Fundamentals of Fluid Mechanics for Chemical and Biomedical Engineers

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Turbulent Flow in a Pipe

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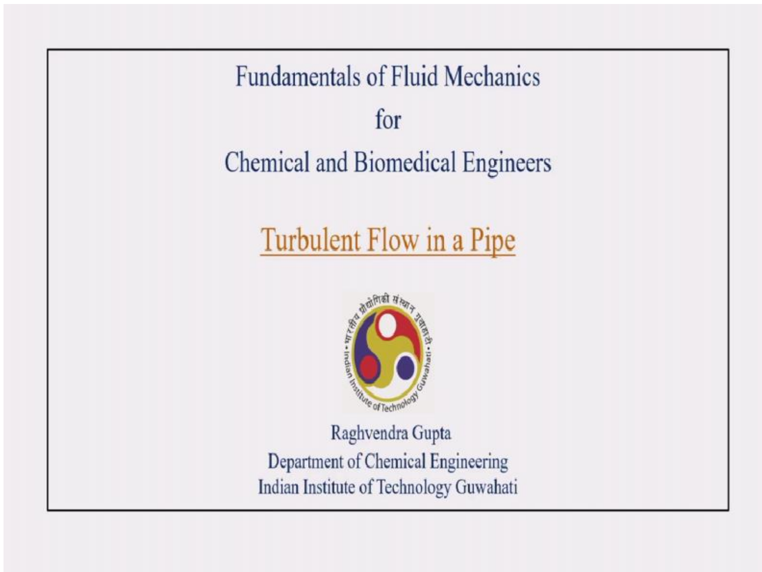

Hello. So, in the last class we discussed some basic features of turbulent flows where we discussed that turbulent is primarily defined by randomness, there are fluctuations in the flow always. Then, we looked at or we could decompose that the velocity components u v w they can be decomposed in a mean velocity + fluctuating velocity and when we do that and average the Navier-Stokes equation and this averaging is called as Reynolds-averaged Navier-Stokes Equation, the averaging is overtime.

So, this, when we do the average of Navier-Stokes Equation we end up getting the mass conservation and momentum conservation equations for the mean flow. And what we got is that for the mean flow in the momentum conservation equation we had an extra term in the form $\rho U' V'$ and So, on. So, it will have nine components and we called it the Reynolds stress or turbulent stress and this stress is the extra term that come in to the Reynolds-averaged Navier-Stokes Equation when we compare with the Navier-Stokes equation for a laminar flow.

So, this turbulent, The problem of turbulence modeling that we discussed that there is closer problem that you need to define this $\rho U' V'$ in terms of known quantities or in terms of mean velocity or mean velocity gradient So, that you can close the system of equations and find the solution for the mean flow in a turbulent flow.

So, now, one of the very often used system in chemical engineering as well as in biomedical engineering or in engineering applications in general is flow in a pipe. Now, we looked at that for a fully developed flow, for laminar flow in a pipe, we can have or we can derive the Hagen-Poiseuille law where we could get a relationship between a pressure drop and flow rate and we could derive a velocity profile. We saw the velocity profile is parabolic and that we could do from the fundamental principles right from solving the continuity and momentum equations and then neglecting the terms depending on the assumptions or depending on the things for example fluid is steady and compressible, fully developed and So, on.

Now, it is not possible to do So, in turbulent flow job in almost all the turbulent flows we cannot find an analytical solution because the flow is random and it is changing with time continuously, only thing we could do is a statistical analysis, it is such kind of analysis what we have been doing for laminar flow is not possible is not possible for turbulent flow.

So, over the years or probably over the decades or maybe even centuries what has been done to understand turbulent flow people have done a lot of experiments, collected the data and from those data and depending on the physical understanding they tried to use dimensional analysis and in recent years lot of computations simulations which is called direct numerical simulations, So, one solves these Navier-Stokes equation in such a manner that all the possible length and time scales can be captured.

So, using that, a lot of understanding has been developed. What we will do here is we will briefly discuss turbulent flow in a pipe and So, the first thing we will look at, that velocity profile, what is the velocity profile when we look at a turbulent flow?

