Fundamentals of Fluid Mechanics for Chemical and Biomedical Engineers
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Turbulence Boundary Layers

Hello, so, in this lecture we are going to talk about turbulent boundary layers. Now, in this module we have been talking about turbulent flows, whereas we looked at boundary layers and in particular, laminar boundary layer in the previous module. So, when we talk about turbulent flows, in the previous two lectures, first we talked about what are the characteristics, properties of turbulent flows. And then we said that it is characterized by the randomness and therefore, the velocities will have a lot of fluctuations.

So, to define turbulence flow or to analyze turbulent flow, we can decompose into the mean velocity and fluctuating velocity. And then, we showed the equations for mean flow, we did not derive it, but we said that, if we average the Navier-Stokes equations over a sufficient length of time, then we will get what is called Reynolds-averaged Navier-Stokes Equation for mean flow. And it will have an additional term when you compare with the Navier-Stokes equation and this term was of the form ρu´, u´ over bar, where bar represents the averaging.

Now, this ρ u´, u´ or ρ u´ v´, this is Reynolds stress or the stress caused by turbulent fluctuations or a form of a stress. And then in the next lecture, we talked about flow in a pipe. And we saw that, based on the experimental observations, one could develop an empirical relationship of the form of power law, where u bar = u × $(1 - r/R)^{1/7}$ or raised to the power of 1/n. Such kind of power-law was developed for velocity profile in turbulent pipe flow.

Now, we could see that this power-law profile was not valid in the near wall region. So, that is why in the near wall region, people studied the velocity profile carefully and it was observed that, for all the cases, it may be a turbulent flow in a pipe or a turbulent boundary layer, there was a universal behavior in the near wall region and one could define a viscous sub layer, where viscous effects are important. And then a outer turbulent layer, where the turbulent effects are important.

So, now, with all this background, we will try to understand a bit more about the turbulent boundary layers and try to use momentum integral equation. So, remember, when we talked about laminar boundary layers, using Reynolds transport theorem, we derived a momentum integral equation which could be used to analyze the integral quantities for a boundary layer. So, let us look at this first.

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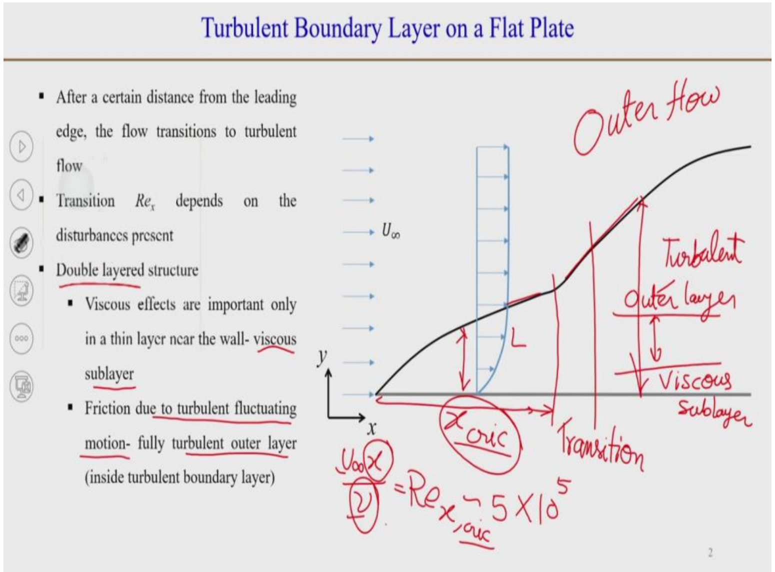

So, this is typical image for a boundary layer over a flat plate that we have been discussing. So, this is the laminar boundary layer and here you have a turbulent boundary layer. Of course, you can see that the turbulent boundary layer is significantly thicker than what you have, say for laminar boundary layer. So, turbulent boundary layer is thicker. The slope you can see here, the slope is smaller and slope in the turbulent flow is significantly larger.

So, as you move along the plate with x, the growth of turbulent boundary layer is faster with x. So, as we move along the flat plate. Now, unlike, say in pipe flow when we talk about, we say that when you increase the velocity if you are talking about same fluid, if you increase the velocity or if you increase the flow rate, then the flow becomes turbulent.

Whereas, in a, when you are talking about flow over a flat plate then, what happens that initially up to a certain distance from the leading edge, let us say this distance is you call it x critical. The boundary layer is laminar. And then after that it starts becoming or it transitions to turbulent. So, it will depend for example, if your plate is long enough then for every flow velocity, you might have a turbulent boundary layer over a flat plate.

