Fundamental of Fluid Mechanics for Chemical and Biomedical Engineers
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Lecture 31
Inviscid Flow: Equation of Motion in Streamline Coordinates

Hello, so in today's lecture we are going to talk about Inviscid Flow. We derived Navier-Stokes equations where the assumptions was that flow is incompressible, flow is viscous and follows Newton's law of viscosity; viscosity is a constant. Then we looked at in the previous classes about the Stokes regime, where we assumed that the Reynolds number is very small. So that means the flow is what we call Creeping Flow, the velocity is very small and the viscous forces are dominant.

Now at the other extreme, so at one extreme we had Reynolds number tending to zero and the flow velocity is very small, viscous forces are dominant. So, at the other end of the spectrum or other end of Reynolds number values we have Reynolds number tending to infinity, so which means that the viscous forces are negligible and inertial forces are what are dominant. So, if we do that, then in the Navier-Stokes equation, we can neglect the viscous term.

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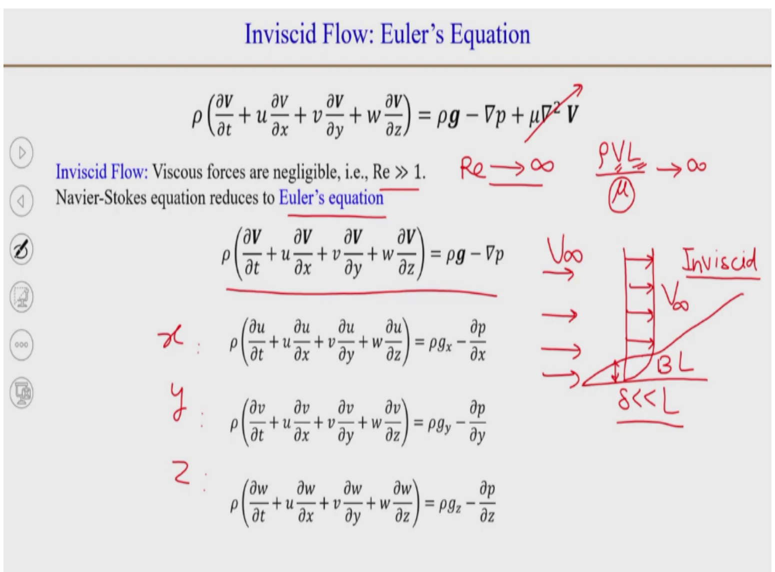

So, if we look into here, this is our momentum conservation equation or Navier-Stokes equation for a viscous Newtonian and incompressible fluid. So, if we say that Reynolds number is very large Re tending to infinity. And again, we can say that Re $=\rho VL/\mu$. So, if this is tending to infinity then either L is very large or generally what you will have the V is very high or μ is low, for example in the case of air.

So, in such cases when the Reynolds number is large, we can neglect the viscous term and what we will end up with the Navier-Stokes equation - viscous term and this is called Euler's equation. You can write this in x, y, and z coordinate or you can also write these equations in r θ z coordinates. The equation has become simpler in the sense that there are no viscous terms there.

But remember that we saw that near the wall viscous forces are important because that is where the boundary layer exists. So even if we have flow to be inviscid, if there is, the flow is happening near a wall, then near the wall viscous forces are dominant. We struggle to find the solution of Navier-Stokes equation because of the nonlinear term there, but as we will see later on that it is easier to find the solution of inviscid flow.

Now what people had done, that they divided the flow regime into two parts, so the inviscid regime, so if you have say flow over a flat plate then there is a boundary layer here and there is free stream flow. So, the flow comes with a velocity say V_∞ and the velocity profile was something like this.

So, you could assume the flow to be inviscid because the effect of viscosity is negligible in this region and you could assume the viscous flow in boundary layer and solve the boundary layer equations with some assumptions in terms of that the thickness of boundary layer is very small than the than the longitudinal direction or the direction along the plate. So, we could have two solutions there. So, in that sense it is important to understand inviscid flow.

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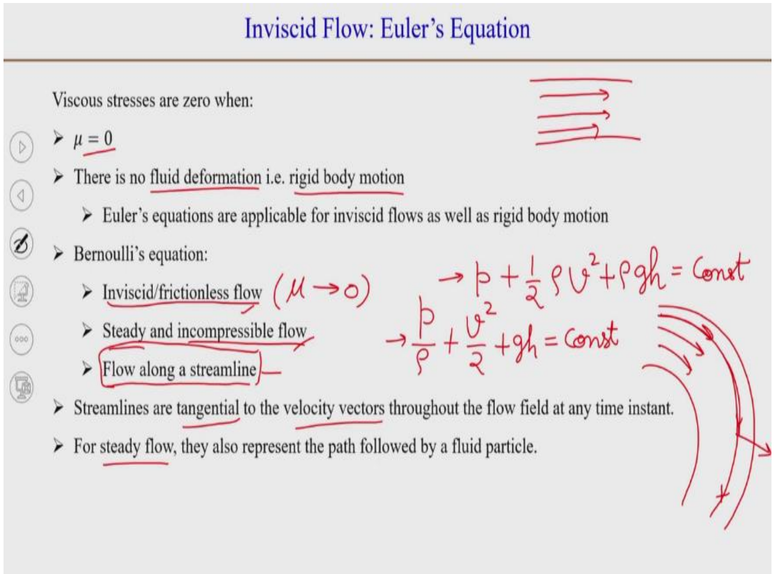

So, when viscous stresses are zero, we can also say that μ is 0 and remember how we defined fluid that we said that fluids are those that when a shear stress is applied on them, they have a continuous deformation happening in them. Now if we say that viscous stresses, which are basically which contributes to shear stress, so if there are no viscous stresses present then there is no, there is no shearing in the fluid.