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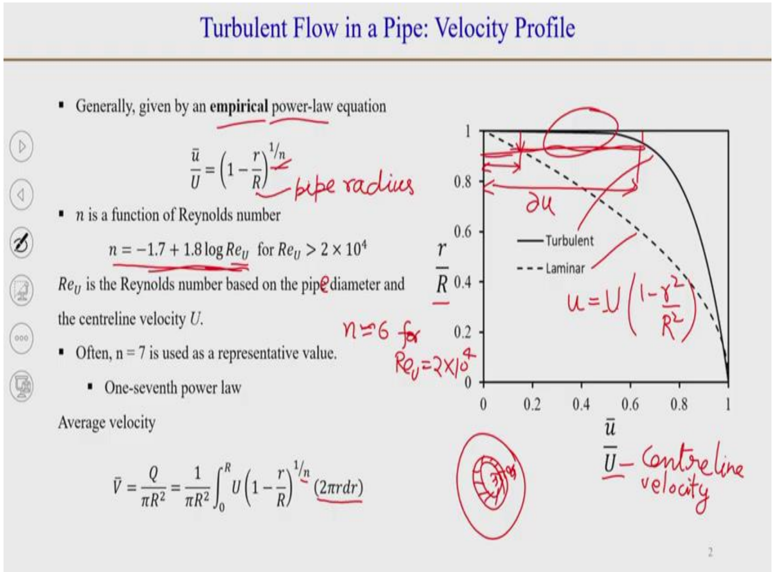

So, in this case what I have plotted here is the non-dimensional radius So, r/R and this is $\bar{u}$/U and U here is center line velocity. So, this is how a turbulent velocity profile will look like and for comparison I have also plotted the laminar velocity where of course $\bar{u}$ is simply u because there are no fluctuations.

So, laminar profile we remember that it was $\bar{u}$ or say u is equal to, do not confuse here because I do not have 2 here. When we have 2 then this u is mean velocity, whereas we have this as center line velocity which is the maximum velocity, which is at the center. So, this $u= U (1 - r^2/R^2)$. It is parabolic velocity profile.

So, what you can notice here is that when you have a laminar velocity profile and a turbulent velocity profile the gradients near the wall $\partial u/\partial y$ is larger, So, if you look at, or $\partial u/\partial r$ in this case that in, there is a lot of change in the velocity in a very radius So, if you take just typically this radius and what is the change?

You see that dr is almost same in this case and the change in velocity here is, this is my du in laminar case and this is my du in the case of turbulent flow. So, the velocity gradient near the wall in turbulent flows are very large and as a result the viscous stresses in the turbulence flows near the wall will be higher when you compare with the laminar flow.

So, this is a qualitative picture that one can have, but if we want to use it for an analysis, let us say if we want to find out the flow rate, like we had such a relationship, So, is it possible to find a relationship for turbulent flows? And by looking at the experimental data an empirical, So, empirical equations are when you have a lot of experimental data or say numerical data and by looking at this data if you can find out a equation which is valid in the range in which the data is present.

So, from that, such equations have been developed So, this is called empirical and it is power-law because as you see the equation is in terms of a power. So, this gives it $\bar{u}$ U=1 - r/R, where R is the pipe radius, whole raise to the power 1/n. So, n is a number. Now the question comes what is this n?

So, we can take n in general if you are talking about that what is the typical velocity profile in turbulent flows then this n can be taken as 7, So, the velocity profile is called 1/7th power-law for turbulent flow and the profile that I have shown here is using 1/7th power-law. But this n is shown to be or it is seem to be a function of Reynolds number. So, the expression that is given in the literature is n=- 1.7 + 1.8 log (Re_u). So, Reu is defined based on the center line velocity. So, we can say / kinematic viscosity u and pipe radius.

So, when it is greater than 20,000 or 2×10^4, we could use this relationship and if we substitute the values then it turns out that n is about 6 for Reu 2 × 10 raise to the power 4 and 2×10^5, the value comes out to be about 7.84. So, that is the reason that the typical value people have taken that if I want to treat it as a constant then this is 7.

Now, we need to remember that this is not an exact equation and it has limited validity. So, we will also look at its limitations down the line. Now, we could calculate once we have obtained this velocity profile or once we have a velocity profile we can try to find out what is the average velocity and to find average velocity we need to calculate flow rate.

So, we can calculate flow rate which will be integral 0 to R, $u(1 - r/R)^{1/n}$ into a strip at radius R So, that will be 2 π rdr is the area. So, if you take a pipe and because it is varrying with the radius So, you can take a typical radius R and of thickness dr a strip and what is the flow rate? Let us say dq, So, 2 π rdr into the velocity and then you integrated from 0 to r.

