their multiplication, we get a scalar quantity, so, that is equal to $ABcos\theta$ so, where, here these A and B represent the magnitude of vector A, magnitude of vector B multiplied by cosθ, where θ is the angle between two vectors. So, the result of course, is a scalar.

Now, if we look at this, we can see that if you have a vector A and another vector B here, and the angle between them is θ. So, you can see that B cosθ, which is this component or this part is the projection of vector B on vector A. So, you can treat dot product as a multiplication of magnitude of vector A multiplied by the projection of vector B on vector A or you can do the other way around, you can write it as well this is equal to BAcosθ. So, you can say that the dot product is equal to magnitude of vector B multiplied by the projection of vector A on vector B.

So, this will also tell you that, if you have a doubt that what is the angle between two? So, you can take angle θ, if you take angle $(2\pi - \theta)$ that will also give $(\cos 2\pi - \theta)$ is equal to $\cos\theta$. So, that will give you the same answer basically. Say if you have mutually perpendicular unit vectors, when you talk about the components, we can have components i, j, k unit vectors or unit vectors e_r, e_θ, e_z in cylindrical coordinate system.

So, we if we write in general that there are three perpendicular unit vectors e_1, e_2, e_3. So, if you have perpendicular vectors then that means θ, when θ is equal to $\frac{\pi}{2}$ we can see from this that $\cos\theta$ is equal to $\cos\frac{\pi}{2}$ is equal to 0. So, A dot will be 0. So, if a vector is multiplied by itself then the angle so, unit vector $e_1.e_1$ the angle between them is 0. So, that will be the product is 1. $e_2.e_1$ the angle is 90 degrees so 0. $e_3.e_1$ is equal to 0.

Similarly, you can have other combinations of these so, when a unit vector is multiplied by itself, the dot product is 1 and when it is multiplied by other unit components of course, we are talking about 3 perpendicular unit vectors, so then in these cases their dot product is 0. Now, for specific, if this take e_1 as equal to i, e_2 is equal to j and e_3 is equal to k of course, again we will have ii is equal to 1, jj is equal to 1, kk is equal to 1 and other products when i.j or i.k they are equal to 0. So, the other thing that we have already noticed here that we take either take A.B or B.A they are equal.

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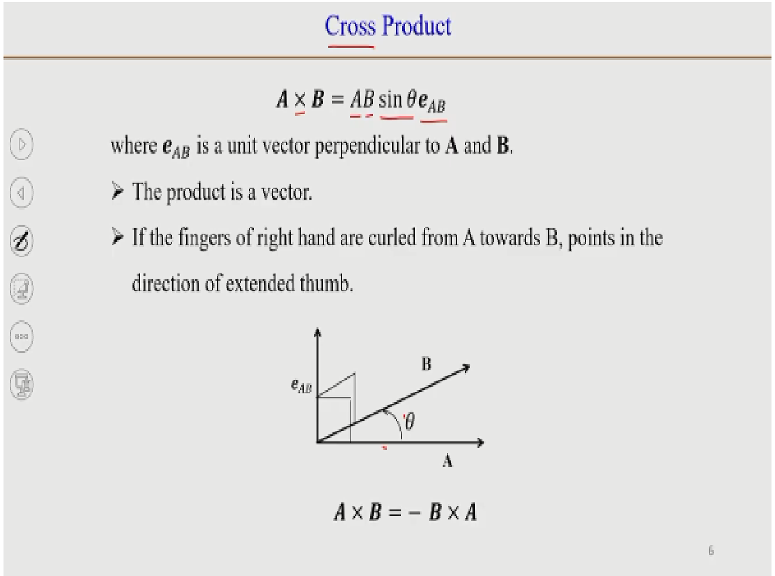

Now, when we take two vectors and multiply it and if the result is a vector, then what we call it? Cross product. So, the symbol here is we use a cross here. So, A×B is equal to magnitude of vector A, magnitude of vector B sinθ into a unit vector e_{AB} which will be normal to this or normal to both the vectors. So, you have a plane in which both the vector will be there and this vector e_{AB} will be pointing normal to it.

