Fundamentals of Fluid Mechanics for Chemical and Biomedical Engineers
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Lecture 30
Creeping Flows

When we started this course, so we looked at three different approaches to understand or analyze the fluid flow behavior, the first approach was dimensional analysis, where we looked at Buckingham pi theorem and then the Ipsen's method and learned about how to find or how to make dimensionless groups by collecting the relevant parameters for a particular flow.

So, this approach gave us some information about the flow and we could reduce the number of experiments or number of CFD simulations in today's world one need to do. Then we looked at the macroscopic balances, so in macroscopic balances we derived the Reynolds transport theorem which related the general rate equation for an extensive variable for system and control volume formulations. So, we could obtain the rate of change of an extensive variable for a system in terms of those for a control volume.

So, using Reynolds transport theorem, we could look at the mass conservation, momentum conservation for a control volume. And there are we could find in general the forces on a system. Now, then we used the differential analysis approach in which we derived the mass conservation or continuity equation and momentum conservation equation in the differential form.

So, these equations, if we are able to find the solution then we can find the most detailed information about any kind of flow. The problem arises that it is not always possible to find the solution analytically for the partial differential equations that g/n the fluid flow which is continuity equation and Navier-Stokes equation for when we consider an incompressible Newtonian and constant viscosity fluid.

So, then once we have derived the Navier-Stokes equation, we tried to find the cases where it is possible to obtain the analytical solution. So, in most of the cases, we were looking at fully developed flow where the fully developed flow, which was one dimensional, the other velocity components were 0, and we could neglect the convective term which has $V.\nabla V$.

So, because we have the multiplication of product of these two terms, 2V here, so this is what brings the nonlinearity in the system of equations whenever viscosity is constant etcetera. Then if we could get rid of this term, then we were able to obtain equation which we could integrate. Now we have done this for a number of example, in the last class we looked at lubrication approximation, and where we used that the flow is two-dimensional, but one component of velocity is less than other component of velocity.

In the lecture today, what we are going to do is look at another class of flows which is called Creeping Flows where the flow velocity is very small or if we generalize it, we can say that the Reynolds number of the flow is very, very low and that is why we call it Creeping Flows.

Now we will also look at some other class of flows, for example, what happens when the tube is or when a channel is not straight, there is some amount of curvature present or if the flow is pulsatile, so the analysis that we are going to look at or the information that we are going to discuss today is more of a qualitative in nature because the solutions of the equations will be slightly complicated and lengthy, so we will not discuss those in this and a first undergraduate level course in fluid mechanics.

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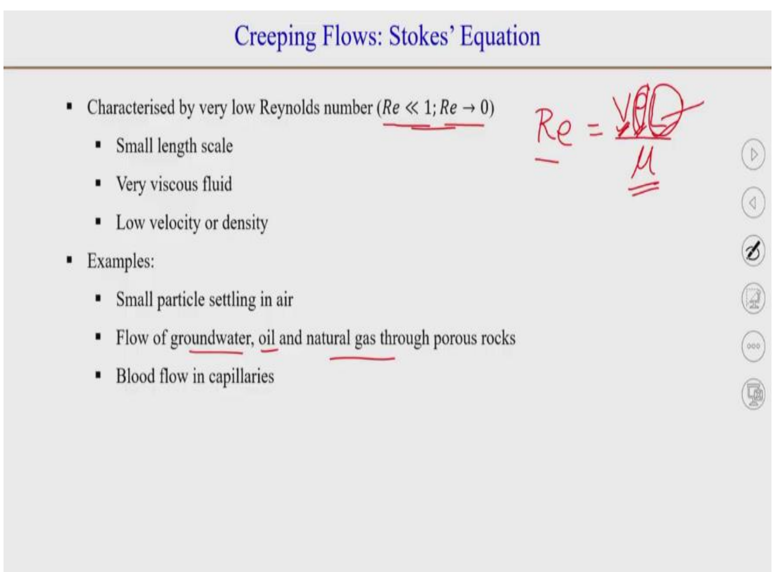

So, the Creeping Flow, the assumption for Creeping Flow is that the Reynolds number is significantly less than 1, or you can say that it is approaching 0. Now if you remember, the

Reynolds number is defined as Re =V ρL/μ, where V is a velocity scale and L is length scale and depending on the problem we can choose the values.

So, this can be achieved that the Reynolds number is very small that can happen if our L is small, so if we have a small length scale, for example, when we talked about lubrication approximation or the flow in thin films. So, the length scale is small in such cases or the viscosity is very high, when the viscosity is very high then the Reynolds number will become low.