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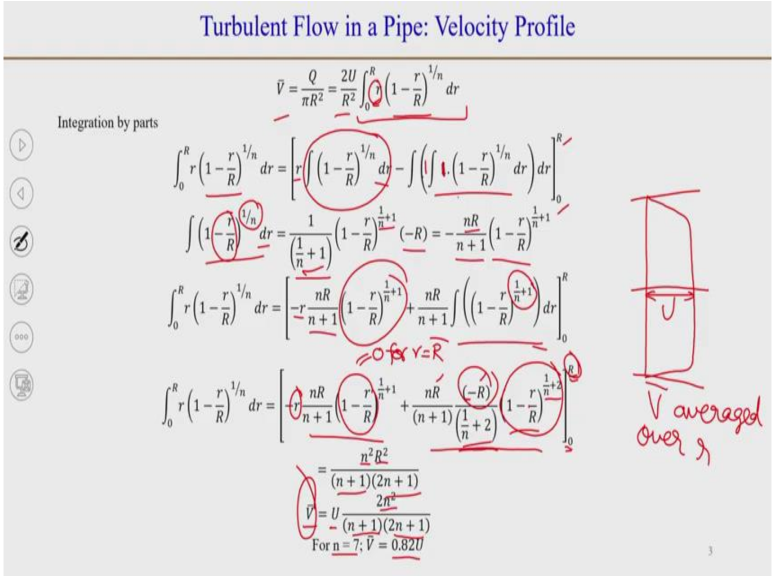

Okay So, this integration is a bit involved So, we will just do it. If you go back we can cancel out π and π here, this 2 can come out and U can come out and R^2 is already out So, we will have $2U/R^2$. As you can see here, So, this is $2U/R^2$ and the integral sign is r(1 - r / R) $^{1/n}$ dr. And we have two fictions here r and (1 - r / R) $^{1/n}$.

So, we can treat it or we can treat it or we can solve this by considering integration by parts, first function and second function. So, we can consider r as a function whose derivative is to be done because when you consider this the derivative will be 1 and things will become simpler.

So, we will do integration by parts. So, we will take just this part and try to write down the integration by parts formula So, r is as it is and we will write down the integral of 1 - r/R whole raise to the power 1/n dr -, within the integral sign this should have been outside. So, differentiation of r, which is 1, and the integral of this; and then, we can substitute the limit once we find it. So, first thing we need to solve this we need to find this integral because it is required here and here as well, So, let us do that.

So, when you have 1 - r/R whole raise to the power 1/n dr we can integrate it. So, when you integrate it, it will be 1 - r/R whole raise to the power 1/n + 1 / 1/n + 1. But you will also need to divide by the differentiation of it. So, we will have / - 1/r or 1 over, - 1/r So, that will give you - r here.

And when you simplify this will be - nR / n + 1. So, we have found out this integral and we can substitute it. So, this becomes - r n R / n + 1 and this whole term, we can substitute this now. Now, once we have done this the next task is to simplify this integral and the only thing, the only difference from here is that the exponent here was 1/n and now we have the exponent 1/n + 1 So, we can simply do the integration again.

So, if we look at what we could do $(1 - r/R)^{1/n + 1 + 1}$ So, 1/n + 2. Here, we will have / 1/n + 2 and 1 more - R there in the numerator. Everything else remain same. So, we can simplify this by substituting the limit. So, when we substitute the limit R and if you see that when I substitute the limit r=R then this term 1 - r/R=0 for r=R, because then it becomes 1 - 1.

So, both the terms have 1 - r/R raise to the power some exponent. So, both these terms are going to be 0 when we substitute the value of R. Now that will be -, when we substitute r=0. So, when you substitute r=0 the first term will again be 0 because you have this multiplied by r. So, the only term when we expand is left with this term. When we substitute r=0 So, when you do that, you will substitute r=0 here, So, this term will become 1 and we will simply get, because this will be - sign again, one more - sign will come here when you substitute the limits.

So, that will be - and - and - will become + and when you simplify this, you will write 2n + 1 divided n, So, you will have one n coming into the numerator. So, you will have $n^2 \times R^2$ / n + 1 × 2n + 1. So, that is your, when you substitute this in the formula here you will get what is $\bar{V}$ or the average mean velocity.

So, we will have two means here. One is the mean turbulent velocity. The mean velocity in turbulent flow, where looking at the time average velocity and this $\bar{V}$ is the average velocity because the profile is wearing with respect to radius. So, this is the average velocity of turbulent mean flow in the pipe. That= U × $2n^2$ / n + 1 and 2n + 1.