So that means the viscous, or the Euler's equations that we just saw where the viscous term is 0, then they can also represent rigid body motions. So that means the Euler's equation can also be used to solve for rigid body motion. Now we, all of us are familiar with Bernoulli's equation. We have read at some point in our school the Bernoulli's equations, sometimes we studied in terms of that the total energy of the fluid is constant $p + 1/2\ \rho\ V^2 + \rho\ gh = \text{constant}$.

So that means that the energy in terms of pressure plus kinetic energy and the gravitational energy or the potential energy, sum of all this is constant. And the assumptions, if you remember, the assumptions there were the first assumption was that fluid is inviscid or frictionless. So that means the viscosity is assumed to be 0. So, the assumptions that we had there that the fluid is inviscid or we sometimes also used to call this fluid to be ideal, so there is no viscosity in this fluid and the flow is steady and flow is incompressible.

So, we had the assumption of inviscid, steady and incompressible flow and then very important point here was that the flow happens along a streamline. So, we can use the Bernoulli's equation

$p + 1/2\ \rho\ V^2 + \rho\ gh$ =constant only when the flow is happening along a streamline. So, if we have a streamline, we could use Bernoulli's equation between point 1 and point 2.

Now in some places you might also see this equation in terms of $p/\rho + V^2/2 + gh$ =constant because the flow is incompressible, so you could divide it by density or the other representation you might see in terms of that the unit is in terms of meter, so $p/\rho\ g + V^2/2g + h$ =constant. So, when gravity is constant, so this term again will become a constant, so you could use either of the forms of Bernoulli's equation.

Now just to remind ourselves, what is the streamline? So, streamlines are the lines which are tangential to the velocity vectors. So, if you take a streamline and plot a tangent at a particular point then that tangent represents the velocity vector at that point or that in time instant. So, and we saw earlier that if the flow is steady then streamlines, path lines and streaks lines, they are all same and they represent the path that a fluid particle follows.

So, until now we have derived the equation of motion or Navier-Stokes equation for in a Cartesian coordinate system and we also looked at the Navier-Stokes equation in the cylindrical coordinate system. Now we will just see the equation of motion in the streamline coordinates. So sometimes streamlines, for example, if you have flow in a pipe then streamlines are straight, but if you have a say curved duct or curved pipe, then the flow will be following the streamlines will be along this. So, streamlines will be curved.

And in some cases, it is beneficial that you can write down your equation of motion along the streamline coordinates. So, you can write down the coordinates which are following the streamline and normal to it. Now because we just saw that we can apply the Bernoulli's equation when the flow happens along a streamline. So, we can, if we, because the other assumptions are the flow is inviscid, flow is incompressible.

So, if we consider an incompressible steady inviscid flow and derive the equation of motion in the streamline coordinates, and then we can take two points on a streamline and integrate the equation of motion and can derive Bernoulli's equation. So that is what we are going to do today.

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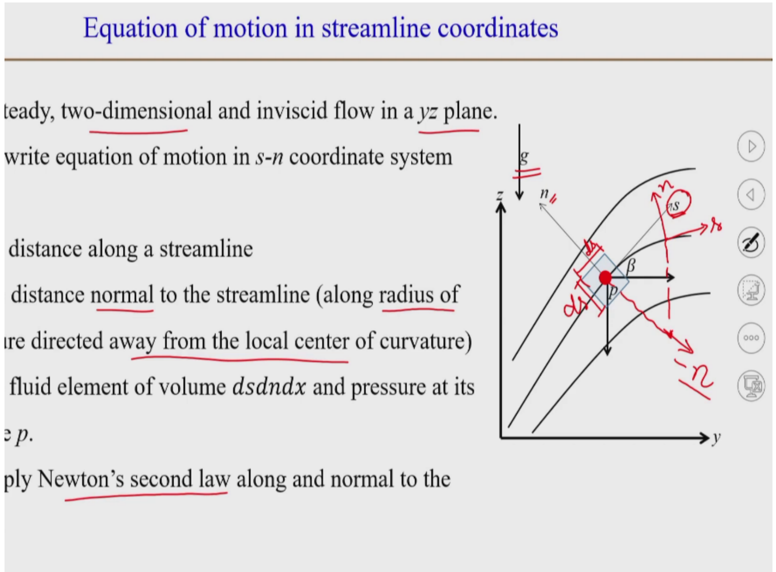

So, we will consider a two-dimensional flow which is in a yz plane, so yz plane will be the plane in the screen and x will be normal to it. So, this is yz plane and x will be pointing normal to it and the gravity as shown here acts in the negative z direction. Now we will consider two orthogonal coordinates, so the first coordinate is of course following the streamline.

So, the coordinate along the streamline we represent by s, so at this point this coordinate is along this line, s, and other coordinate which is n, normal to it is here. If you see the coordinates at a different point, then you will have s pointing here and n pointing out here.

Now the convention for the second coordinate, so s will be along the direction of velocity and along the streamline whereas the other coordinate n, it will be normal to the streamline and the positive direction will be of course the radius of curvature will be normal to the streamline at that particular point and it will be pointing away from the local center of curvature.