So, for very viscous liquid, we can have such things or we can have either of this, either ρ is very small or V is, or the velocity of the flow is 0, or very low, velocity of the flow is 0, then there is no flow, but if the velocity is very small then we can achieve what is Stokes flow. So, for example, when you have a flow in capillaries, where the diameter, when we can I talk about capillaries, it can be a small diameter channel or the capillary that are present in our body where the blood flow is from arteries to arterioles in capillaries.

So, the capillaries that are present in our body, they are of the dimension of about of the order of 10 micron or so, so the velocity and the velocity there is very small, so the Reynolds number is of the order of 10^{-3} or smaller even. So, that is one case where the flow is creeping. Other example is when you have a very small say a dust particle, which is settling in the flow or settling in air, so this particle will have very small length scale.

So, when the length scale is small and as a result the Reynolds number will be very small, the density of air is also very small. Now the other example is, you have porous media, so in the porous media, for example underground, there are, when you look at the structure of the soil, in between the soil particles or in between the different particles present in the Earth there are pores and these pores are of very small dimensions of the order of few hundred microns to few microns and this is where the groundwater flows, where the oil and natural gas is present, so in porous rocks, in porous media.

So, in such cases against the length scale is very small, the velocity scale is small, so the flow is going to be a Creeping Flow. Another example is, these days we talk a lot about microfluidics. So, microfluidics is the flow is small scale flow in the channels or in the grooves of the order of few hundred microns or at the max 1 millimeter. So, such channels also the velocity is very small and the Reynolds number is low, so as a result the flow in micro-channels is also often Creeping Flow, especially in the channels of few microns, say 10s or and 100 of micron dimension.

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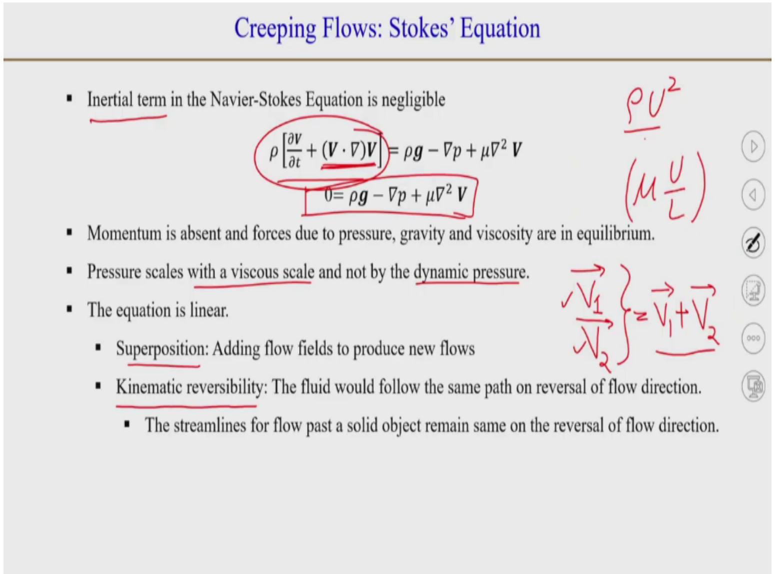

So, then as we know that the Reynolds number is a ratio of inertial and viscous forces, which is basically inertia is ρ U ^{2} where U is a velocity and viscous force will be μ U/L and we can obtain the definition of Reynolds number from there. So, when Reynolds number is low, that means the inertial force or if I write inertia as an I, is very, very less than the viscous force.

So, the inertia is negligible. Now if write down the Navier-Stokes equation in the vector form here, so this is the term which represent inertia, so that means this term is neglected and what we will get in a Stokes flow is that the viscous forces and pressure and gravity forces, they balance each other. So, basically the nonlinear term that we had is again eliminated. So, if you are able to eliminate the nonlinear term then this equation becomes linear.

One thing that we need to or we should remember here is that when we talk about non-dimensionalizing the flow or non-dimensionalizing the Navier-Stokes equation to non-dimensionalize pressure we often use the term ρ U ^{2} where U is velocity scale and this is basically dynamic pressure also, or we can, we also represented just now inertia in terms of ρ U ^{2}, but as we said that the inertial term is negligible in Navier-Stokes equations and you can see here that the pressure is being balanced by the viscous forces.

So, in such flows the pressure should be scaled by μ U/L which is a scale for the viscous stresses. So, in the Stokes flow, the pressure will be balanced by the viscous stresses and not by the dynamic pressure. So, when non-dimensionalizing such equation we need to remember that. Now as you can see here that there is no $V.\nabla V$ term, so this equation is linear and linear equations they follow the principle of superposition.