So, we can substitute let us say value of n if we put it say about 6, n=6, that is what we get at r=20,000 So, the value will be about 0.79 and for n=7 the value is about 0.82 and So, on. So, the value of $\bar{V}$/U will be 0.8 or 0.86 or 87 around that range. So, just to remind ourselves, what is $\bar{V}$ and what is U? So, if I plot my profile something like this, So, you U is the velocity here and $\bar{V}$ is the average velocity.

So, $\bar{V}$ is mean velocity=0.8 two times of the center line velocity and when you compare with the laminar velocity profile, the average velocity was half of center line velocity because center line velocity was the maximum velocity. So, when you substitute r=0, you will get that $\bar{V}$=0.5 × U. So, the mean velocity is half of the center line velocity in laminar flow whereas here it is 0.82 or 0.8 of that order.

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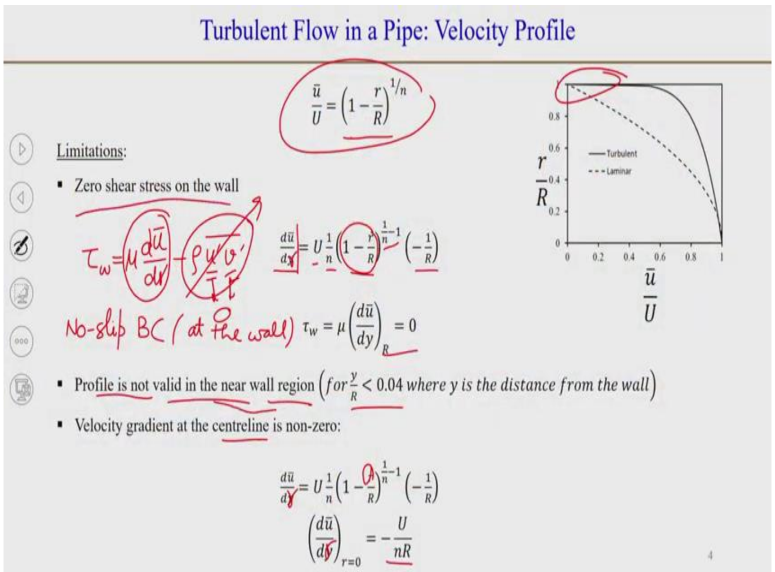

So, as I said earlier that this is an approximate relationship and it has some limitations as well. So, we can see or we can check some of the limitations. So, of course, once we have a profile and we will tend to find the stresses. Now, the shear stress on the wall we can calculate the viscous shear stress by tau w=mu d$\bar{u}$/dy.

So, to do that we need to find d$\bar{u}$/dy, So, that d$\bar{u}$/dy when you differentiate this $\bar{u}$=u into this whole expression So, when you differentiate with respect to, it should have been r actually, because we are talking about the radial coordinates So, du/dr=u × 1/n × 1/r/R whole raise to the power 1/n - 1 and differentiation of 1 - r/R which will give us - 1/R.

So, if we want to find out at r=R du/dr at r=R then this term will be 0. So, this shows that the viscous shear stress at the wall is 0, but that is not the case. If you remember the wall shear stress in a turbulent flow when we consider the mean velocity it will be, now this ρ u′, v′ will be 0 because these fluctuations are 0 at the wall because of no slip boundary condition.

So, that means, near the wall the turbulent shear stress or Reynolds stress gets eliminated. There is now Reynolds stress near the wall. So, the only term that we are left with, the viscous shear stress and this profile gives us that the viscous shear stress is 0, which is not true, if this would have been true, then I would have, I would not need to do any work to push the flow through a pipe in the laminar or in the turbulent flow region.

And we can see from here, But, that is not exposed here So, we will not be able to see the gradients that what is the gradient there? Actually, the gradient in this is 0, but, sorry a gradient in this is 0, yeah, because I have plotted 1/7th power-law. So, what is there that this profile is not valid in the near wall region? So, we cannot use this velocity profile to find the wall shear stress. Well So, that means that we cannot find, we have got a velocity profile which we can approximate everywhere except in the near wall region.

So, when we say that it is near wall region when y/R is less than 0.04 where y is the distance from the wall. So, that is my one limitation that this profile is not valid in the near wall region. Then it is not So, clean even at the center line. If we look at the center line and again, find the gradient at the center line, So, when I put r=0 here for the center line then this term will become - U/nR.

So, at the center line the gradient is not 0. You can consider it as du/dr. Now, that says, that at the center line the gradient of velocity is not 0. So, which is also another limitation of this profile. Nevertheless, this is quite successful in predicting say flow rate, etc. So, we can use this profile but knowing what are its limitations. The main limitation or what we need to remember that it is not valid in the near wall region.

