Fundamental of Fluid Mechanics for Chemical and Biomedical Engineers

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Boundary Layers: Momentum Integral Equation

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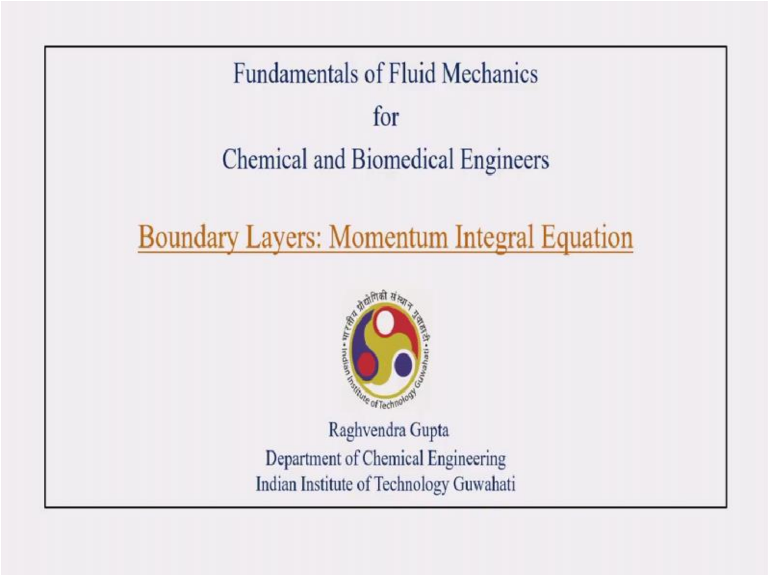

In the previous lecture, we introduced the concept of boundary layer. The boundary layer is the region near the wall in which viscous effects are important. And because of which we see large gradients in velocity in the boundary layer region. Even if the Reynolds number of the flow is very high we consider a small layer near the wall where viscous effects are important or viscous effects and inertial effects are of similar magnitude.

So, using boundary layer in the previous class we saw that some of the physical phenomena such as the presence of non-zero drag on the summer surfaces or on the bodies could not be explained by assuming potential flow for the cases where the Reynolds number is high or the approximation of inviscid potential flow is closely valid or its approximately valid there.

So, we introduced the concept of boundary layer and then we defined the thickness of this boundary layer region in different ways boundary layer thickness or what we call disturbance thickness, momentum thickness and displacement thickness. Then, we looked at the Navier-Stokes equation for an incompressible steady two-dimensional flow over a boundary layer. And simplified the Navier-Stokes equation so that we can find or the solution of these equations to find the velocity and flow profile in the boundary layer.

Now, what we are going to discuss today is the momentum integral equation in the boundary layer. So, we will consider the boundary layer region and apply macroscopic mass and momentum balances so that we get a equation.

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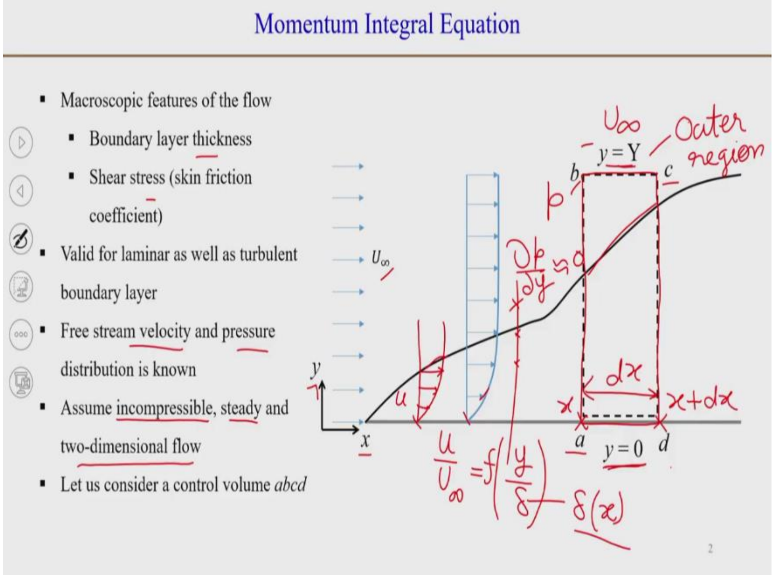

And that equation can be used to find the macroscopic properties. So, the equations that we wrote in the previous class when we solve these equations you will be able to find the differential or what would be that that would be differential analysis, because we will be able to find the velocity at every point.

However, the mathematics for the solution of such equations its slightly involved. So, we will not do that in this course, but we will what we will do is do the macroscopic balance and that of course will give us macroscopic properties. For example, we will be able to find out when we solve this equation, we will be able to find out thickness of the boundary layer and the shear stress.

So, we can find the microscopic features for example boundary layer thickness and shear stress or skin friction coefficient. Now, this will require that we need the velocity profile in the flow. So, we need to either we will need to make an assumption about the velocity profile or we will need to know the velocity profile in the fluid in order to be able to apply this equation.

Because when we will do this analysis we will not make any assumptions about if the flow is laminar or the flow is turbulent. So, this analysis will be valid for laminar as well as turbulent flow. Now, when we do differential analysis the equations that we use can be valid for both, but we will need to be able to resolve all the features for example when we talk about velocity profile 1/7th law of velocity profile for turbulent flow that is not valid near in the near wall region.

So, the differential analysis that we could do accurately is for laminar boundary layers but this analysis will be valid for laminar as well as turbulent boundary layers. And this will give us approximate values of boundary layer thickness and shear stress and from shear stress integrated over the area we can find the drag, the drag force on any surface.

Now, what we will need is as an input, we will also need free stream velocity and pressure in the free stream. So, when we solve the flow in this outer region, where the potential flow is valid then we will be able to obtain what is U_∞ and what is pressure distribution in this region.

Now, remember that we said that $\partial p/\partial y$ is approximately 0. So, the pressure in the outer region we can use the pressure in the outer region as well as in the boundary layer at any cross section as x. Now, there is another important feature that the velocity profiles in the boundary layer they are self similar.

So, if we write down the velocity profile in the form u / U_∞ where U_∞ is the free stream velocity and u will vary as a function of y. As you can see from here that the velocity at the wall is 0 and then it reaches to U_∞ at the boundary layer. So this profile at any x that will be self similar. So, that will be function of y or in non-dimensional form y/δ where δ is the film thickness.