Now, so, next question comes, when do we say or what is this number for x critical? Now, it will depend on a number of factors. The first and foremost is of course, the plates should be smooth. So, all the analysis that we have been talking about, everything we have and inherent assumption that the pipe is smooth or the flat plate is smooth. If you have a roughness on the plate, then that may trigger turbulence earlier. So, it will depend on the roughness and it will also depend the other disturbances that may cause, lot of disturbances to the flow and the turbulence may trigger at an earlier length.

In general, the number that is accepted are quite often quoted is Reynolds number, say Re x of 5×10^5 is considered to be the critical Reynolds number, based on x. So, this Re x is basically, $U_\infty x/\nu$, where ν is the kinematic viscosity. So, that is the number generally used. Now, if your flow is relatively free from disturbances then this number could be say, 3×10^6, or if there are a lot of disturbances then this number could be 1×10^5 or so.

So, this is just a number that you could use for analysis. So, that is the number and from this, you could calculate, what is the value of x for a given value of U_∞? What is the value of x at which the flow of a particular fluid, so fluid will come or property of fluid will come in ν. So, for a particular flow, at a particular fluid velocity U_∞, what is the number or what is the length at which the boundary layer will become turbulent or will not remain laminar anymore?

Now, unlike laminar boundary layer, so in the laminar boundary layer, we said that in the boundary layer viscous effects are important. Whereas, in the turbulent boundary layer, we will have layered structure, so remember, when we talked about near wall turbulence in pi flow, we said there are three layers, the viscous sub-layer and the outer layer, and in between the buffer layer or what is called overlap layer.

So, similarly, we have three layers or you can say two layers in turbulent boundary layers. It is same in both the cases. The only thing is that we say that, in the viscous sub-layer, the viscous effects are important and in the outer layer, the viscous effects are not important or the turbulent effects, turbulence effects are important. Now, because we need to have an overlap region, so there is a buffer layer or the overlap layer, so we call it three-layer structure.

So, there is no inconsistency, so to say. You could even say that this is a three-layered structure. But from the physical point of view, there is a very thin layer near the wall in which viscous effects are important, which we call viscous sub-layer. And the outer layer, where turbulent fluctuating motion is important and it causes the friction and where the Reynolds stress will be dominant. So, this is fully turbulent what is called, outer layer. Remember that this is not, this is not this outer flow.

When we talk about, we have a viscous sub-layer and a outer layer. And a outer layer and in order to match these two, we will have a buffer or to have same nomenclature, we should call it overlap layer. So, that tells us that the, the logarithmic law, the linear velocity profile in the viscous sub-layer u+ = y+ and logarithmic law that u + x =some constant × ln y + another constant, that logarithmic law was valid in the overlap layer. So, we could still have those laws valid for the turbulent boundary layers and we will use them.

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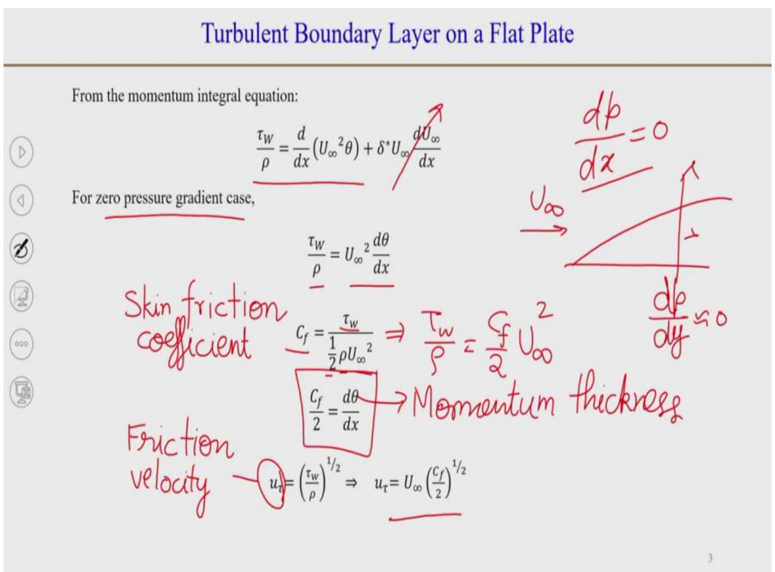

So, let us look at the turbulent boundary layer on a flat plate. And we will consider that this plate is smooth, there is no pressure gradient along the flow direction. So first, we will write down the momentum integral equation. So, this momentum integral equation, we derived when we looked at the boundary layers. So, in this, if we say that the pressure gradient is 0, so from because in the boundary layers, when the pressure gradient is 0. So, that means, dp/dx = 0.