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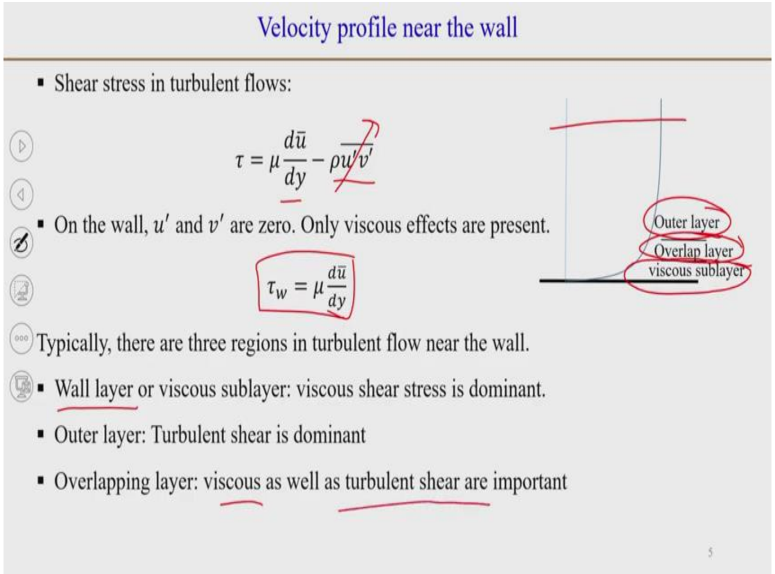

Now So, what we need to look at is the question comes what to do in the near wall region or how we can find out what is the velocity profile in the near wall region. And it turns out that it is not only whatever analysis has been done is valid not only for flow in a pipe, it can be valid for most of the turbulent flows where the flow separation is not happening. So, it can be valid for turbulent flows in a pipe. It can be valid So, which is internal flow, it can be flow over a flat plate, external flow, flow over a cylinder or a sphere.

So, what we could see is that as we saw that the turbulent or the stress, shear stress in the turbulent flow, we can have laminar as well as turbulent component and this will vanish in the near wall region or next to the wall, because u´, v´ are 0. So, that is my wall shear stress mu d$\bar{u}$ y, dy. Now there are typically three regions we can divide in the near wall region, the near wall region can be further divided.

One, where viscous effects are important, So, this is called viscous sub-layer and because it is next to the wall, So, it is also called a wall layer. And in this, only viscous effects are important as we saw here, that near the wall, even for turbulent flows only viscous effects are important. Then when you go away from the wall, sufficiently away from the wall there is an outer layer region and see here that the outer layer region is now where the shear is 0, there is shear present, there are gradients and velocity here, So, you have an outer layer region. So, when you talk about say turbulent boundary layer the outer layer will be inside your turbulent boundary layer. So, in the turbulent boundary layer also you can have such three regions.

So, viscous sub-layer region which is where the viscous effects are important, outer layer region where the inertial effects are important or the turbulent effects are important, So, this is also called inertial sub-layer. And then, in between them where both the effects, viscous effects, they are dominant near the wall and as you go away from the wall viscous effects are decreasing and in the outer layer, inertial effects are important but they start gradually increasing.

So, between them you have a layer which is called overlap layer. So, there both, the viscous as well as turbulent shear both are important.

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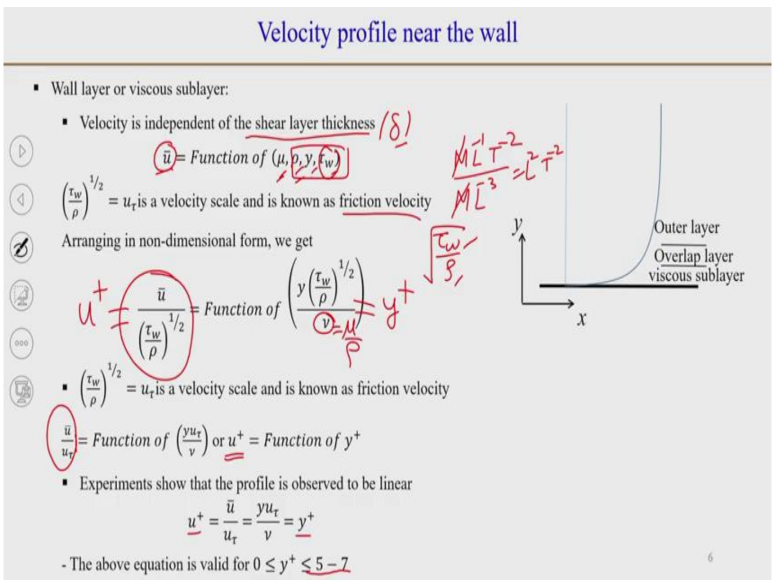