So, this product is your vector and then of course, it will be normal to it but which direction because you can have two directions. So, right hand thumb rule comes here, that if you, if the fingers of right hand are curled from vector A towards vector B, then the resultant vector or the unit vector, it will point in the direction of thumb, so, when you go from A to B it will be pointing in this in this direction.

So, A×B will be normal to board or B×A will be entering or inside the board, this will be pointing inside the board if we take B×A and A×B will be pointing away from the board. So, that also tells us that A×B and B×A not, are not equal because they, their magnitude are equal, but their direction, they are different. So, that is cross product.

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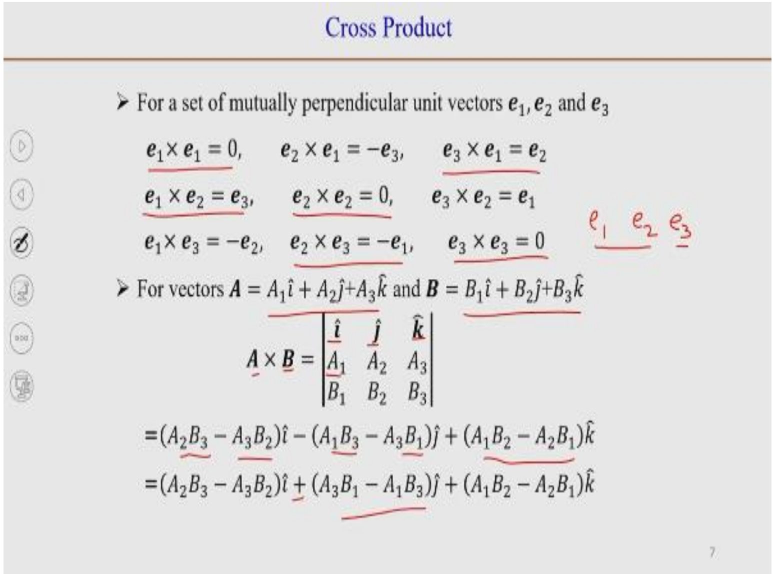

Now, if we take in terms of unit vectors that we can write generally, when we work with vectors, we will be writing all the vectors in terms of its components. So, it is important to understand or important to know and remember the cross product of unit vectors of these x, y, z direction or r θ z direction. So, again $e_1 \times e_1$ because when sin 0 is equal to 0, so, $e_1 \times e_1$ will be equal to 0, $e_2 \times e_1$ is equal to minus e_3, $e_1 \times e_2$ will be as we see here $e_1 \times e_2$ will be e3 again when you multiply or when you have a cross product of a vector with itself you will get the result is 0.

If you multiply two vectors, two-unit vectors you will get the third one and the direction will depend, it may come with plus sign or minus sign. So, you can see that, if you write e_1, e_2 and e_3 so, when you have e_1 crossed e_2 you will get e_3, if you take $e_2 \times e_1$ you will get minus e_3, if you take $e_3 \times e_1$, then you will get e_2, if you take $e_2 \times e_3$, what you will get, $e_2 \times e_3$ you will get minus e1.

So, if you have vectors, A_1i + A_2j + A_3k and B_1i + B_2j + B_3k, then their cross product can be written in terms of a matrix, where the first row is the unit vectors i, j and k. The second row is the magnitude of the first vector A so, A1 the components of first vector A_1, A_2 and A_3 and the third row will be components of the second vector B_1, B_2, B3.

So, when we simplify this we can write i multiplication of A_2 B_3 - A_3 B_2, so that is first component, - j(A_1 B_3 - A_3 B_1), the third term will be + k (A_1 B_2 - A_2 B_1). So, if we want to have the sine for all the terms as plus so, we can change the order of these terms that will be when it goes inside then it becomes A_3 B_1 - A_1 B_3. So, that is about cross product.