So, if you look at this streamline the center of curvature of this streamline will be somewhere here. So, you are pointing away from it. In this direction, what you will have is - n coordinate. So, we are looking at here in s-n coordinate streamline and normal to streamline two coordinates we consider here.

Now so we will consider a fluid element here as you can see here, this is a point shown by the red dot here and along or surrounding this red dot we take a small fluid element, which has dimensions of ds. So, ds is the dimension along the streamline, dn is the dimension which is along the n direction, and ds is the dimension normal to the screen. And as we have done to derive Newton's to derive the equation of motion, we apply Newton's second law, of course with consideration that we are solving in Euler frame of reference or in the Eulerian framework not in the Lagrangian framework.

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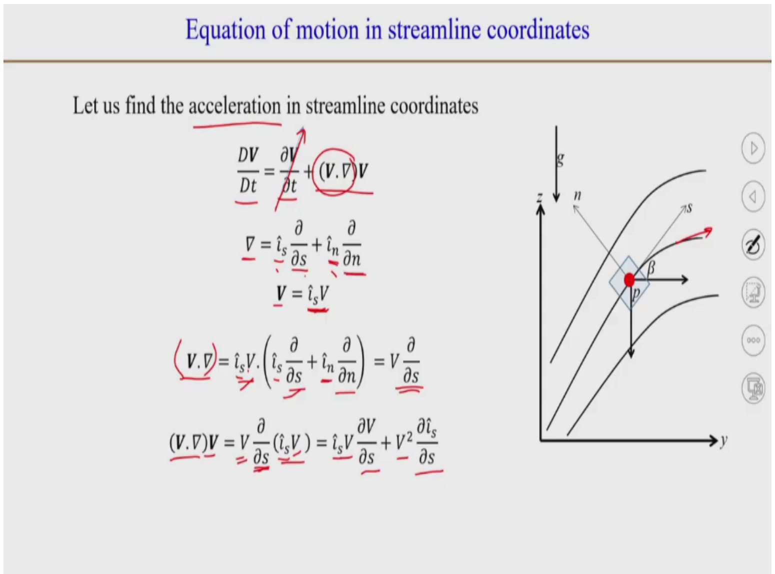

So first we need to do is and we apply Newton's law of motion we will write the acceleration and then we can write f =ma along the streamline coordinates and f =ma along the normal direction. So, for that we need the acceleration because we know and we know the volume of this fluid element, so the mass of this fluid element dm will be ρ ∂ x, ∂ y, ∂ z and what we need to know is its acceleration.

So, we know now that acceleration in the Eulerian framework can be written in terms of substantial derivative which where we have a transient term which represent the local acceleration when the velocity field is changing with time, but the flow is steady. So, this term will go to 0. And we have convective acceleration V dot del V.

Now this is what we need to find. So, we can write because we are using streamline coordinates. So, we can write del in the streamline coordinates, unit vector i s∂/∂ s + unit vector i n + ∂/∂ n. So, we are looking at a two-dimensional flow. So, we will have two components along the streamline, normal to the streamline ∂/∂ s along here and ∂/∂ n along here.

Now we can write down V in terms of these unit vectors along streamline coordinates. So, the velocity is along the streamline. So, Vn component or the velocity along the normal direction will be 0. So, there will be only one component of velocity, so we can say this magnitude is V and the direction is i s which is along the streamline.

Now we can substitute this here. So, when we substitute, we get, first we will find out what is V dot del. So, V is i sV, small, this V is just magnitude, when we have bold then it is a vector. So, i s. we can write down what is del, i s ∂/∂ s + i n ∂/∂ n. And we can take the dot product, when you take dot product of i s with i n they are normal to each other. So, cos time t is 0. So, this term will go to 0 when you multiply with i s V.

Now the only thing remain will be the dot product of the first term in the bracket and the term outside the bracket. So, i s dot i s will be 1 and V ∂/∂ s is what is left. So, we have V dot del =V ∂/∂ s. Now we can apply this operator on the vector V. So, we can write V dot del of vector V which means V ∂/∂ s of vector V which is i sV here.

Now because i s is a unit vector, its magnitude is constant. But as you can see here the direction of i s will change at different points. So, ∂ i s or ∂/∂ s of i s is not 0. So, we will have to consider while taking the derivatives, we will have to consider it. Remember when we did the similar exercise for cylindrical coordinates unit vectors or unit vectors in the cylindrical coordinate system.

So, we can write here V and when we differentiate i s V so we can differentiate this product, so we can write i sV and the derivative of V with respect to ∂/∂ s, so ∂ V/∂ s. Next, we can write VV, so which is V^2 and derivative of i s with respect to s, so we can write ∂/∂ s of unit vector i s.

Now we need to find what is ∂/∂ s of unit vector i s it will be along the similar lines what we did when we wanted to find out the derivative of a unit vector in the r direction with respect to θ direction. So, ∂/∂ θ of er. So, you could write the unit vector in terms of i and j and take the derivative or we could do here a different thing and see the other method, how can we find it. So, we can, we can plot it here.