Now, u / U_∞ will be same at all x, but u will be a function of x you can see from here this δ is not constant δ is a function of x. So, the boundary layer profile that we talk about is self similar it is called that u / U_∞ is same at every x in terms of function of y / δ.

So to[*]t this analysis, we will again consider the flow to be steady. That means it is time independent, it is not transient, flow is incompressible and we will consider a two-dimensional flow in x and y coordinates. So, we will consider the Cartesian coordinate system. And we will consider a small volume.

So, we will consider a control volume a b c d here as shown by the dotted line. So, this control volume is encompasses the fluid which is next to the wall so it*ts from y = 0 and goes on up to a region y = Y. So, we consider here that the b c surface is parallel to this wall.

And a b is normal to the free stream direction of velocity and then you have c d. The distance between a and d is dx, so we can say that the coordinate at a is x and coordinate at d the x direction coordinate is x + dx at point d. So, x and x + dx and the y = 0 to Y = Y.

Now, this is necessarily this need to be in the outer region, so that the control volume covers the boundary layer or it is beyond the boundary layer surface. The velocity of course in this region will be U_∞. Now, this U_∞ can be a function of x or not it will depend on what case we are looking at.

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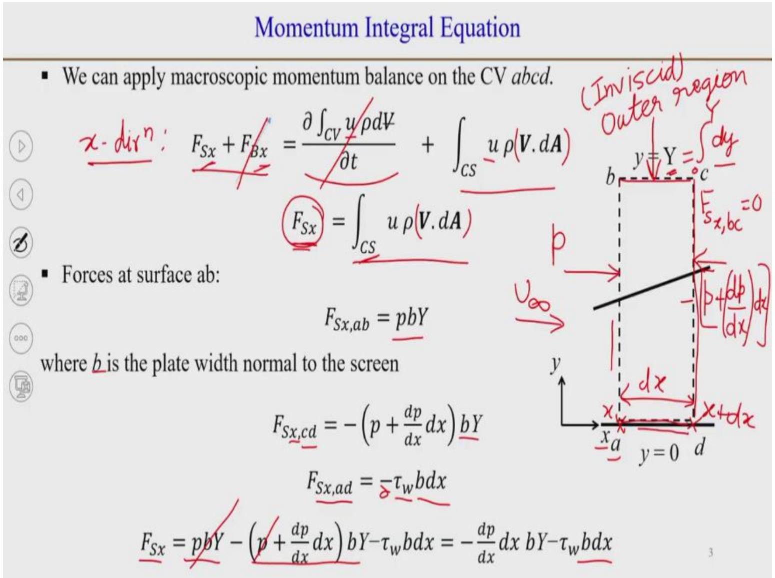

So, we will take U_∞ as a function of x. So, now, what we will do we will write down the macroscopic momentum balance on this control volume. So, let us remind ourselves the Reynolds transport theorem where we have a forces of surface force and body force and then ∂ / ∂ t of u ρ dV which is rate of change of momentum in the control volume and the momentum flux at the control surface.

So, for control surface we have here a b c d. So, we consider this force for x direction. So, the equation is for x direction, we have the subscripts as x and for velocity component we have u here. Remember, this V.dA is a scalar quantity. Because, the flow is steady so this term will be 0 and there is no body force present in this direction along the x direction.

So, we will consider that this x direction is horizontal direction. So, FBx = 0 or otherwise we can neglect the body forces. Now, so that will give us the momentum conservation equation will become that the sum of surface forces will on the control volume will be = integral u ρ V.dA over the control surface.

So, we will evaluate each of these terms one by one so first we will evaluate the forces on this control volume FSx. So, on the surface of this control volume we will have four surfaces a b b c c d and dA so we will consider one by one. Now, if we look at surface a b the force on this will be because of pressure along the x direction.

So, if we consider that pressure at x is p and it will be a compressive, so it will act along positive x direction. Now, let us also assume that the width normal to screen is b, so the area of this surface a b will be = b into a integral 0 to Y dy or we can write it simply as y, but we will later on come back to write it in this form.

So, the force on surface a b will be p b dy it may also have it will have a shear stress but the direction of the shear stress will be not be along the x direction, because the surface normal edge along the x direction. So, the x direction force on another surface c d that will be = now the force will act in this direction the normal force.

So, from Taylor series we can write the pressure here is p + dp / dx, we can write it as dp / dx because p is not a function of y, so we can write it as an normal derivative multiplied by dx. Remember, that this distance is dx and at a what we have coordinated x and at dx + dx.

So, a pressure on the force due to pressure on surface c d = -, it will be acting in the negative x direction so that is why there is a - sign here. So, - p + dp / dx into dx and this multiplied by the area of surface which is again b into Y. Now, in the x direction on these two surfaces a b and c d the forces will be because of normal forces.

Now, if we look at surface b c on the surface b c the pressure will act in the y direction or in the negative y direction, so we will not need to consider the pressure force on this surface. Because, what we are considering is FSx or force along the x direction. The other surface force is shear because of shear or viscous forces.

Now, this is in the outer region where the flow is inviscid. So, when the flow is inviscid, the shear stress is 0. So, the force on surface b c is 0 or we can say FSx, b c = 0. Now, the force surface which is near the wall so on this again, force due to pressure will be normal to it so it will be along the y direction. So, what we will have is only the viscous force.

So, let us say the wall shear stress which is a function of x so we can say just τ w will be multiplied by the area of this wall. So, the depth normal to the screen is b and this distance is dx so τ w b dx. So, we have written down forces on all four surfaces where force on b c surface is 0.

Now, we can combine all these and find the final expression for FSx. So, when we add them together, the first on a b surface p into b into Y this is the force on surface c d so p + dp / dx into dx whole multiplied by bY and - τ w b dx. Remember, this - sign is because the shear stress on the fluid will be acting in the opposite direction of the flow.

The flow happens along the positive x direction. So, pbY pbY will cancel out because one is positive another one is negative and what we will end up with two terms - dp / dx into dx bdY - τ wbdx. So, we have obtained FSx and now we will have go ahead and find out the fluxes on all the four surfaces of the control surface.