So, by looking at the experimental data what people saw and they did a bit of dimensional analysis. So, they saw that in the wall layer we know that the viscous effects are going to be important and one thing that they observed that this is independent of the shear layer thickness. So, when we talk about a pipe flow we can say the shear layer thickness is my radius on the pipe. If I talk about boundary layer then my shear layer thickness can be the turbulent boundary layer thickness, which is, let us say we will use a term for shear layer thickness here, delta.

So, this is because it is a very region near the wall and it does not matter for this region, in the viscous sub-layer region for the velocity profile that what is my shear layer thickness. And it is dependent on the following quantities, mu, ρ, y and tau w, because when we do measurements, people could probably measure tau w. So, they have been able to observe or see that $\bar{u}$ is a function of mu, ρ, y and tau w.

Now using dimensional analysis, we can use say Buckingham pi theorem and try to formulate or try to find the dimensional as groups here. So, one can do that analysis, what we could do or by looking at these variable, we can try to see what my dimensional groups can be, because what we are looking at this is my dependent variable. So, if I want to non-dimensionalise it, u, So, to non-dimensionalise it we might need some kind of velocity scale.

So, that velocity scale let us say if we take ρ y and tau w, as my repeating variables, So, we could construct a velocity scale from these, So, if I look at only by say tau w is basically M L T - 2 / L^2, So, - 1. And ρ is M L raise to the power - 3 and if you derive M will cancel out, So, you will have only L and T and from that we will be able to find that=L^2 T raise to the power - 2. So, if we take its^2 root, So,^{2} root of tau w/ρ will be a velocity scale.

So, this velocity scale is known as friction velocity. So,^{2} root of tau w/ρ, we know is or we term it, this velocity scale is friction velocity, So, basically it is constructed from the fluid property which is density and wall shear stress tau w and So, we have got one non-dimensional term $\bar{u}$/u tau or frictional velocity.

Now another term you can get, because you have got a velocity scale and you have mu, ρ and y, So, we can construct a Reynolds number, So, we can write this as mu/ρ or nu So, that will be y tau w/ρ or friction velocity / nu. So, that is my Reynolds number. So, we have constructed two non-dimensional groups and these two non-dimensional groups are known as, So, this one is known as u + and this one is known as y +.

So, that is the non-dimensional velocity in the near wall region and that is a non-dimensional length scale or non-dimensional measure of the distance from the wall, because this is y non-dimensionalized by nu/u tau also this is basically y / nu/u tau So, this is a length scale.

So, what we could see that when we arrange these in the non-dimensional form that my u + is a function of y + and after looking at this data or when people got this data they saw that the profile is linear and it comes in the form that u +=y + So, it is a simple linear profile and the beauty of this result is that this is valid for almost for most of the turbulent flows internal as well as external flows in the boundary layer region as well as in the pipe flow or internal flows.

So, that is velocity profile we have obtained in the viscous sub-layer. Now we can look at the, and this is valid, the validity of this expression is y + up to 5 to 7, So, only in this range it is important. But it is very important to include it either if you are resolving your simulations up to that, then because this will give you the correct prediction of turbulent wall shear stress.

So, this is a very important part if you are looking at say your simulations and when you do turbulent flow simulations you need to be certain that what you are, are you resolving the flow to this scale So, that, and what you are putting, are you using a wall function there. So, the discussion of y +, u +, etc. frequently comes in to turbulence models when, you do CFD simulations for example.