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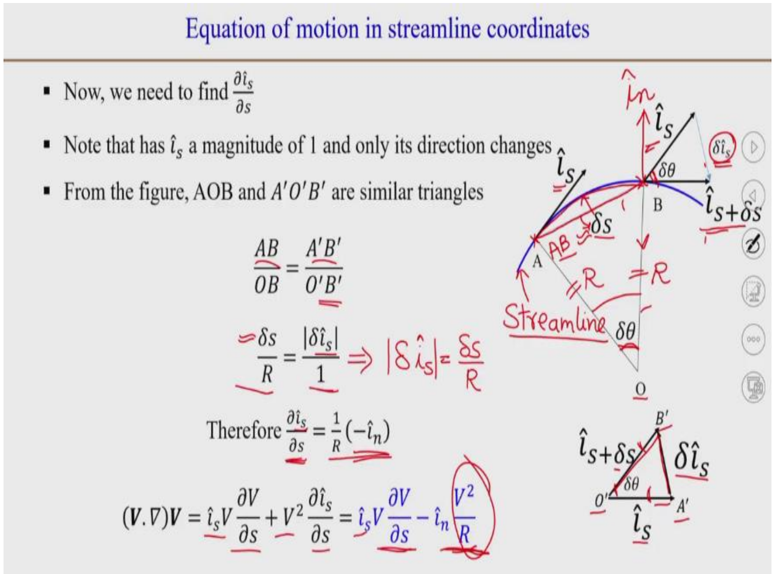

So, if you have a streamline which is shown by the blue color here and consider two points, point A and point B. The small displacement, so the distance between the two points is not very large and the distance, the distance it travels is δ s. And the center of curvature of this curve AB is at point O and the angular displacement here, let us say is δ θ. The unit vector at point A is i s and unit vector at point B is i s + δ s.

Note that they are unit vectors, so their magnitude is going to be 1 and what we need to find is the difference between these two-unit vectors, i s + δ s - i s. So, we can put here and this is the difference between the two vectors. You can say that i s + δ i s will become i s + δ s.

Now, we could draw a triangle here which is the same triangle, so we can write this as O', A', B' and the angle between line OA and OB is ∂ θ and OA and OB are equal because that is radius of curvature, so they are equal to the radius of curvature R here. And similarly, the angle between O', A' and O' B' between i s and i s + δ s, this angle is θ here. And their difference, A', B' is δ i s.

Now the magnitude of these two is 1. So, the angle, one angle is same and two sides are same so we can say that the two triangles are similar triangle OAB and triangle O' A' B'. Note that we can consider this, this is the triangle. And because the displacement is small so we can say the length of this is also δ s, while this is actually δ s, but we can approximate that this length is also δ s. So, we can say that AB can be approximated as δ s.

Now because they are similar triangle so we can write the ratio of their corresponding lines. So, AB/OB, the ratio of line AB, this one and OB which will be equal to R, that will be equal to A' B' /O' B'. So, A' B' /O' B'. Now AB is we saw that we can approximate it to be δ s and OB is radius of curvature R and A' B' is δ i s that is what we want to find.

And so, this is magnitude because it is just magnitude of the δ i s and O' B' is the magnitude of i s vector which is 1. So, from this we can find that δ i s =δ s/R. Now this is the magnitude and as we can see from here that this will point out or this will point in the, if this is i n direction, then this will, the vector will point in - i n direction, it is not drawn properly, so they do not seem appear to be parallel.

So, this will be 1/R - i n. So - i n is the direction in which this will point out. So, we have ∂/∂ s of unit vector i s =- i n /R. So, we have found this out now and we can substitute in the acceleration there. So the acceleration we saw that it is i s V ∂ V ∂ s + V^2 ∂ i s/∂ s. And we can substitute the value of ∂ i s/∂ s here, so we will have i s V ∂ V/∂ s - i n V^2/R.

So, you can see here first thing that though the velocity along the normal coordinate was 0, but we have in the acceleration there is one term which is non-zero along the normal direction. And if you look at this term, V^2/R which is centripetal acceleration, so this is the same thing comes here. It is like rigid body motion we saw, so V^2/R is the centripetal acceleration. So, we will have the acceleration component along the streamline, V ∂ V/∂ s and acceleration along the normal coordinate is - V^2/R.

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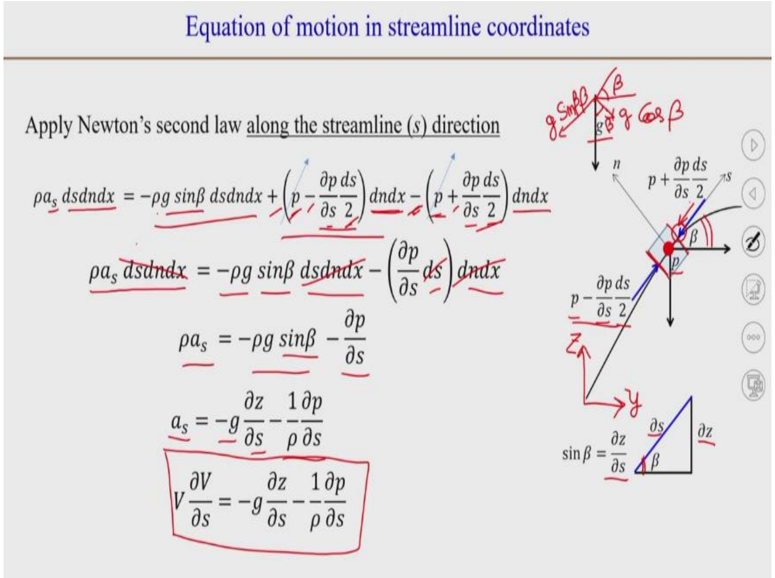

So let us Newton's law, Newton's second law along the streamline, direction first. So, the only force, the only surface force we will have here is pressure force and there will be one body force, which is because of gravity, the viscous force is of course 0. So, we can write down ma where m =ρds dn dxa s is the acceleration along the streamline direction which was V ∂ V/∂ s.