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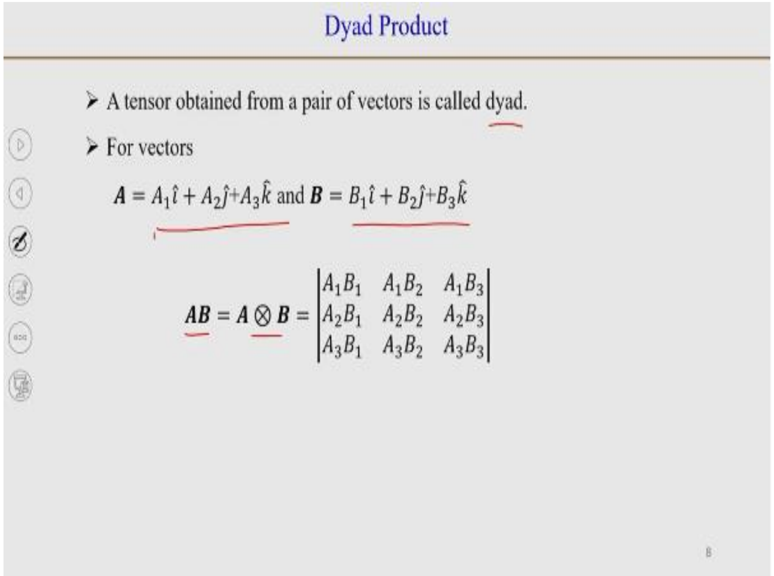

Now, the third operation, which you might have not been familiar with is that if you can also take a Dyad Product. So, when you multiply two vectors and you get a second order tensor, then it is called a dyad product. So, if you have a vector A_1i + A_2j + A_3k and another vector B, B_1i + B_2j + B_3k, you can take its dyad product and this is the symbol so, when you see this symbol A cross under is inside a circle, then you know that it is a dyad product that when you multiply these vectors, you get a tensor and the magnitude of this tensor will be A_1 B_1, A_1 B_2, so, this is the matrix that you get by the multiplication of this tensor.

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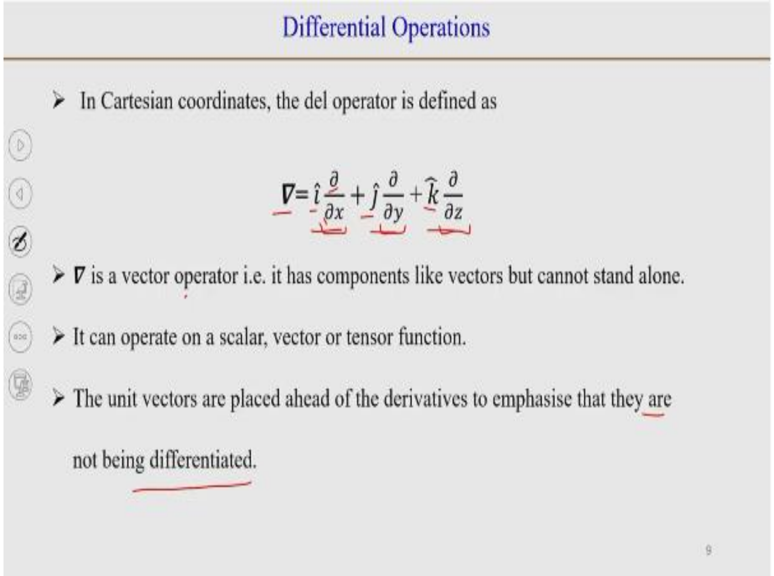

So, the other operations or other operation that we will be interested in is of Differential Operation on vectors. So, in Cartesian coordinate we define a operator which is

$$\nabla = \hat{\imath}\frac{\partial}{\partial \mathrm{x}} + \hat{\jmath}\frac{\partial}{\partial \mathrm{y}} + \hat{k}\frac{\partial}{\partial \mathrm{z}}$$

So, as you will notice that this looks like a vector or ∇ operator, it is also called Nabla. So, you might notice that this symbol, the inverted triangle or triangle upside down we call it del and this symbol is also I generally keep pronouncing it del.

So, because of habit we, I use del for both, but you can also to differentiate between the two you can also pronounce as, pronouns this symbol as dou. It can be pronounced as dou, del and sometimes it will also called as curl D. So, this is as you can see that there are 3 components i, j and k so, this appears to in along the x direction, y direction and z direction there are 3 components, but we will have these differential operator $\frac{\partial}{\partial \mathrm{x}}, \frac{\partial}{\partial \mathrm{y}}, \frac{\partial}{\partial \mathrm{y}}$.