And in a boundary layer, when we derive boundary layer equation, we said dp/dy is approximately 0. So, pressure is not a function of y. So, the same pressure field, what is there in this outer flow will be valid in the boundary layer. So, we could write that, $p + 1/2\ \rho\ u^2$ in the inviscid region. And from that, if $dp/dx = 0$, we can deduce the dU_∞/dx is also going to be 0. So, this term will go away.

Now, just to remind ourselves, we looked at the governing equations for laminar boundary layers where we took the two-dimensional equations in Cartesian coordinate for flow over a flat plate. So, the similar equations one can write down for turbulent flow over a flat plate. The only difference we will have that we will have an additional term in terms of Reynolds stress. And these equations will be for the mean flow or mean velocity and mean pressure field.

So, this equation simplifies to $\tau\ w/\rho = U_\infty^2\ d\ \theta/dx$. Now, we will also bring into picture, the definition of Cf. Which we call, a skin friction coefficient. So, it is nothing but a non-dimensional form of wall shear stress. So, you have τ w and it is being non-dimensionalise by $1/2\ \rho\ U_\infty^2$, where U_∞ is the free stream velocity.

Now, if we look at these two equations, then we can write that $\tau\ w/\rho, = Cf/2 \times U_\infty^2$. And we can substitute this here, U_∞^2 will cancel out and we will get. What we will get is $Cf/2 = d\ \theta/dx$. Remember that this θ is, momentum thickness. So, this form of momentum integral equation is valid for laminar as well as turbulent flow, the only assumption here we had is that the pressure gradient is 0.

Now, we also bring into picture, the definition of friction velocity, u τ which we defined to define the velocity profile near the wall. So, u τ is called friction velocity, in many places you might also see it, written as u*. So, this $= (\tau\ w/\rho\ )^{1/2}$ and this will give you a unit of velocity.

So, we can write from τ w/ρ, we can use $Cf/2\ U_\infty^2$. So, u τ will become U_∞ Cf/2. Again, this has an assumption that $dp/dx = 0$. So, we cannot use it always, only when $dp/dx = 0$, then we can use, $u\ \tau = U_\infty, \sqrt{Cf/2}$.

Now, the question comes that what we need to do is solve this momentum integral equation. And we solve the momentum integral equation for laminar flow over a flat plate, where the pressure gradient was 0. Now, in this case, when we talked about laminar flow or a flat plate, we assumed a velocity profile and that velocity profile was parabolic velocity profile, probably in analogue with what we observed in flow in a, say laminar fully developed flow in a pipe. So, if we, what we could do? We could assume a power law kind of velocity profile or say 1/7th power-law kind of velocity profile.

But the problem with this profile is, that it is an approximate profile and it is not valid in the near wall region. We saw in the previous class, that if you use power-law velocity profile in the near wall region that you will, then you will get the gradient of this mean velocity to be 0 at the wall, and that will result τ w being 0, which is not true. So, we could not use because, in that case our Cf or τ w useful results. So, what we could do?

The another velocity profile we have, especially in the near wall region is logarithmic velocity profile, which is for the overlap layer region. Now, overlap layer region means, it matches down to the viscous sub-layer and it matches to the outer layer. So, we could approximate that the logarithmic or we can assume, that the logarithmic velocity profile is valid across the turbulent boundary layer. And using that we could derive or we can analyze the momentum integral equation. So, we will do that.

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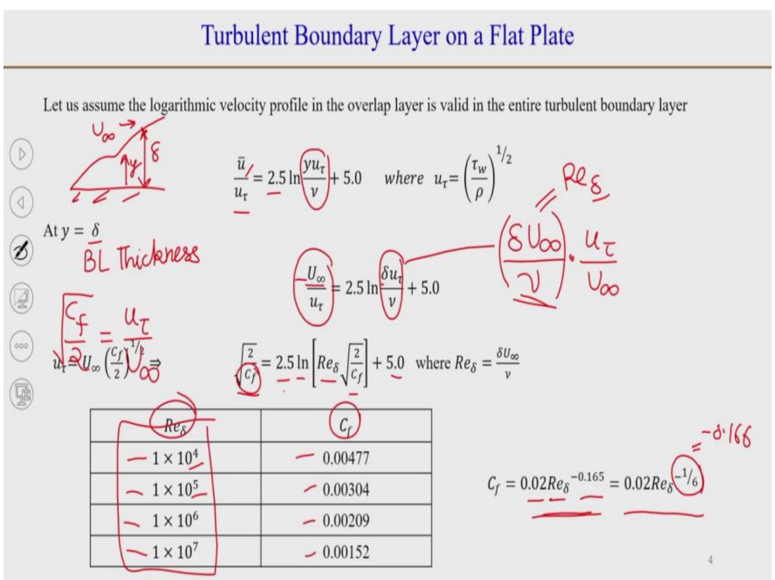