Now we can write down the body force term. So, if we take this angle as β, the angle from, if this is y and this is z direction. So, we can see that g is pointing out in the - z direction, so we can take the component of g, because this angle is β. So, this will be 90 - β and that will be β. So, this angle will also be β and a component of g in this direction will be g cos β, and it is in the negative streamline direction.

So, we will have - mg cos β and m is again ρ ds dn dxg sine β, g sine β. And this direction we will have g cos β. So then come the pressure forces. So, if you look at the surface here, on this surface the force, the pressure here at this point is p and at a distance ds/2 in the negative direction, the pressure force will be p - ∂ p/∂ sds/2 from Taylor series expansion and this will be pointing out in the positive direction.

So, we will have p - ∂ p/∂ s, ds/2the area of this distance is dn and the distance normal to the screen is, or the dimension of this fluid element normal to the screen is dx, so the area is dn dx. Now on the other surface, we will have a pressure because this is along the positive x direction, so the pressure will be p + ∂ p/∂ sds/2 and the pressure points out, it acts as a compressive stress on the fluid. So, it will be negative and the area again will be area of the face will be dn dx.

Now we can see that these two terms of one p is - and another one is plus, so they will cancel out and these two terms we can combine, they are same and you have ds/2 here, ds/2 here. So, you will make it ds. So, when you write down you will be able to write ρ a s ds dn dx =- ρ g sine β, ds dn dx and two negative term of equal magnitude will give you - ∂ p/∂ s ds dn dx. So, you can cancel out ds dn dx, from the three terms and you will end up with the having ρ a s =- ρ g sine β - ∂ p/∂ s.

Now we can write down sine β in terms of the coordinates. So, if we consider sine β in terms of the streamline coordinate ∂ s and ∂ z, so we can write sin β, you can see from here sin β =∂ z/∂ s. So, we can substitute that and we get a s =- g ∂ z/∂ s - 1/ρ ∂ p/∂ s. So, we get, we can substitute the value of a s that we obtained, V ∂ V/∂ s =- g ∂/∂ s of z - 1/ρ ∂ p/∂ s.

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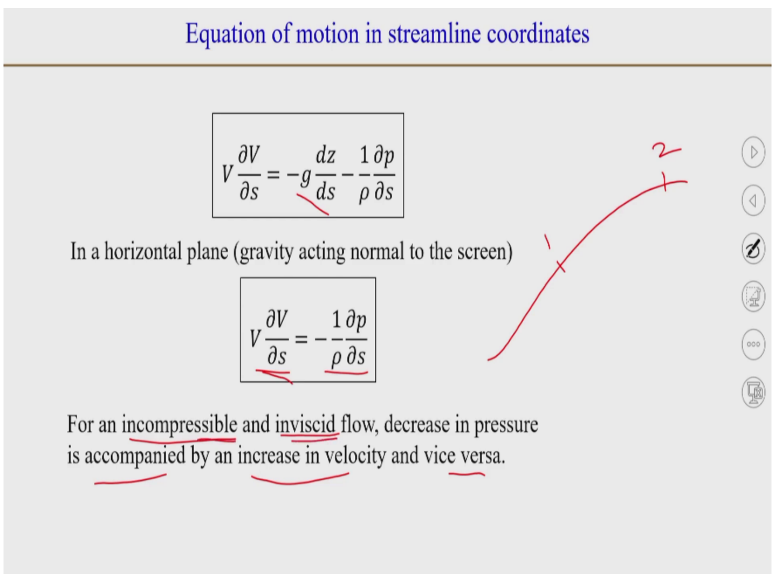

So, this is the equation when we have a g acting along the negative z direction. If we consider that g is not acting, if yz is a horizontal plane then g, of course, will be normal to it along the x direction and this term will go to zero. So, in that case what we will have is V ∂ V/∂ s =- 1/ρ ∂ p/∂ s. So that means that with our assumptions which are the incompressible, inviscid and steady flow, there is a decrease in pressure that is accompanied by increase in velocity or vice versa.

So, if you go along a streamlines say from point 1 to point 2, if there is a change in pressure so that change in pressure will be negative of $V\,\partial V/\partial s$. So, change in pressure will be accompanied or will have a change in velocity and that will be opposite for an incompressible and inviscid flow. So that is our momentum balance equation that we obtained in the streamline coordinate.

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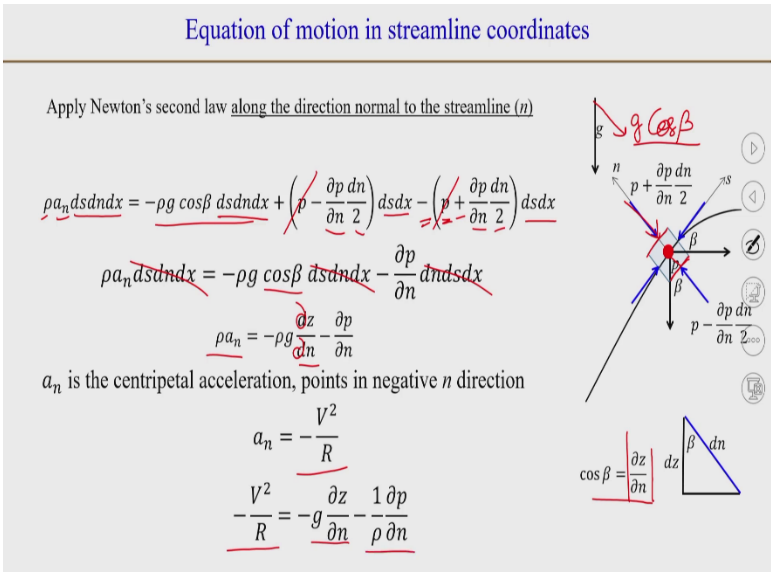