So, this operator does not stand or does not hold good or does not mean anything alone, but when it is operated, we get a vector, if we operate on a scalar as we will see. So, this is a vector operator, it has components along the 3 directions, but it cannot stand alone. So, we can operate it on a scalar, we can operate it on a vector or we can operate it on a tensor and that is what we will see.

So, the other thing that I would like you to notice here that I have written these components i, j and k just before the derivatives, you can also write this after the derivatives, but you need to remember the fact, that these unit vectors are not being differentiated here. So, $\frac{\partial}{\partial x}$ is not $\frac{\partial}{\partial x}$of i, it is i multiplied by $\frac{\partial}{\partial x}$ j multiplied by $\frac{\partial}{\partial y}$ plus k multiplied by $\frac{\partial}{\partial z}$.

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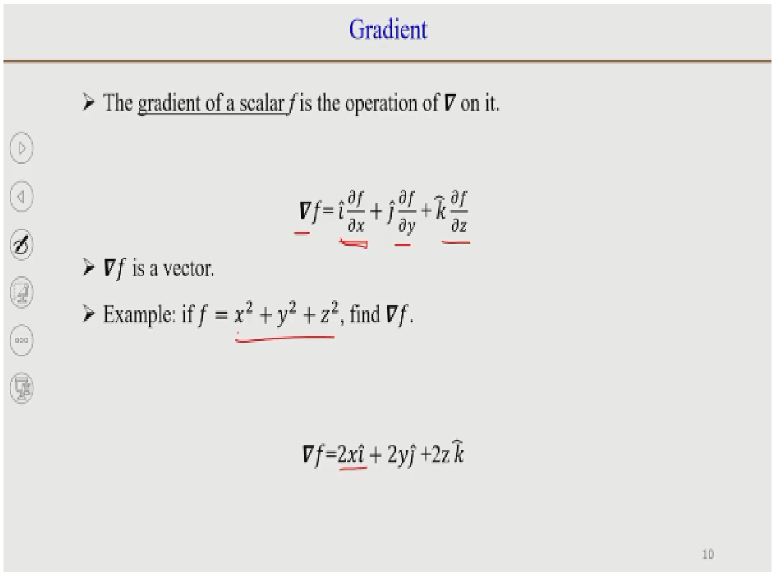

So, if we operate the del operator on a scalar, let us say this scalar is f. So, if you operate it on a scalar f, what you will get is x component is $\frac{\partial f}{\partial x}$, the y component will be $\frac{\partial f}{\partial y}$and the z component will be $\frac{\partial f}{\partial z}$. So, what you have done when you operate a del operator on a scalar you get a vector. So, if you talk in terms of tensor, when you have a gradient operator or when you operate it in terms of gradient, the order of the tensor is increased by 1, a scalar becomes a vector and you operate on a vector, the vector becomes a tensor.

So, when you take the gradient of a vector, you remember that velocity gradient, I said that the velocity gradient is a second order tensor. So, let us look at a simple example, that if we say or if we have a function f is equal to or a scalar function which is x square plus y square plus z square and we need to find its gradient. So, $\frac{\partial f}{\partial x}$, so, $\frac{\partial}{\partial x}$ is a partial derivative of f with respect to x. So, we can treat when we are taking partial derivative with respect to x we can treat or we need to treat y square and z square as constant.

So, $\frac{\partial f}{\partial x}$ will be 2x so, the x component will be 2xi, the y component similarly, will become 2yj and the z component will be 2zk. So, the gradient of $x^2 + y^2 + z^2$ scalar will be 2xi + 2yj + 2zk.