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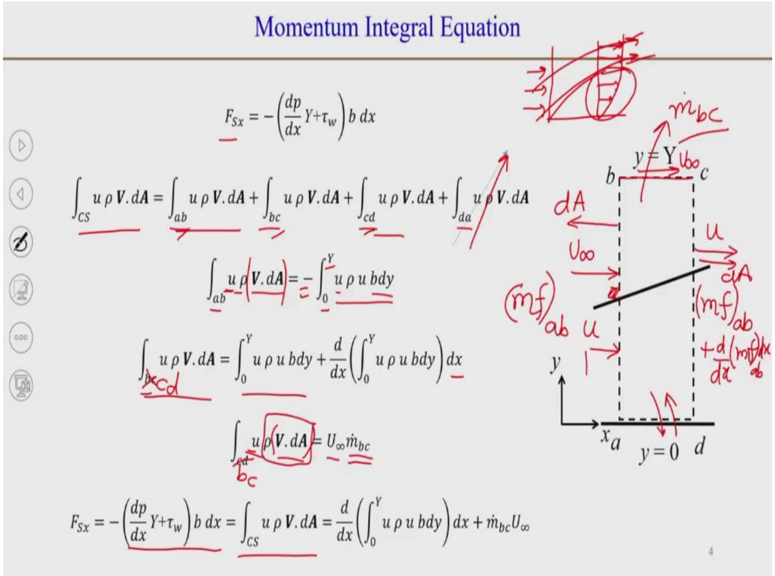

So, that is the equation that we obtained for forces. We have taken b dx out of the bracket. Now, we will write down the fluxes on the surfaces, so we can write down individually for all the four surfaces a b, b c, c d and d a.

Now, we can see that on surface dA the there is no flow from this surface so this flow is going to be 0. And what we will have is let us[*]t with surface a b so on the surface ab. So on the surface ab, we will have u ρ V.dA.

Now, u is the velocity ρ into V.dA so on this surface the velocity is in this direction or if you look at u so we will use the term u here because this region has u as well as U_∞. So, we can say that u which is a function of y and V.dA the normal to this the area vector will be pointing as outward normal.

So, as a result, we will have a negative sign here and we can integrate it from 0 to Y. So, u ρ u bdy is the area dA of the surface if we take an element surface along the y direction. So, if we take small strip of thickness dy. So, b dy and integrate it from 0 to Y.

Now similarly, we can write the flow rate on the surface bc the same expression what we are going to write is because this is momentum flow. And momentum flow because of if we write that this is momentum flow at ab then at dc or cd we can write momentum flow at ab + d / dx of momentum flow at ab into dx.

We can write this from Taylor series so that is what we are going to do here. The momentum flow at ab is the sign will be changed because here the outward normal and the velocity u both point in the same direction. So, the sign will be positive, so the momentum flow will be 0 to Y u ρ u b dy.

So, that is first term mf at ab + d / dx of this so integral 0 to Y u ρ u b dy multiplied by dx. So that is the momentum flow at surface bc. Now, we will consider the third surface which is c d. Sorry, We have already considered the surface cd this should have been cd. Now, we consider the surface bc. Because, the flow here it is not necessarily a stream line, so flow which is going from here to here there is some change in the flow rate.

Because, when you consider say a boundary layer and it will be clear if you look at that at the leading edge the velocity is uniform but when you go inside the boundary layer or somewhere in the plate region then there is a velocity gradient.

So, the flow has to divert in this region. So, in the last class we also plotted the stream lines. The stream lines will be slightly diverted away from the region and the distance between them will be increasing. So, the mass flow rate from ab to bc in this region in the boundary layer will decrease. So you will have a net flow which is $\dot{m}$bc.

We can write down that what is the flow here through surface bc = u ρ V.dA. Now, ρ V.dA, V.dA is the volumetric flow rate and ρ V.dA is the mass flow rate from this surface. So, we can write this in terms of mass flow rate.

So, ρ V.dA is mass flow rate through the surface bc and u because this region is in the outer region. So, the velocity here will be U_∞, so $u = U_\infty$. Now, remember, these coordinates are in terms of the boundary layer, so if it is a flat plate then this is along the flat plate if it is a curved plate then x is along the curve plate and the y will be normal to the curve plate.

So, in any case, if the flow is not horizontal then it is along or it is the U_∞ is the velocity free stream velocity which is parallel to the plate direction. So, this is in any case it is U_∞, so this is U_∞ into $\dot{m}$bc. So, we have obtained the fluxes or the momentum flows from ab or at the surface ab, bc and cd. And at the surface dA there is no flow either coming and going out and there is no slip boundary condition there, so this term is 0.

Now, we can substitute these values in the momentum balance equation. So, the force on the right hand side and the momentum flux on the, force on the left hand side and momentum flux on the right hand side and we can substitute these values and try to simplify it.

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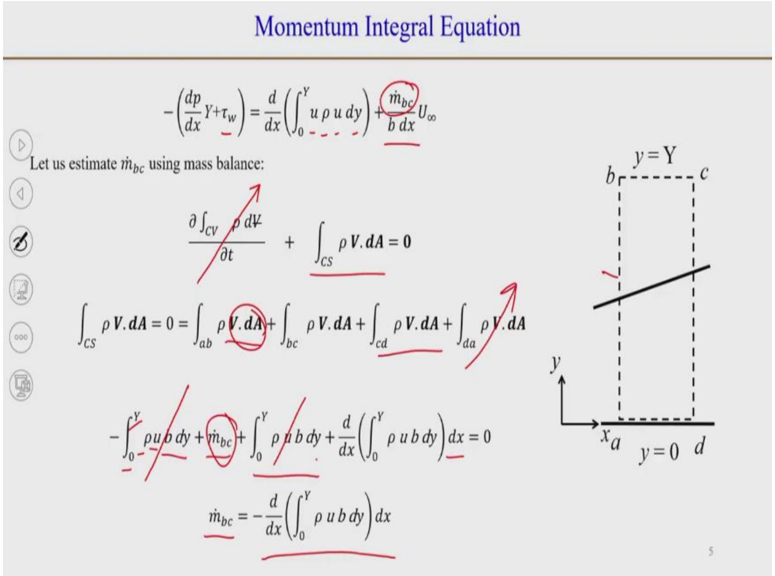

So, we see from here, so we can see that here we have a term b dx and in this also we have b dx. So, we can divide throughout by b dx because this term does not have b dx so this will be /b dx. So, when we write this and simplify, you will get - within bracket dp / dx into Y where Y is the distance between surfaces ad and bc + τ w which is wall shear stress = d / dx integral 0 to Y u ρ u dy + $\dot{m}$ bc / b dx into U_∞.