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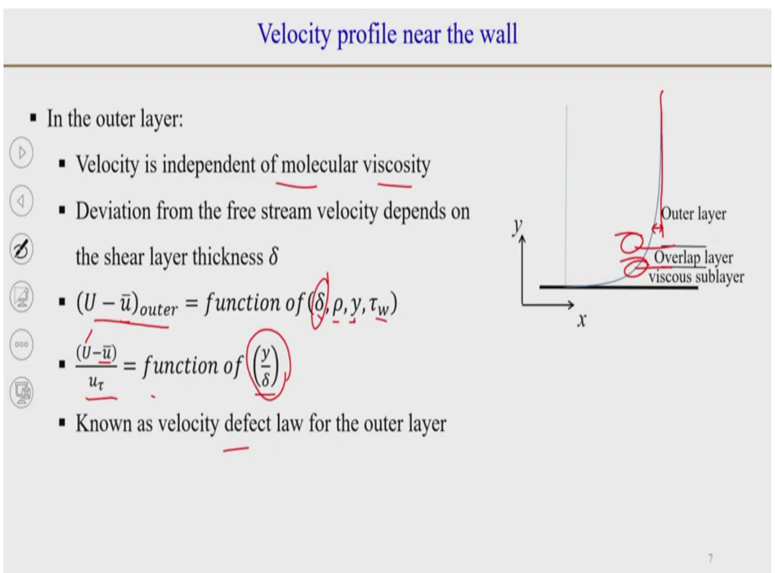

And in the outer layer region the observation was or when we divided this category, when we said that there is no viscous effect, importance is not there in the outer layer region then we can say that it is independent of molecular viscosity, but it is a function of delta now.

So, that depend or what they saw or what they suggested that the difference between U which is the center line velocity and $\bar{u}$ which is the velocity in the outer layer region, So, their difference is a function of delta, ρ, y and tau w. So, when we compare from the viscous sub-layer, mu has gone and delta has come in because it depends on the shear layer thickness.

We can again construct the same friction velocity So, root of tau w/ρ and what we could do is we could get one relationship or U - $\bar{u}$ / friction velocity and another non-dimensional group we can form from here y and delta, So, y/delta. So, that says that it is a function of y/delta or this is difference of the free stream velocity when we talk about boundary layer flows or the center line velocity for pipe flows, So, U - $\bar{u}$ and it is known as or we call it velocity defect. So, this is called velocity defect law.

So, what it tells us that U - $\bar{u}$/u tau is a function of y/delta and we do not have the form of function here. Now we need to have overlap region in such a manner that it can combine the two dependents. So, U +=y + or U/u tau=y, u tau / nu which is valid in the viscous sub-layer and U - $\bar{u}$/u tau=this as a function of y/delta, this should be valid in the other limit.

So, if you look at overlap layer it should be matching the viscous sub-layer in this region and it should be matching with the outer layer in this region. So, an expression has been derived or has been developed which is logarithmic.

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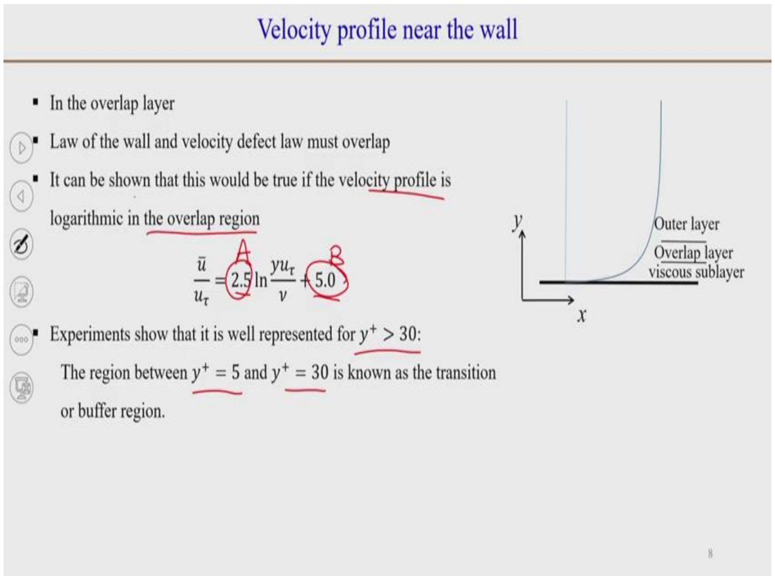

So, it can be shown that this profile or we can have overlapping if the velocity profile in the overlap region is logarithmic. So, this is the logarithmic profile and you see the two constants here. So, first one somebody said or somebody observed that this matching is possible only when you have let us say a constant A + B. So, A, y +=A ln or u +=A ln y + + some constant B.

And then people found or said that the constants from the experimental data comes out to be 2.5 and 5 respectively. So, that is called log law near the wall and it appears to be valid in y star is greater than 30, there is the reason between y star=5 or 7 where the up to which the viscous sub-layer is there, where the linear profile is valid and between y + 5 and 30, So, this is called transition or buffer layer where the profile changes.