This is the logarithmic velocity profile, which is basically u bar/u τ, which we call u + also, is equal 2.5 times ln, y u τ/ν + 5. So, this is also called y + yu τ y ν. When y = δ, so that means, where δ is the disturbance thickness or boundary layer thickness, so at this region, at this point, if the free stream velocity is U_∞, then u bar will be $=U_\infty$. So, we can write this equation, we can substitute u bar or mean velocity = U_∞ and y = δ.

Now, in the previous slide we looked at Cf/2, we could write that $\sqrt{Cf/2} = u\ \tau/U_\infty$. So, we could substitute this value here, that will be $\sqrt{2/Cf}$. Similarly, we could rearrange this. So, we can write this $\delta\ U_\infty/\nu \times u\ \tau/U_\infty$.

So, this is a kind of Reynolds number, where the length scale is δ boundary layer thickness. So, we will call it as Re δ, which represents that the length scale here is δ boundary layer thickness. So then, we can substitute $U_\infty/u\ \tau = \sqrt{2/Cf} = 2.5 \ln Re\ \delta$, which is this. And $u\ \tau/U_\infty$ again$\sqrt{2/Cf} + 5$.

Now, we have brought in the form of Cf. This equation has come in the form of Cf and n δ, which is basically hidden in this Re δ as you can see from here. Now, what we do is, we could write using this equation, we could find out the value of Cf for different Re δ. You will need to do a bit of iteration, but that is very simple. You could code this in an excel and try to find out the values. So, what I have done, I have calculated it for different Re δ, the values of Cf.

And as suggested say, by F M White that we could fit in this data and try to find out a equation. So, that equation, if you find the best fit curve for this data for using these four data points only, then you will get a simpler relationship. Because this relationship, though it is accurate one or relatively accurate, but it has a logarithmic form. So, to simplify the mathematics, what they suggested that we could write down a simpler relationship between Cf and δ and which will be valid for at least these values of Re δ.

Remember that this is Re δ, not Rex and δ is less than x. So, that is why this, 1 raised to the power, 1×10^4 and 10^5 are coming into picture here. Even in these values of Reynolds number δ, Re δ the flow will be turbulent, because Rex will be much, much higher.

So, we will have Cf, when we fit into this. We can write this $0.02\ Re\ \delta^{-0.165}$ or which is close to Re δ this number is - 0.166. So, we can write in this form $0.02\ Re\ \delta^{-1/6}$. Now, what we have is a simpler relationship between Cf and δ.

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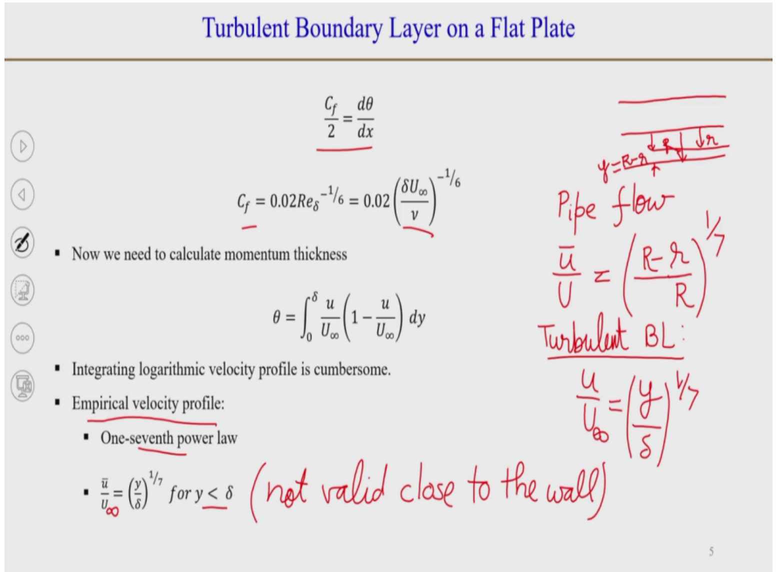

So, our momentum integral equation, when we wrote down it in terms of Cf, it was Cf/2 d θ/dx. Now θ, remember that it was that we will need to know the velocity profile integral 0 to δ u/U_∞ - × 1 - u/U_∞ × dy. So, we will need to know u as a function of y or u/U_∞ as a function of y/δ. So, we need to know the velocity profile and we could use, because in the entire boundary layer, we have assumed the logarithmic velocity profile. So, we could integrate that velocity profile, but again, the integration will be quite involved.