Now, what we need to find is $\dot{m}$ bc so that can be found using the mass balance through this control volume. So, we can write down the mass balance equation and in this mass balance equation because the flow is steady so this term is going to be 0. And we can consider the mass through all the surfaces here.

Now, when we do that we need to write the mass flow all the surfaces. So, at the surface a b which is this one we can write a mass flow is ρ into u V.dA which will be bdy and we can integrate it from 0 to Y. Similarly, on the surface cd the mass flux will be like we did for the momentum flux that the mass flux integral 0 to Y ρ u b dy + d / dx into a mass flow multiplied by dx.

So, integral 0 to Y ρ u b dy multiplied by dx. So, the mass flow through bc is let us say it is $\dot{m}$ bc and there is no mass flow through surface dA so that will be 0. Now, when we combine these terms these two terms will cancel out and what will we get is $\dot{m}$ bc = - d / dx integral 0 to Y ρ u b dy. So, we can substitute this for $\dot{m}$ bc in this equation here. We can substitute it here.

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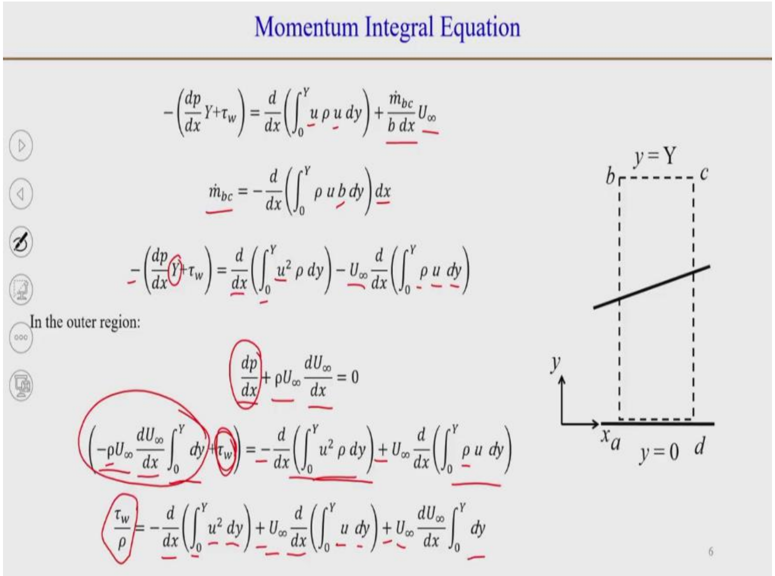

So, we have this equation which is a momentum balance equation and in this we can substitute the value of $\dot{m}$bc that we have just found out. So, when we substitute it we get the following equation here, we have written u into u so this has become u^2, so we get the equation - dp / dx into Y + τ w = d / dx integral 0 to Y u^2 ρ dy - U_∞ comes here and dx will cancel out and this b will also be cancelled out so we will have inside the integral d / dx of integral 0 to Y ρ u dy.

Now, we can write down the Bernoulli's theorem in the differential form of from potential flow we can write down that in the outer region dp / dx which is the pressure gradient along the x direction + ρ U_∞ d / dx of U_∞ = 0. Of course, in this the gravity is not considered so we will consider this to be horizontal surface.

Now, from this, we can substitute the value of dp / dx in terms of - ρ U_∞ dU_∞ / dx. So, we will do that and when we do that we can write this down equation as dp / dx, we can replace it by - ρ U_∞ dU_∞ / dx and in place of Y we will write now it as 0 to Y dy y.

Because, we have everywhere those integrals except in τ w, everywhere we have integrals from 0 to Y. So, we will write this down in terms of integral 0 to Y and see if we can simplify or combine these terms to get something which is familiar to us. There was this - sign so we have changed the sign on the other side.

So, that is why we have this sign here negative and this sign has become positive. Everything else remains same. So now, we can divide by ρ and rearrange it, so we can write this equation in terms of ρ, so when we divide this term by ρ this will become τ w / ρ. And the first term on the right hand side will be - d / dx.

This row will cancel out so you will have integral 0 to Y u^2 dy + the second term will be U_∞ d / dx integral 0 to Y ρ will cancel out because we have divided the equation throughout by ρ. So, integral 0 to Y u dy + this term will come on right hand side. So, ρ will be divided and it will become + U_∞ d / dx of U_∞ integral 0 to Y dy.

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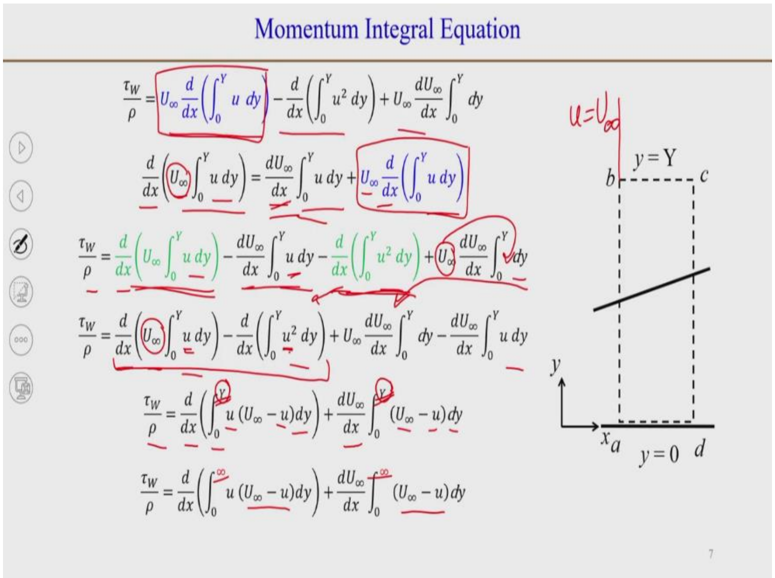

So, let us look at this equation if we can do something here. So, this term the first term we can write down in a different form. So, if we take d / dx or if we differentiate U_∞ into integral 0 to Y u dy, so U_∞ is also in general it can be a function of x. So, when we differentiate it by product, we will get a differentiation of first function which is U_∞. We can say this is first function and this is second function.