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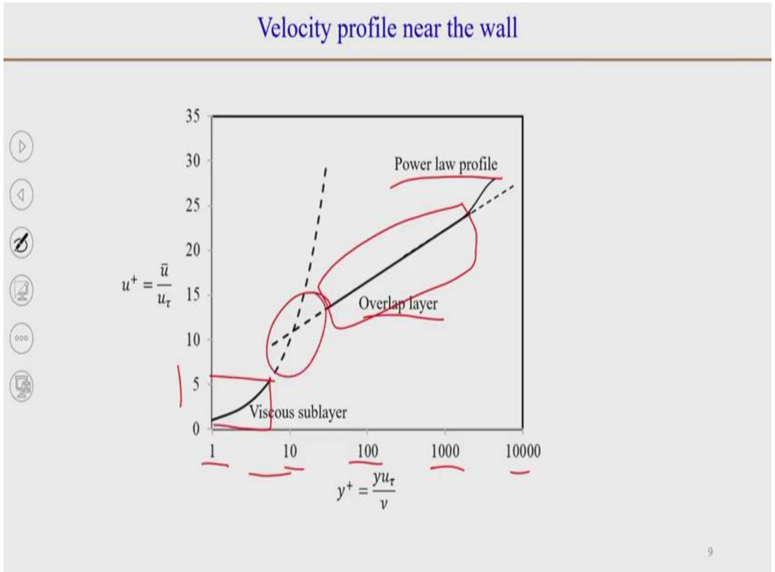

Now if you plot it, So, this is your viscous sub-layer where you have, notice that this is a logarithmic profile, So, the linear profile that is why it does not. It semi-log plot So, x-axis is on long scale; and y-axis is on the linear scale.

So, you have viscous sub-layer in this region and this is the profile, will look like line on this plot So, this is overlap where you have log law valid and then you have outer layer and when we talk about pipe flow then power law profile can come even in the, as we will see that in the turbulent boundary layer also one can use a similar profile. So, this is where power-law profile is valid. So, you will have matching coming into these places, but apart from these regions you could use these laws.

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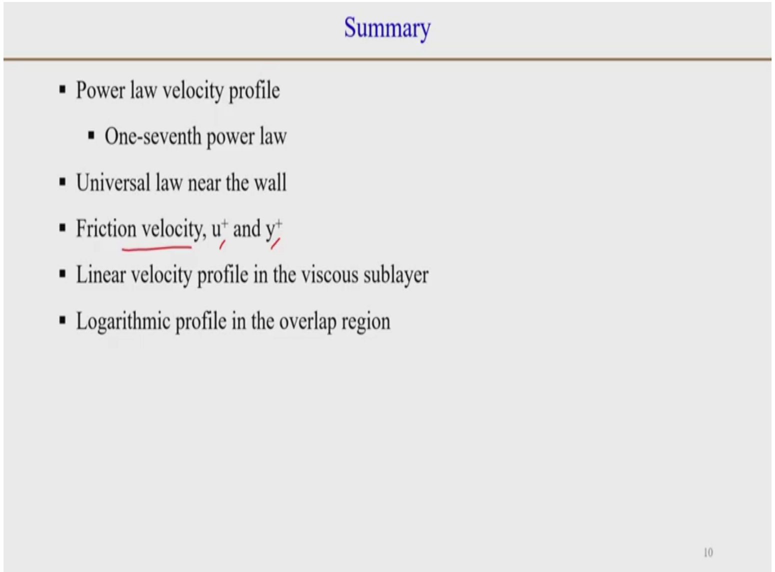

So, what we have seen today is we looked at the velocity profiles in a turbulence pipe flow and we found that a power law velocity profile which is obtained empirically can approximate the velocity profile in turbulent flows. But its validity is limited into the regions away from the wall, and that profile is when that exponent is 7, we can also call it one-seventh power law for turbulent velocity profile.

Then we saw that this profile is though not valid in the near wall region, but we could derive from the scaling arguments from the dimensional analysis, the velocity profile in the near wall region, which is, we saw that there are some important scales there, which we called say, friction velocity which is^{2} root of wall shear stress / the density of the fluid and u + which is ration of u and friction velocity and y + which is basically a Reynolds number where the length scale is the distance from the wall and the velocity scale is u tau or friction velocity

And then, we saw that there are three layers, viscous sub-layer, outer layer and overlap layer and viscous sub-layer in which the velocity profile is linear. So, linear velocity profile means that u +=y + and then you have logarithmic velocity profile in the overlap layer region.

So, we will stop here, thank you.