So, what one could do is, one could use the power-law velocity profile to find out θ. So, if we do that, we already have Cf in terms of δ. And if we calculate the θ, in terms of δ, then we will have a differential equation for δ. We can solve this differential equation, with certain limits. And then, we will be able to find what is the expression for δ. So, to calculate momentum thickness, we can assume empirical velocity profile, which is one-seventh power-law velocity profile.

Now, for a pipe flow, we had one-seventh power-law velocity profile was u bar/U = 1 - r, where r is the radial coordinate, r/R, which is pipe radius raised to the power 1/7. Now, in analogue, with that we could write for turbulent boundary layer, u bar/U, this basically is, you could write it R - r. And what is R - r for a pipe flow? So, this is r, this is R, and basically this value is R - a r, which is the distance from the wall. So, we can write if this is y, so, distance from the wall y. And in place of R we will use the turbulent boundary layer thickness.

So, y/δ $^{1/7}$. So, that is what we will have, that in the turbulent boundary layer u bar/U it actually, u should have been u bar/U_∞ = y/δ $^{1/7}$, because we are using U_∞ as the free stream velocity. Of course, again, it will not be valid close to the wall. That is the region, that we have used logarithmic velocity profile to calculate wall shear stress.

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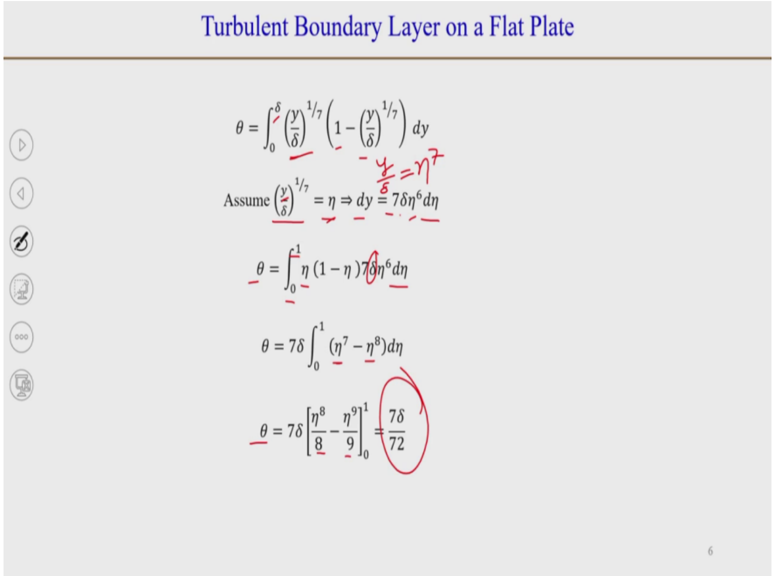

So, let us calculate momentum thickness θ, is by substituting the velocity profile. Now, we could simply solve this integral by assuming say, that $y/\delta^{1/7} = \eta$. So, that will give us that $y/\delta = \eta^7$ or from that we can write that $dy = 7 \times \delta \times \eta^6 \times d\eta$. So, we can substitute this for dy.

And the limits will be that when y = 0, η will also be 0 and y = δ, then η will become 1. So, we will have our θ = integral 0 to 1 the limits and the first term $y/\delta^{1/7}$ will become η. The next term within the bracket will be, $1 - \eta \times 7\delta\eta^6 d\eta$. Now, δ is the boundary layer thickness. So, it is constant with respect to y say 7 is constant, so we can bring them out. And that will basically become, when you bring all those terms together that within bracket you can have η raised to the power $7 - \eta^8 d\eta$.

And when you integrate it, this will become $\eta^{8/8} - \eta^{9/9}$. And after substituting the limit, you will have 1/8 - 1/9. So, that will be 9 - 8/72 or 1/72. So, you will have θ = 7 δ/72.