So, dU_∞ / dx into integral 0 to Y u dy and then we will write U_∞ into differentiation of second function. So, d / dx of integral 0 to Y u dy. Now, we get this term it is same as here so we can replace this term by d / dx of U_∞ into integral 0 to Y u dy - d u / dU_∞ / dx into integral 0 to Y u dy.

So, we can replace this term and see τ w upon ρ = when we replace this term by the difference of these two terms. So, first term is d / dx of U_∞ integral 0 to Y u dy - dU_∞ / dx integral 0 to Y u dy and then, the remaining two terms here. So, which will be - d / dx integral 0 to Y u^2 dy + U_∞ dU_∞ / dx integral 0 to Y dy.

Now, we can also consider this, we can bring in because this is U_∞ is constant with respect to y, so we can even bring it inside the integral because it does not make a difference. Now, if we look at the two terms which are in green, so both of them have differential d / dx.

And when we combine these two terms we will have U_∞ into u - u^2 dy. And if you remember the definition of displacement thickness, which was similar to where we had U_∞ into u - U_∞, so we can combine these two terms and we can write them together.

So, d / dx U_∞ integral 0 to Y u dy and the other term d / dx of integral 0 to Y u^2 dy. Similarly, we can write down this dU_∞ / dx into U_∞. So, this term comes here - dU_∞ / dx integral 0 to Y u, dy the remaining term.

So, when we combine this, the first two terms it will become τ w / ρ = d / dx of integral 0 to Y u. Because, you have u^2 here, so one u can come outside the bracket and within bracket you will have U_∞ - the remaining u into dy. And the other term will be dU_∞ / dx integral 0 to Y U_∞ - u dy.

Now, we can we have limit Y here but because when we go above Y u = U_∞. So, both of these term have U_∞ - u into them. So, the value of this integral from Y to ∞ will be 0 because U_∞ - U_∞ will become 0. So, we can as well replace this limit with ∞. So, we have change this limit to ∞.

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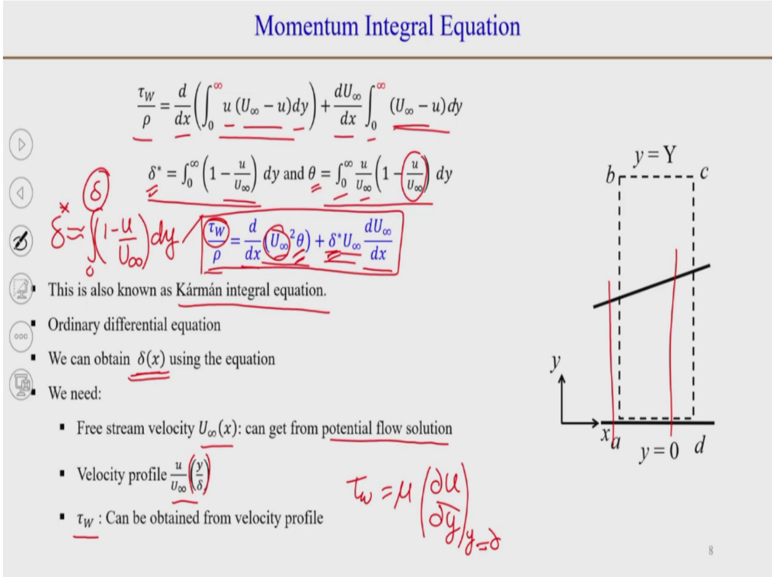

Now, we have τ w / ρ = dy dx integral 0 to∞ of U, U∞ - u dy + dU∞ / dx integral 0 to∞ U∞ - u into dy. Now, we can write down the definitions of displacement thickness and momentum thickness and compare these two.

So, when we see momentum thickness integral 0 to∞ u / U∞ into 1 - U∞. So, if you write down or simplify it, it will be u / U∞ into U∞ - u /U∞. So, this term is basically θ into U_∞^2. Similarly, the term here U∞ - u dy integral 0 to∞ is basically δ^* which is displacement thickness into U∞.

So, we will write this, so we will get τ w / ρ = d / dx of U_∞^2 θ + δ^* U∞ dU∞ / dx. And this equation is what we call the final form of momentum integral equation or after Karman who derived this equation it is also known as Karman integral equation.

Karman was a PhD student of Prandtl who derived boundary layer. Now, this is an ordinary differential equation and this equation is a function of δ boundary layer thickness which comes from θ and δ^*. We have written this in terms of 0 to∞, but if you remember we said that this can be approximated δ^* integral 0 to δ 1 - u / U∞ dy.

And similarly, for θ the limit will be 0 to δ because beyond δ, δ to∞ u = U∞ and this term becomes close to 0. So, this has implicitly δ into it there is τ w, there is U∞ and both δ^* and U∞ are function of u / U∞, which is basically the profile of velocity in the boundary layer.

So, to solve this equation, let us say, if we want to find out the variation of boundary layer thickness along the x direction we can use this or equation or for a known δ x we can find out τ w. Now, if we want to find out δ x we need to know first U_∞ which is easier. Because, U_∞ is the velocity in the outer region so we can use potential flow solution to find what is the variation of U_∞ with x.

Now, we also need the velocity profile, so as we discussed earlier that u / U_∞ is self similar solution. So, it the value of u / U_∞ in terms of y / δ will be same across all the x. So, we need the velocity profile u / U_∞ as a function of y / δ. And we also need τ w, but when we know the velocity profile we can find τ w because τ w is nothing but μ ∂ u / ∂ y at y = 0.

So, once we know the velocity profile we can also find out τ w. So, we can make an assumption about the velocity profile if that see that how we can find the profile of boundary layer.