Let us do this in the n direction or the direction normal to the streamline. So, if we consider again the force balance then ma, so dm here ρ ds dn dx and the acceleration let us say is a, n. The body force is the g term here will be the gravity g cos β. So - ρ g cos β, ds dn dx and the pressure forces here on the on this surface pressure will be acting in the positive direction, the area of this surface will be dsdx.

So, dsdx the pressure at this point will be $p - \partial p/\partial n ds/2$. And on the other surface the pressure will be acting in the negative direction, so - and the pressure at that point will be $p + \partial p/\partial n$, dn/2ds dx. Again, we can cancel out the terms, so p and p will cancel out and we can write this down, we can cancel out ds dn dx and we can again write cos β in terms of $\partial z/\partial n$.

So, we can substitute this cos β =dz/dn or $\partial z/\partial n$ here. So, this will be $\rho a n - \rho g\, \partial z/\partial n - \partial p/\partial n$ and n is the centripetal acceleration which we obtained as $- V^2/R$. So, we can substitute $- V^2/R$ =- ρ z, $\partial z/\partial n - 1/\rho\, \partial p/\partial n$.

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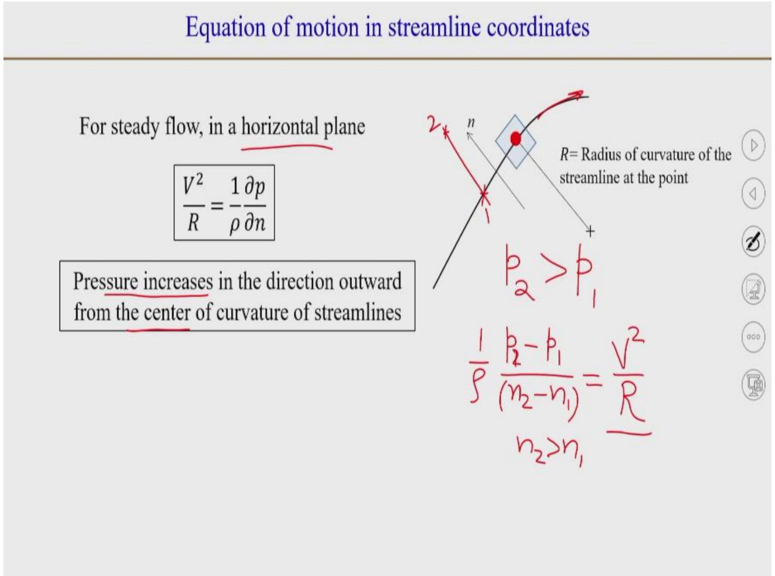

So, if the flow is steady and the plane is horizontal that means the gravity does not act in the z direction but normal to the screen then the g term will be 0 there and you will have simply from here, -, - will cancel out and you will have $V^2/R = 1/\rho \, \partial p \, \partial n$. So that means the centripetal acceleration that will be balanced by the pressure force here.

So, if you have a flow happening along this direction then in a curved channel when the flow is inviscid, then the pressure is higher in the direction outward from the center of curvature, so if you take two points here and the flow is happening in this direction, this is the curvature here. So, if you take point 1 and point 2, p2 is greater than p1, that is what you see here, that $\partial p/\partial n$, which is that p2 - p1.

So, if we write $1/\rho$, p2 - p1 /n2 - n1 and here n2 is greater than n1, because we are moving along the positive direction. So, this term is positive and that $=V^2/R$, V^2 is of course going to be a positive term R is the radius of curvature. So, we can see from here that p2 - p1 will be positive. So p2 is greater than p1. So, we have derived the equation of motion in the streamline coordinates for an incompressible, inviscid flow and remember why we wanted to do it, because we wanted to derive the Bernoulli's equation. So let us do it.

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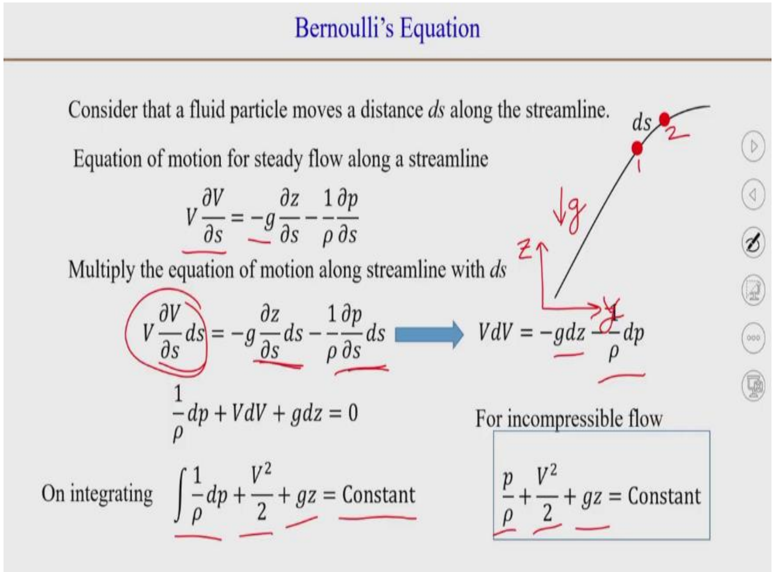

So, Bernoulli's equation is that we apply the equation of motion for a fluid particle which is moving along the streamline. So, we take a streamline and say that if fluid point move from point or fluid element or fluid particle moves from point 1 to point 2 and the distance it travels along the streamline is ds, a small distance and we can consider the equation of motion along the streamline here, which we just derived.