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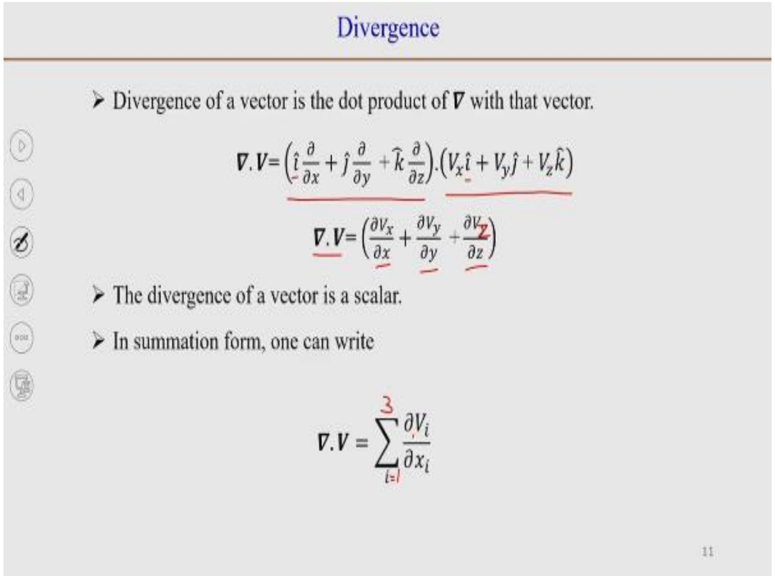

Now, the second thing is Divergence. So, divergence when we operate the del operator on a vector in terms of dot products, so, when we take the dot product of del operator with the vector, we get its divergence. So, del operator we know that is in Cartesian coordinate system $\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$. We take its dot product with a vector Vxi + Vyj + Vzk and because it is a dot product so, only the component where the, that is multiplying or the unit vectors are multiplying with itself they will be nonzero, other terms will become 0.

So, you will have a $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$. So, that is the divergence. So, as we can see here when you take the divergence of a vector and it is a dot product, so, when you operate it on a vector what you get is a scalar quantity so, divergence of a vector becomes a scalar. So, its order decreases by 1.

We write it in summation form, you will see that

$$\nabla . V = \sum_{i=1}^{3} \frac{\partial V_i}{\partial x_i}$$

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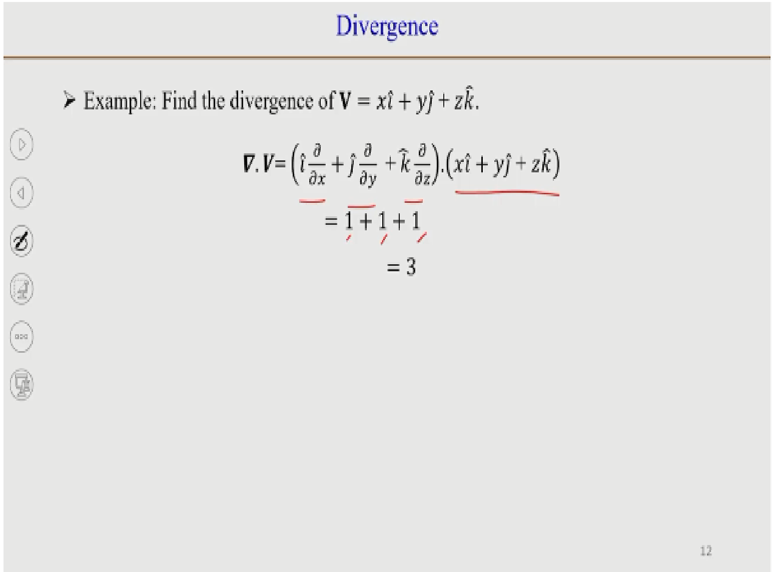

So, again let us take a simple example, where we need to find the divergence of a vector V, which has which is

$$V = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\nabla.V = \left(\hat{\imath}\frac{\partial}{\partial \mathrm{x}} + \hat{\jmath}\frac{\partial}{\partial \mathrm{y}} + \hat{k}\frac{\partial}{\partial \mathrm{z}}\right)(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$= 1 + 1 + 1 = 3$$