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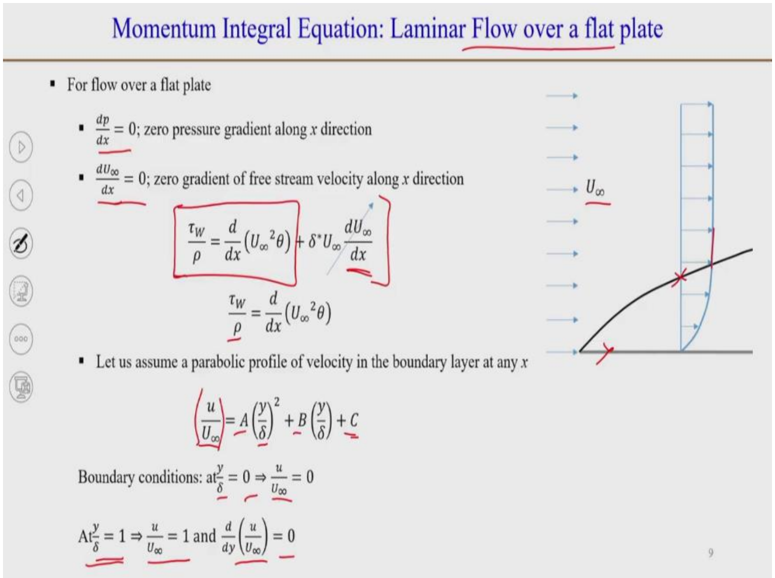

So, what we are going to consider now an example and see how we can use the momentum integral equation to solve or to find out the different parameters of boundary layer. For example, the boundary layer thickness.

So, we will consider a case of laminar flow over a flat plate. So, when we have a flat plate the velocity is coming and the flow is uniform, so U_∞ is constant across the y. And we will have U_∞ not a function of x and pressure gradient there will be 0. So, we will have a 0 pressure gradient along the x direction as well as 0 gradient in the free stream velocity along the x direction.

That is valid for a flow over a flat plate. So, it might be a laminar or a turbulent, but what we are going to consider is laminar flow over a flat plate, because we need to make an assumption about the velocity profile.

So, let us assume that the velocity profile is parabolic here. But, before we do that, we can write down this momentum integral equation and see that the second term here which has dU_∞ / dx is 0, because we have considered the flow over a flat plate.

So, in many books you might see only this form of momentum integral equation because in that they consider the flow over a flat plate so this term becomes 0. So, for a flat plate you will have a simplified form of momentum integral which is τ w / ρ = d / dx of U_∞^2 θ.

Now, as I said, we can assume that the velocity profile is parabolic so or we can assume a quadratic equation here where we have three constants A B and C. Now, we can found find these constants that at y = 0 which is at the wall u = 0 we have written this in non-dimensional form.

So, u / U_∞ = A y / δ^2 + B y / δ + C. We have three constants here A B and C. So, we have first boundary condition, no slip boundary condition on the wall. The next boundary condition is at y / δ = 1, which is the surface at the boundary.

At the boundary layer at y / δ = 1 you will have u = U_∞ or u / U_∞ = 1 and the gradient of velocity here will be 0. So, we can also say that d / dy of u / U_∞ = 0 at y / δ = 1.

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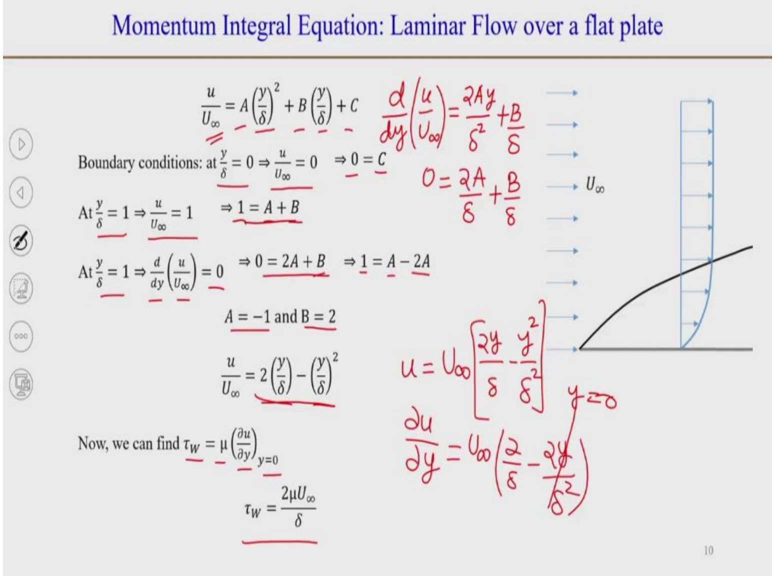

So, when we do that so from this first boundary condition that y / δ = 0 at that location u / U_∞ is 0. So, when we substitute that we will get u / U_∞ which is 0 and this term will be 0 because y = 0 the second term will also be 0 and we will have C = 0.

Now, the next boundary condition that at y / δ = 1 which is at the boundary layer interface u / U_∞ = 1. So, we can write this, u / U_∞ = 1 y / δ = 1, so that gives us A + B = 1.

The next is at y / δ = 1. The differential of u /$_\infty$ = 0, so we can differentiate this with respect to y and that will give us 2 into y / del or A / δ^2. So, we can write this d / dy of u / U_∞ that will be 2 A y / δ^2 + B / δ.

So, when we have y = δ, then we will get from here this = 0. So, this becomes 0 and 2A/δ + B/δ = 0 or we can simply write 2 A + B = 0.

Now, when we simplify, from this we can say that B = - 2A and we can substitute in this equation. So, we will get 1 = A - 2A or - A = 1 or A = - 1 and B = - 2, so B = 2.

So, that will give us our equation will become when we substitute the values of A B C in this, the equation becomes u / U_∞ = 2 y / δ - y / δ^2. So, if we assumed a parabolic or quadratic velocity profile, using the boundary conditions we get this velocity profile.

Now, from this we can also find out the shear stress on the wall which is μ into ∂ u / ∂ y at y = 0. So, when we do that ∂ u / ∂ y will be, so we can write this u = U_∞ into 2 y / δ - y^2 / δ^2 and ∂ u / ∂ y = U_∞ into 2 / δ - 2 y / δ^2.

Remember, that δ and δ^2 and U_∞ are not function of y, so we can treat them as constants here. So, when we substitute that y = 0 then this will be 0 when y = 0 this term will be 0. So, you will have a μ multiplied by ∂ u / ∂ y at y = 0 which is 2 U_∞ / δ. So, you will have τ w = 2 μ U_∞ / δ.