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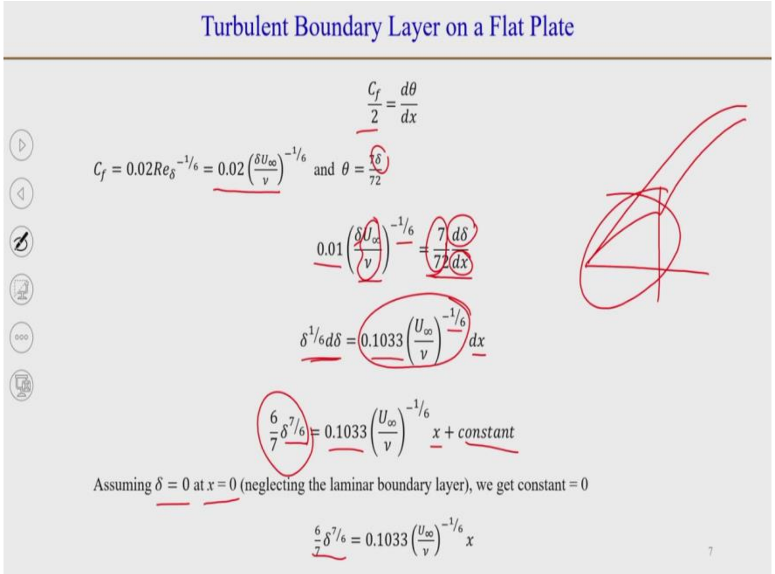

So, now we have Cf in terms of δ, we have θ in terms of δ. We can substitute both. So, when we substitute in this then this del, Cf/2 will become simply 0.01, Re $\delta^{-1/6}$ or we can write Re δ in the expanded form, where we have δ explicitly, U_∞ and ν are constant. This = 7/72 d θ/dx. So, this will be d δ/dx.

Now, we can integrate it, we can take the variables on each side. So, we can take δ on one side, so, that will be δ × 1 $\delta^{1/6}$ × d δ = 72/7. So, that will come here, 72/7 × 0.01, which will give us 0.1033 and U_∞/ν. So, and this is multiplied by dx. So, dx × $U_\infty/\nu^{-1/6}$. And when you integrate it that will be δ 1/6 + 1, so 7/6, $\delta^{7/6}$. So, it will become 6/7 δ 7/6 and all this is constant. So, we will have simply x here, + a constant of integration.

Now, the next question comes, how do we find this constant? So, a gross assumption probably has been made. But that is neglecting the laminar boundary layer over the leading edge. So, the assumption is that, when you talk about turbulent boundary layer, there is, in the certain region, there is a laminar flat plate. So, it does not start at x =0, but what we assume here that, this turbulent boundary layer starts from x = 0 itself. So, that is the assumption that, at δ, at x = 0, δ = 0. So, that will be 6/7 δ raised to the power 7/6. So, this constant basically becomes 0.

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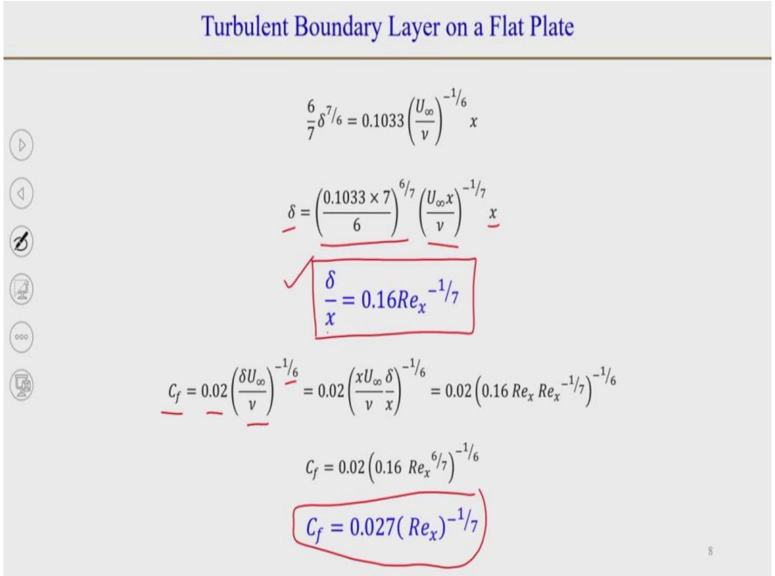

Now, we can rearrange it in a simpler form. So, we can do this and we will get $\delta = Re\ x^{-1/7} \times x$ multiplied by some constant. So, that will give us $\delta/x = 0.16\ Re\ x^{-1/7}$. So, we get a relationship for turbulent boundary layer thickness.

The assumptions here are the plate is smooth. The pressure gradient over the plate is 0. The other assumption we have assumed, the velocity profiles. We have assumed, remember, we have assumed two velocity profiles, we have assumed the power law profile, to calculate momentum thickness. We have assumed the logarithmic velocity profile to assume τ w.

Now, if the integration is cumbersome, probably we could use the logarithmic velocity profile for both the cases. And use the numerical integration to simplify things. So, this gives us a relationship for turbulent boundary layer thickness.