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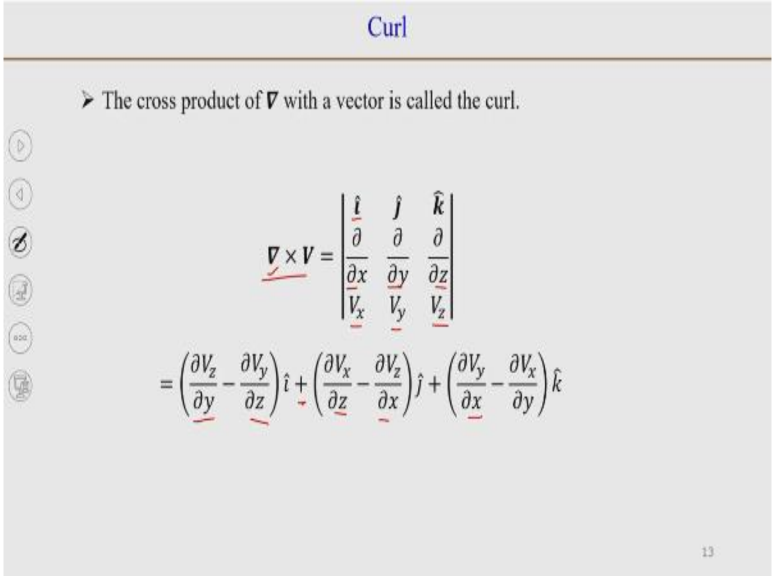

Now so, we have looked at that what happens when the del operator is operated on a scalar quantity and we get a vector and which is called gradient. When the del operator is operated on a vector and we take the dot product, we get divergence, when we operate the del operator on a vector and we get a cross product then we call it curl. So, cross product of del operator with a vector is called curl.

$$\nabla \times V = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

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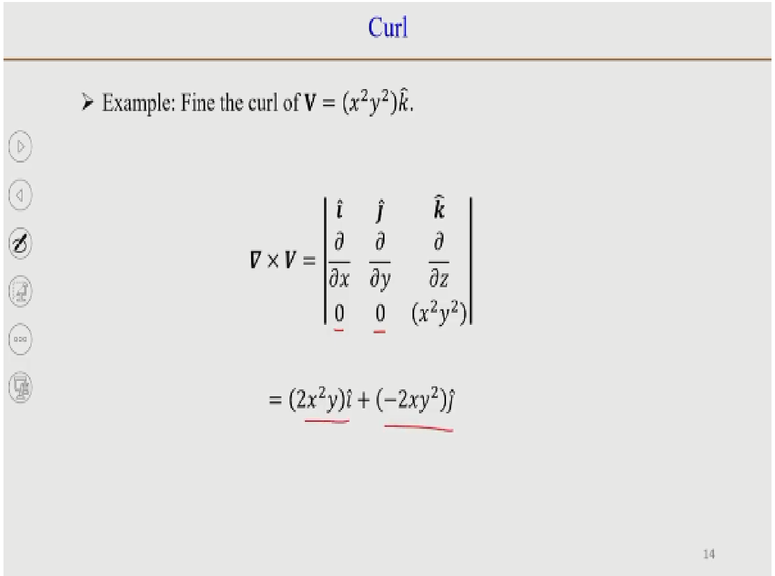

$$V = x^2y^2\hat{k}$$

$$\nabla \times V = 2x^2y\hat{i} + (-2xy)\hat{j}$$

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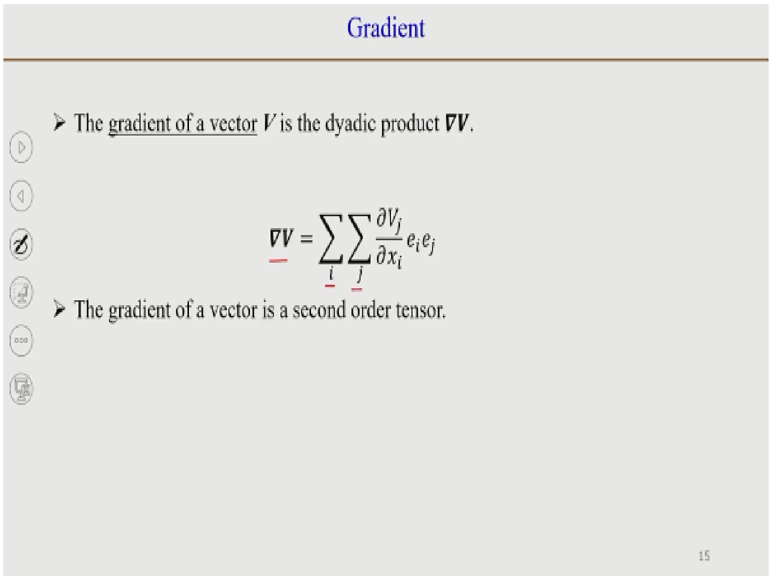