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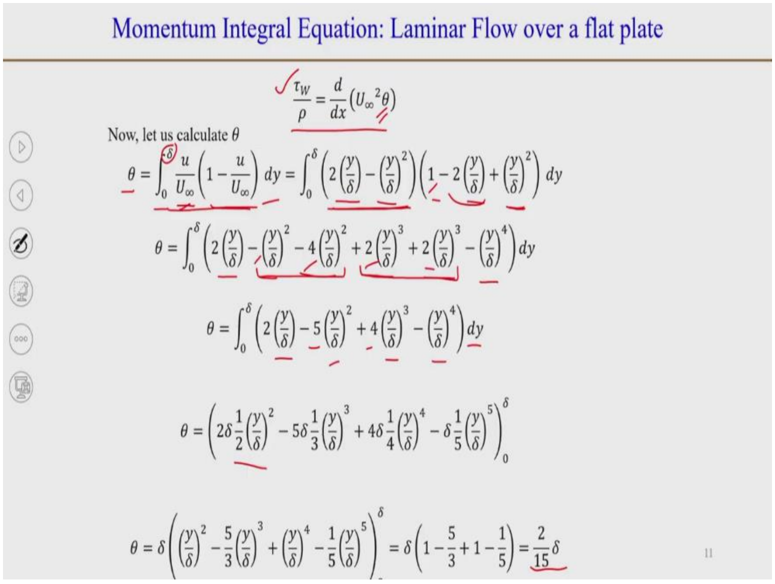

Now, we can substitute the value of τ w there. In the momentum integral equation, so we have the value of τ w and what we need is the value of θ, U_∞ is a constant here in this case. And so we can write down the expression for θ which is integral 0 to δ u / U_∞ into 1 - u y_∞ dy.

Now, just to remind again, that we use the limit δ or$_\infty$ interchangeably, because when the fluid is or when we are in the outer region u = U_∞ so this term 1 - u / U_∞ is 0. So, we can use either limit δ or$_\infty$ here.

So, that is the definition and we can substitute the values of u / U_∞ in this equation so u / U_∞ is 2 y / δ - y / δ^2. And 1 - u / U_∞ will be 1 - 2 y / δ + y^2 / δ^2 into dy. So, we need to simplify it. We can multiply the terms.

So we can multiply this with 1 first, so when you multiply 1 with so you get 2y / δ - y^2 / δ^2. Then, you can multiply 2y / δ with the negative sign in the first bracket, so that will give you - 4y / δ^2 + 2y / δ^3.

And finally, we can multiply the last term or third term in this second bracket with the terms in the first bracket. So, we will get + 2y y / δ^3 and the last term will be - y / δ^4. So, we can combine the terms here we have 2y / δ^2 and 2 terms we have with y / δ^3.

So, that will give us integral 0 to δ 2 y / δ - these 2 term when you combine this will become - 5y / δ^2 these 2 terms when you combine this will become 4y / δ^3 - y / δ^4. So, that is your θ and when you integrate it so you will get the after the integration, so you will get you are integrating with respect to y.

So, 2 δ into 1 / 2 y / δ^2 and so on. And when you simplify this further and doing a bit of algebra you will get value of θ = 2 / 15 δ. So, we have θ in terms of 2 / 15 δ. So remember, we talked about that you can use the momentum integral equation to find out the boundary layer thickness here.

So, we can replace now θ in terms of boundary layer thickness

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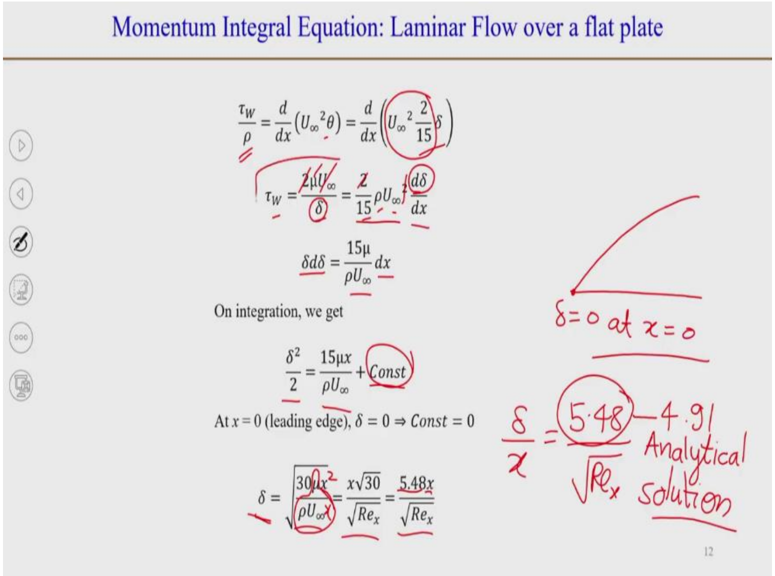

So, we will do that that $\theta = 2 / 15\ \delta$ and we also have τ w which is $2\ \mu U_\infty / \delta$. So, we will have this equation simplified because U_∞^2 into 2 / 15 is constant. So, we can bring this out here ρ has come on the other side so we can write τ w which $= 2 / 15\ \rho\ U_\infty^2\ d\ \delta / dx$.

Now, remember, this δ is a function of x. So, we can combine the terms or separate the variables each side and can try to integrate it. So, we can combine this term δ into d δ, 1 side and this 2 and 2 will cancel out and we will have a 1 U_∞ also cancelling out so 15 into μ this term and this term we have taken care and divided by $\rho\ U_\infty$ into dx.

So, that is our equation, we can integrate it so that will be δ^2 by 2 15 $\mu / \rho\ U_\infty$ into x + a constant. Now, to get this constant when we have a flow or a flat plate at x = 0 at the leading edge the velocity or the boundary layer thickness will be 0, so $\delta = 0$ at $x = 0$.

So when we substitute, we will get this constant to be 0. And we can rearrange this, so we will get $\delta^2 = 30\ \mu\ x / \rho\ U_\infty$. Now, we have $\rho\ U_\infty$ here so we can multiply x at the denominator and numerator, so we will get^2 $\sqrt{30}$ multiplied by $\sqrt{x^2}$ which is x and this term here is basically 1/Rex.

So, we get δ = x into $\sqrt{30}$ /$\sqrt{Rex}$ or $\sqrt{30}$ is 5.48. So, 5.48 x in /$\sqrt{Rex}$. So, δ / x = 5.48 /$\sqrt{Rex}$. And we discussed in the previous class that this number is comes out to be 4.91 from an exact analytical solution.

So, we are not too far off and the assumption that we made here is that the velocity profile in the boundary layer is parabolic. So, if you can make an educated guess about the profile in the boundary layer you can use this methodology to find what is the boundary layer thickness as a function of x?