So, if you have obtained a velocity field, let us say V1 and there is another velocity field V2, and you can combine these two and you will get a new field V1 + V2. So, if V1 and V2, they satisfy the Stokes equation then V1 + V2 will also satisfy the Navier-Stokes equation and that is what the principle of superposition is and the linear equations follow the principle of superposition. So, that is one good thing about the linear equations, I mean there is whole lot of theory about the linear partial differential equations, most of the properties of linear partial differential equations can be used to understand the Stokes flow.

Another very interesting property is that the flow kinematically reversible which means that if you have, if you have obtained a solution for a flow as V1 then if you replace V1 -V1 that means if the flow is happening from left to right, and if you change the flow direction from right to left then the flow will be exactly reversed on you can find, or if you have a mirror here, you will find the mirror image of it. So, if you know the flow in one direction you will know flow or the flow properties in the other direction.

So, if you, for example, when you have flow on such a wedge and flow streamlines when the flow is happening from left to right it will be something like this. Now if the flow direction is from left to right, then also you will get the same bunch of streamlines or if it is reverse from right to left then also you will get the similar streamline pattern. This is let us say a wedge here.

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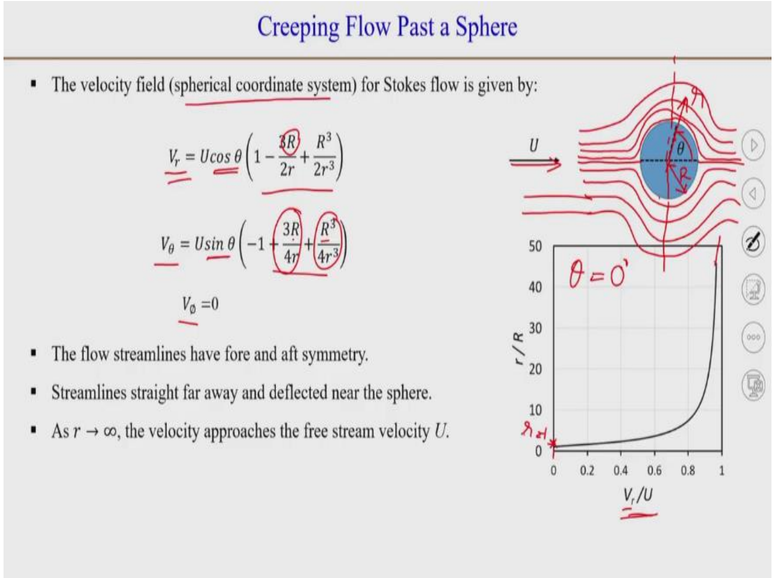

Now we can consider or one of the most common example of Stokes flow is flow Creeping Flow or flow past a sphere. In our school, we have learned about drag on a sphere falling in a fluid. And if you remember, we, you would have derived the drag on such a sphere is 6 Π μ R × V, where μ is the viscosity, R is the radius of sphere and V is velocity of the sphere. The underlying assumptions for such flow was or such an equation was that the Reynolds number is very low.

So, when we consider the Creeping Flow around a sphere, it needs to be derived when we look at it, it needs to be done in the spherical coordinate system. So, we are not going to derive the Creeping Flow around a sphere, but we will look at directly the solution. So, in a spherical coordinate system, you will have flow, you will have three normal directions, so in a Cartesian coordinate you have x, y, and z, the three normal directions, in cylindrical coordinates you have R θ, z, whereas in a spherical coordinate you have one radial coordinate and two angles, θ and φ.

So, if you look at a sphere and or this is let us say the plane that passes through the middle of the sphere, so one direction is of course force the radial direction, the other direction is θ and the flow is happening from left to right on this screen. And the third dimension will be φ, so that will be the angle that is made in this direction, so that will be φ, but because flow is happening in this direction, so the velocity in this in the or Vφ will be 0. So, we will have the velocity field that there is velocity component Vr, Vθ and Vφ.

So, you can see that Vr and Vθ both of them are function of θ and r, capital R is the radius of this sphere and small r is the radial coordinate. So, you can see that there are two terms in both of these, which has 1 / R and other term has 1 / R^3, of course r and capital R^3, capital R and capital R^3 both of them are constants. And you have a cos θ and sin θ here.

So, if we can plot the streamlines for such a flow around a sphere, it will be that the streamline that is passing through the diameter of plane, that will be going straight and then you will have the streamlines turning around the sphere and the flow remains attached. So, barring my drawing skills, if you look at, the flow is, there is a symmetry about this plane.

So, the streamlines on the left and right side, they are same. So, if the flow is happening from left to right as shown here or if the flow is happening other way around from right to left then in both the cases the streamlines will look same. And you can see that far away from the sphere, so the streamlines are straight and parallel to each other because the velocity is uniform, or you have free stream flow where the flow is uniform.