So, the acceleration, if you remember was along the streamline, $V \partial V/\partial s =- \rho g \partial z/\partial s - 1/\rho \partial p/\partial z$ and here if you remember that this was y direction and this was z direction and g was acting in the negative z direction. So, we need to remember this configuration because then only we will be able to write the correct things for g.

Now if we multiply this with ds, so we will get these terms here and we can write this as VdV. This as g dz, so - ρ - g dz and this will be - 1/ρ dp. And if you just see this equation and you can integrate it, when you integrate it, you will get integral 1/ρ dp + $V^2/2$ + gz, that =constant. And we say density is a constant then we can get that p/ρ + $V^2/2$ + gz =constant. So that is our Bernoulli's equation for an incompressible, inviscid flow.

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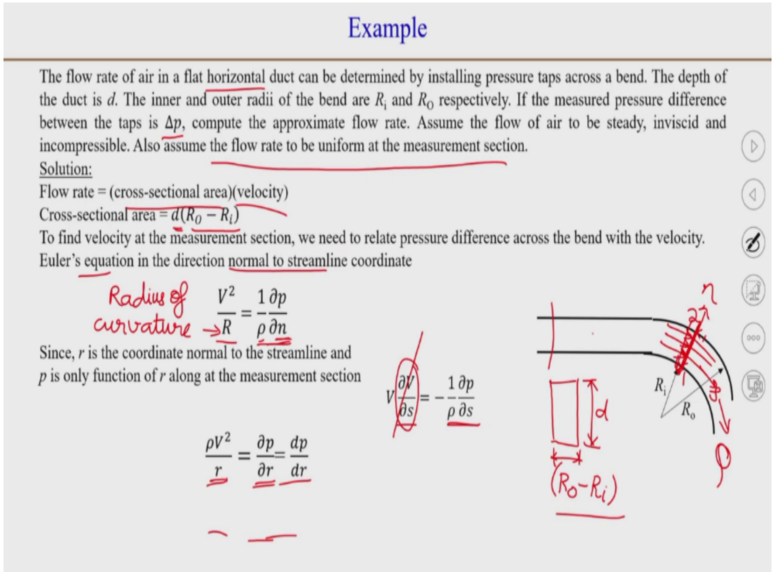

Now we can look at a simple example where we are looking at because along with the Bernoulli's equation, we have also derived a equation for, equation of motion in the normal direction where we have a streamline curvature. Remember that if if the flow is straight then the curvature of streamline will be infinite and V^2 R will be 0.

But if you have a curved channel or the flow in by whatever manner, if the streamlines are curved then this equation is useful to us especially when the flow is inviscid. So, if we have a flat horizontal duct where there is a band, so there is band here and what they are trying to do is they are installing. So, if the flow is happening, the flow can be considered inviscid here.

And the streamlines are along this direction. So sorry the streamline is along this direction and this is the direction normal to a streamline here. So, they are putting two probes here at point 1 and point 2 and the depth of the duct, so if you look at any point and take its cross-section. So, it is a rectangular duct and at the cross section the depth is d, and this distance of course everywhere is the difference of two radii, so Ro - Ri.

Now if the pressure difference which is measured between two points, we just discussed that p2 will be greater than p1, so p2 - p1 =Δ p, and we need to find out the flow rate. So, this can be a way to measure the flow rate. So, by measuring of pressure difference or by measuring pressure at two points and they can connect both the taps by a manometer, so they measure the pressure difference and using that pressure difference, so they can find out Q. So, we need to derive a formula for that.

So first we need to find out flow rate which will be the cross-sectional area multiplied by velocity and we can see from this figure that cross-sectional area will be, it is a rectangular duct. So, the cross-sectional area will be rectangle. So, it is dRo - Ri, which is the area of the cross section.

Now we need to find the velocity. So, we can use the equations along the streamlines or normal to streamline which we have just derived. We can write down the Euler's equation in the direction normal to streamline coordinate because we are looking at change in pressure along the, so if this is s direction and this is n direction. So, because we are looking at the pressure difference along along the n direction, so we will consider the Euler's equation in the normal to streamline or n direction.

So, the gravity acts in the, because it says that the duct is horizontal. So, the gravity will be acting normal to it, so the gravity term will be 0 here and we will have simply $V^2/R = 1/\rho\ \partial p/\partial n$. Now this R or n, so because you can have the radius varying, so you can consider this, the change in radius as r, so you can write this in terms of that $\partial p/\partial n$ in place of n, you can write r because that is the radius that you are considering.

And $\rho V^2/r$, so at any point in the streamline, you will have this as r. So, this is radius of curvature, and it will vary depending on where the streamline is located. So, we can write this as a general coordinate, small r. So, this will be $\rho V^2/r$, $\partial p/\partial r$, so n we have replaced with r and ρ, because the flow is incompressible, it can just go here. So, it is $\rho V^2/r = \partial p/\partial r$.