Now, we have found the boundary layer thickness, we can also use this to find τ w.

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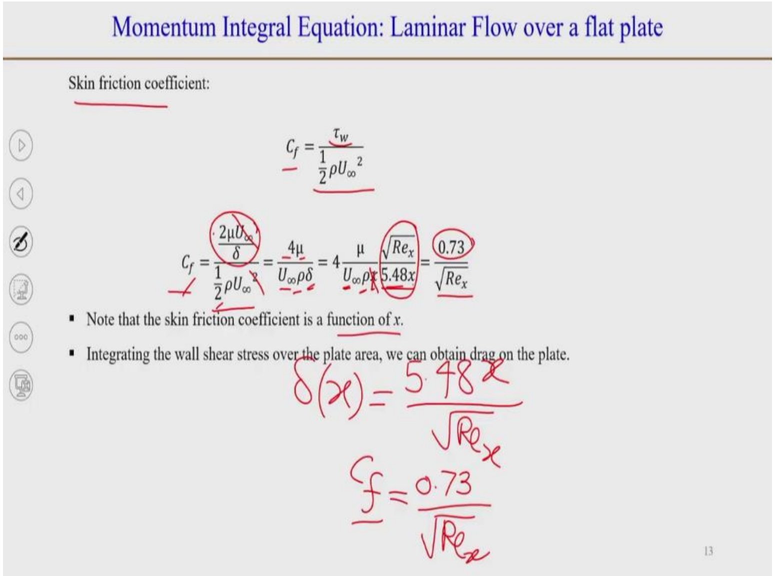

So, we will do that to find τ w or we define a non-dimensional parameter which is called skin friction coefficient. So, that is defined as Cf or it is represented by the symbol Cf and it is defined as τ w non-dimensionalized by dynamic pressure in the free stream 1/2 ρ U_∞^2. So, that is called as skin friction coefficient.

So, skin friction coefficient will be τ w which is 2 μ U_∞ / δ /1/2 ρ U_∞^2 1 U_∞ will cancel and you will have 2 into 2 multiplied, so you will have 4 μ /U_∞ ρ into δ.

Now, δ is a function of x, so Cf is also going to be a function of x. We can rearrange this a bit and so we can write that δ is 5.48 x / √Rex, so that is our 1/δ and this multiplied by μ into U_∞ into ρ and this x probably is extra here. So, we will have this so we can remove 1 of those x's.

So, U_∞ ρ into x so you will have a 4 divided 5 / 4 5.48 that will give you 0.73 and √Rex divided by this number again will be Rex. So, μ /U_∞ ρ into x that will be 0.73 /√Rex so that is the value of Cf. So, we get this skin friction coefficient as a function of x here.

So, we have found the value of δ as a function of x which is 5.48 x /√Rex. We also calculated Cf here, which is 0.73 /√of Rex. So, when we integrate this Cf x or τ w over the flat plate then we can obtain the total drag on the flat plate.

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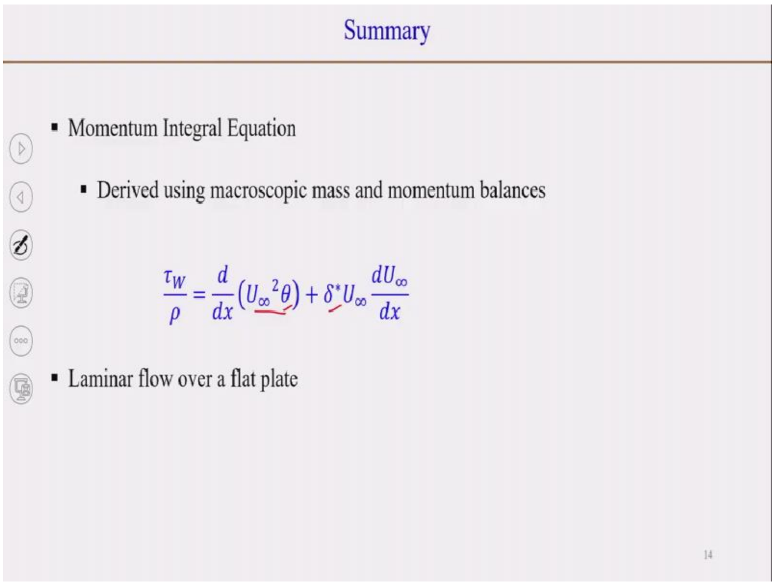

So, in summary, what we have done today is we did a lot of substitution integration differentiation, but what we got or what we did today is derived a momentum integral equation. So, we considered a steady two-dimensional incompressible boundary layer in Cartesian coordinate.

We applied the momentum balance equation and when we applied momentum balance equation we needed a term $\dot{m}$ bc, which was the mass flow rate that is going out of this control volume or $\dot{m}$bc. Then we use mass balance equation to find that mass balance.

And obtained equation for integral of momentum balance and that was the equation, which was rearranged so that we get in this nice form which is a function of θ momentum thickness and δ^* displacement thickness. So, this also tells us the importance of these 2 thicknesses θ and δ^*.

Probably, the definitions comes because this equation was derived and these equations the ones we saw the mathematical form of it and the physical significance of these terms. Then they were defined the way they are defined now a days.

And then, we consider this equation to solve the laminar flow over a flat plate. We needed of course, U_∞ we needed θ and we needed τ w to find δ. So, θ or because this term will become 0 as over a flat plate the pressure gradient is 0 and gradient of free stream of dU_∞ / dx is also 0. So, we needed τ w and we needed θ.

For θ we needed the velocity profile, so we assumed a velocity profile remember, this was an assumption it is not necessarily comes on from a physical basis. We assumed that it is a parabolic velocity profile and then, we used the boundary condition to obtain the constants in that.

Once we obtained the constants, so we could get a velocity profile and from this velocity profile, we could also calculate τ w, which is μ multiplied by du / dy at y = 0. And using the velocity profile we also could obtain θ in terms of both τ w and θ in terms of δ.

Once we obtained that, we substituted and obtained a differential equation for δ and integrated it and obtained the value of δ. From that, we also calculated a skin friction coefficient, which is basically non-dimensional form of shear stress τ w / 1/2 ρ U_∞^2.

So, we will stop here thank you.