Now, we could use, this expression to calculate Cf or a skin friction coefficient. So, that is 0.02 Re $\delta^{-1/6}$ and we can substitute the value of δ here. And what we will get after a bit of algebra, we will get $Cf = 0.027\ Re\ x^{-1/7}$. So, that is this skin friction coefficient.

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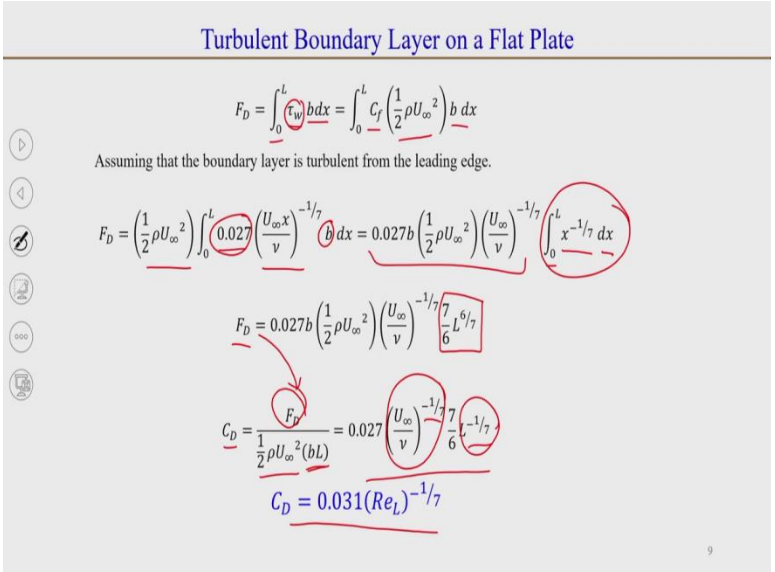

Now, we know the skin friction coefficient, we know the boundary layer thickness, we could also calculate what is the drag on a flat plate. Let us say, where the turbulent boundary layers starts from x = 0 itself. So, that will be τ w × b dx, where b is the plate width normal to the screen. So, we can write τ w in terms of Cf × 1/2 ρ U_∞^2 × b dx and then we can replace the value of Cf there.

So, CF is basically 0.027 × Re $x^{-1/7}$, and 1/2 ρ U_∞^2 is constant with respect to x. So, that can come out of the integral. And b and 0.027 are also constants. So, we basically have a constant into integral 0 to L $x^{-1/7}$dx and when we do that and put the limits, then what we will get from this integral $x^{-1/7}$+, so that will be basically 6/7.

So, after substituting the limits you will get 7/6 × $L^{6/7}$. So, that is your drag force. Now, we can simplify this further and write this down in terms of drag coefficient. So, drag coefficient is drag force/1/2 ρ U_∞^2 multiplied by weighted area, remember that for a flat plate which is aligned with the flow, we use the weighted area. And in this case, we assume that flow is happening only on one side of the plate. If the flat, if the flow is happening over both sides of the plate then this will be 2 bL, but here we assume that is only on one side of the plate. So, area is bL.

And we can substitute the value of FD from here. The ρ_∞^2 will cancel out, b will cancel out and what you will get is this $(0.027\ U_\infty/\nu)^{-1/7}$ × 7/6 $L^{1-1/7}$, so when you combine this and this, you will get Re $L^{-1/7}$, multiplied by 7/6 × 0.027. So, that will give you the drag coefficient, cd = 0.031, Re $L^{-1/7}$.

So, all that analysis has given us that, how we can calculate the drag force in a turbulent boundary layer with an assumption that the turbulent boundary layer starts from the leading edge of the plate, in all the in the calculation of δ as well as in the calculation of Cf.

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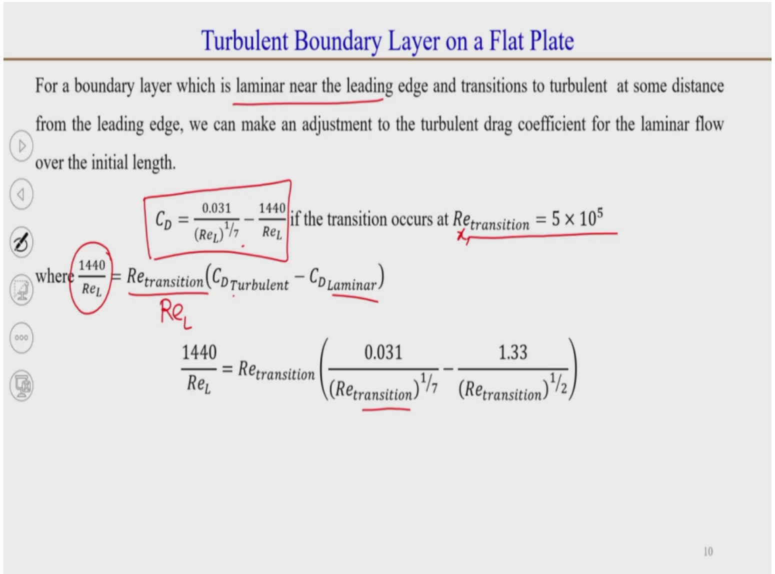