Now if we also consider the equation along the streamline direction and it has been given to us that you can assume the flow rate to be uniform at the measurement section. So, if the flow rate is a uniform and so the velocity is also uniform here. So, if the velocity is uniform, there is no variation in velocity along this direction. So, this will become 0 and this so, because we are looking at one particular s, there is no variation of s.

So, $\partial V/\partial s$ is 0 and we can say that pressure is constant with respect to s. So, pressure is not a function s. So, we can just write dp/dr. Now when you write this you will get $dp = \rho V^2/r dr$ and you can integrate this equation. So, you will get $dp = \rho V^2/r dr$ and V is the velocity, and velocity at that particular section in the duct, the area is constant and the flow is incompressible. So, velocity is going to be constant everywhere.

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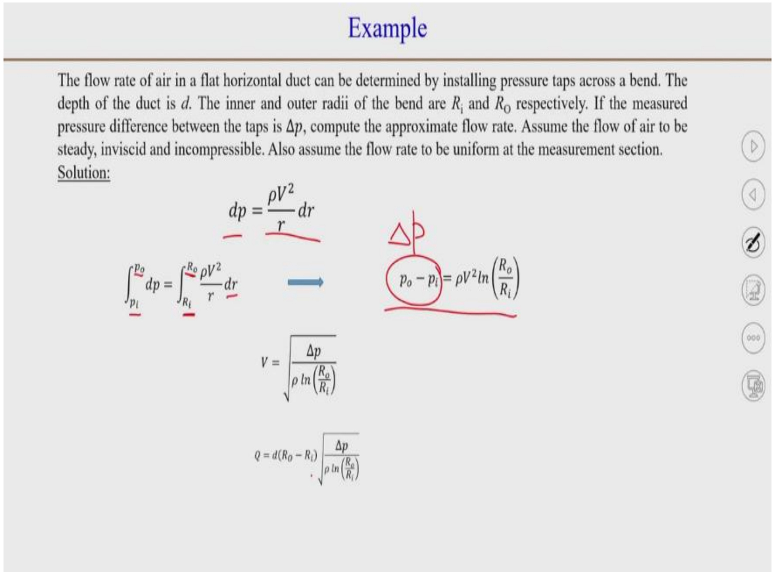

So, we will have dp =ρ V^2/r dr and you can integrate it with respect to R from Ri to Ro. The inner radius of the band and outer radius of the band, let us say pressure at the inner radius is Pi, pressure at the outer radius is Po. So, you can write this and integrate Po - Pi =ρ V^2 which is a constant and dr/r when you integrate, you will get ln R and after substituting the limits you will get Ro/Ri.

So, it will be V =when you do the algebra Po - Pi which =Δ p. So that will be Δ p /ρ ln Ro/Ri, that is the velocity and then you can multiply it by the cross-sectional area. So, you will get flow rate Q here.

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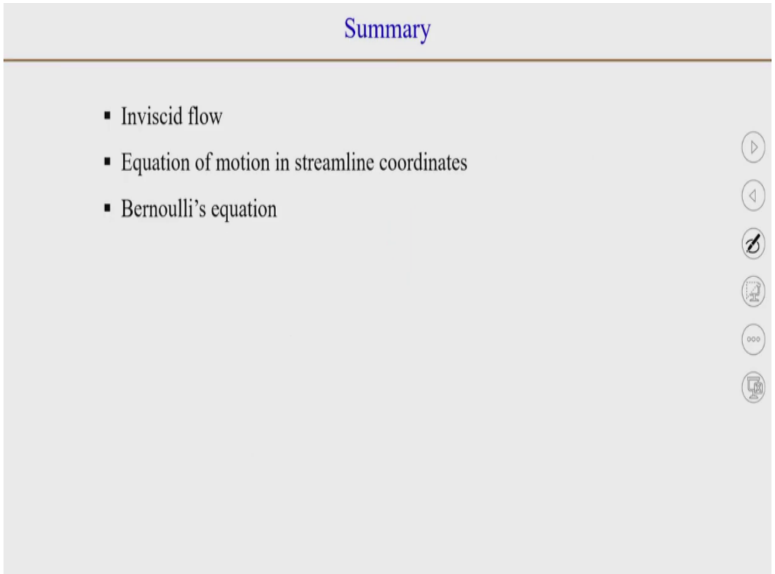

So, in summary, what we have been able to do today is we just looked at what is inviscid flow and we now know that in the inviscid flow we can neglect the viscous terms. And then we looked at the revisited Bernoulli's theorem and the assumptions there, the main assumptions are that the flow is steady, incompressible, inviscid and it can be applied when along a streamline.

So, we derived the equation of motion for an incompressible, inviscid steady flow along the streamline and normal to streamline coordinates in a two-dimensional plane. And from that we could derive the equation of motion. There was another relationship that we got here that $\rho V^2/R = \partial p/\partial r$, which is the relationship that we can use for an inviscid flow to find out the centripetal acceleration or the pressure variation along the radial direction.

If you remember, when we talked about curved flow in the previous lecture, there we also had viscous terms. So, it will not be that, there will be three forces there and that is why we have a secondary vortices coming into play. Here we have only the centripetal force or centrifugal force, it is being in balance by the pressure gradient. So, it is relatively simpler here. We will stop here. Thank you.