So, we have looked at the dot product of a vector with the del operator, cross product of a vector with the del operator and if we operate the del operator and take the dyad product of del operator with a vector V then what we get is a gradient of a vector V. So, gradient of a vector V will be a tensor and you can write this in the summation form easily.

So, which will have again nine components, you can have

$$\nabla V = \sum_i \sum_j \frac{\partial \mathrm{V}_j}{\partial \mathrm{x}_i} e_i e_j$$

you can expand it and write all the nine terms. So, this if V is the velocity vector, this is what we will have a velocity gradient term. So, velocity gradient is the dyad product of del operator and the velocity vector.

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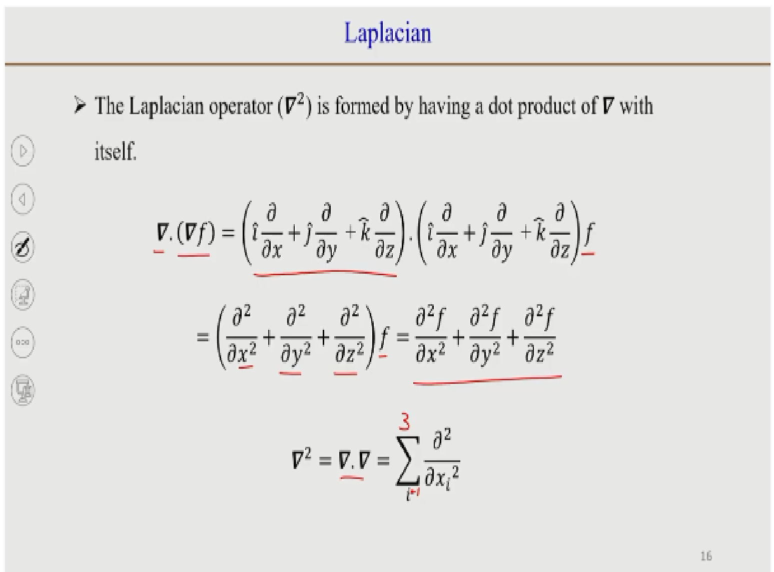

Now, we can have another operator which you would have studied in mathematics already which is called Laplacian Operator. So, if you take a dot product of del operator by itself, then you will get a scalar operator which is ∇^2which we pronounce, $\nabla.(\nabla \mathrm{f})$, if we do then we can write in the expanded form

$$\left(\hat{\imath}\frac{\partial}{\partial \mathrm{x}} + \hat{\jmath}\frac{\partial}{\partial \mathrm{y}} + \hat{k}\frac{\partial}{\partial \mathrm{z}}\right).\left(\hat{\imath}\frac{\partial}{\partial \mathrm{x}} + \hat{\jmath}\frac{\partial}{\partial \mathrm{y}} + \hat{k}\frac{\partial}{\partial \mathrm{z}}\right) f$$

So, when you take this dot product,

$$= \left(\frac{\partial^2}{\partial \mathrm{x}^2} + \frac{\partial^2}{\partial \mathrm{y}^2} + \frac{\partial^2}{\partial \mathrm{z}^2}\right) f = \frac{\partial^2 f}{\partial \mathrm{x}^2} + \frac{\partial^2 f}{\partial \mathrm{y}^2} + \frac{\partial^2 f}{\partial \mathrm{z}^2}$$

$$\nabla^2 = \nabla.\nabla = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$$

So, all of these operators we will later use when we write down the equations in the vector form. So, we will further discuss some of these things but before we start the course I just wanted you to refresh your memory and remember some of these things and practice some problems so when you are looking at the derivation, or when you are looking at the problems you can appreciate in a better manner, specially we will be using curls, divergence and gradients quite frequently, so we need to remember what these terms means specifically, so we will stop here, thank you.