So, it has been suggested that, if one need to take into account that the boundary layer is laminar near the leading edge and the transition to turbulence at some distance, the drag coefficient an adjustment can be made in the drag coefficient and this formula has been given. Now, this is CD = 0.031/Re L $^{1/7}$, so the same formula which we just derived for the turbulent flow over a flat plate - 1/1440/Re L. And here, the assumption is that the transition starts at Reynolds number say this is Re x, at 5×10^5.

Now, this 1440/Re L comes into, actually it should have been/Re L. So, this is basically, this number 1440 comes from that Re transition into CD for turbulent flow - CD for laminar flow at the transition Reynolds number. So, we can actually calculate them, we can substitute these values that Re transition is 5×10^5. The number comes out to be 1438 and so, it has been approximated to 1440. So, we could use that to take into account that in the initial length the flat plate has laminar boundary layer and then, it has turbulent boundary layer and the drag coefficient can be calculated using this.

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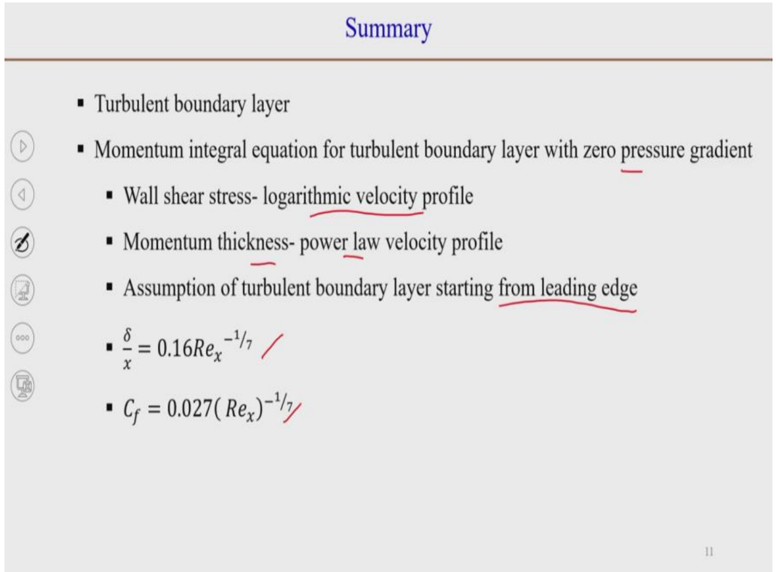

So, what we have done today is, we looked at turbulent boundary layer. We discussed that in the turbulent boundary layer, it grows faster, it is thicker then you have different region viscous sub-layer and the outer layer. The Universal log law of the wall that we talked about in the pipe flow is also valid for the turbulent boundary layers. And then, we used the momentum integral equation, to analyze the integral quantities for a turbulent boundary layer, over a flat plate with 0 pressure gradient.

In doing that, what we did is because we needed wall shear stress in terms of δ or Cf in terms of δ and we needed momentum thickness in terms of δ. So, we assumed a logarithmic velocity profile, which is basically u + = 5 Ln y + 5 for which is valid near wall region. So, we used that velocity profile, calculated wall shear stress. And then, momentum thickness, for that we used power-law velocity profile and then, we also assumed that turbulent boundary layer starts from the leading edge and using all those assumptions, we derived like expressions for turbulent boundary layer thickness and Cf or the skin friction coefficient.

Now, you might see in the books or there is another analysis where one could use the value of τ w with in analogue with the pipe flow and from the, which is in terms of the distance from the wall and so on. So, one could use that expression and one will end up with slightly different equations for δ/x and Cf. So, it is not important to remember these relationships, but it is important to see that how does this vary.

And this, the turbulent boundary layer varies faster than the laminar boundary layer. And how do we do this analysis? The assumptions that we have made, the velocity profiles that we have assumed. So, consider this as a kind of a problem that, if you have a velocity profile given, how you can use the momentum integral equation to find out the integral quantities?

So, we will stop here. Thank you.