Whereas when the flow approaches near the sphere, these streamlines turn and there is a deflection in the streamlines. And then they turn back again when they, when the flow moves past or away from the sphere. Now if we plot these say, velocity, so what I have plotted here is the non-dimensional radial velocity at θ =0^o. So, what you will have there that the this is of course R =1 at this point, and at R =1 that means on the surface of the sphere.

The velocity will be 0, because, because of the no-slip boundary condition on the surface of this sphere. When you are far away from the sphere, the velocity approaches or Vr/U approaches 1 because the velocity will approach to the uniform velocity, but you what you can see here that it takes lot of time for the flow to achieve a uniform velocity.

So, even about 45 r, this is about 0.96, only point 96 percent, of Vr is only about 96 percent of the velocity, uniform velocity. So, that means the effect of the sphere in the Stokes flow is gone to very long distances and that is one of the properties of Stokes flow. So, if let us say if you are doing some CFD simulations and you are supposed to do Stokes flow, then in such a case you will need to take a very, very large domain so that you can assume that the flow has approached or the velocity is uniform in both the sides, in upper stream as well as downstream directions.

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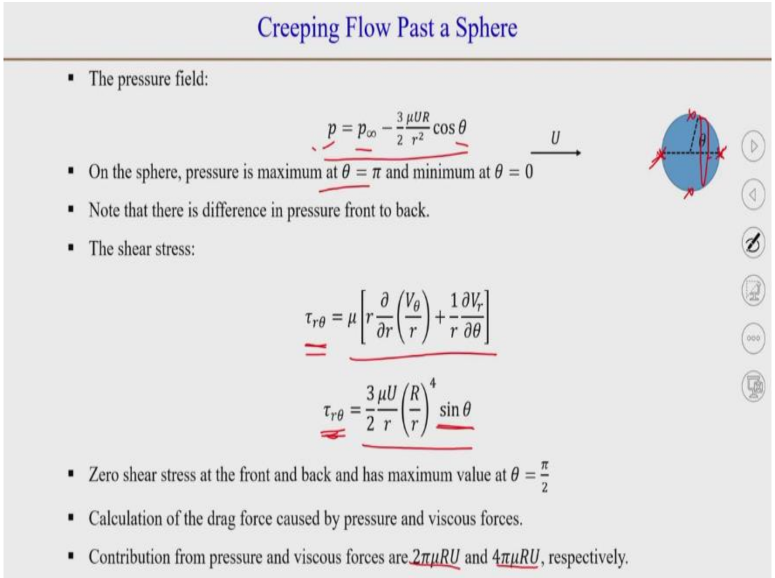

Now if you look at the pressure field, so the pressure field, the solution for the pressure field by Stokes was given as p is equal p infinity -3/2μ UR/r ^{2} × cos θ, where p infinity is the pressure far away from the sphere. So, you can say that p infinity is the pressure when r tends to infinity. And as you can see that p -p infinity or p will be equal to p infinity when θ is π/2 and at θ =0 this term will be positive.

So, θ =0 means at this point the pressure will be minimum and at this point p will be θ =π, so cos θ will be -1 and you will have this term to be maximum. So, at θ =π, pressure is maximum, and θ =0, the pressure is minimum. So, remember that we had streamlines which are symmetric, but the pressure field is not symmetric, you have a pressure difference at the front and back of the sphere.

Now if we look at the shear stress on this sphere and for the spherical coordinates we can give τ r θ using this expression. This will be the only shear stress component that will be non-zero, τ r θ or τθr, they will be equal of course. So, using the expression for Vr and Vθ that we saw in the previous slide, we can find the expression for the shear stress.

And as you can see here from this sin θ term that the shear stress will be 0 at these two values and value of shear stress will be maximum at θ =π/2. So, at these two points. And that is also a bit obvious because at this point which is called stagnation point that θ =π by, θ =π, so at this point the velocity will be 0, so the shear stress will be 0 at these two points, because next to the wall, just next to it at the stagnation point velocity will be 0 and then you have higher shear stress at this.

Now once you have obtained the expressions for pressure, pressure and shear stress you can integrate it / the entire sphere, so you can take a at a particular θ you can take a small strip and integrate pressure as well as $\tau r\theta$ and you will be able to obtain the force caused by the pressure and by the shear stress on the sphere. And when you combine those two forces, you will get the drag on the sphere.

So, you can do that as an exercise and you will find that the force, when you integrate the pressure / the sphere you will get the force as 2 π μ RU and because when you integrate the shear stress, you will get the force as 4 π μ RU and when you add them together what you will get is 6 π μ RU, which is the total drag force. As we will see later on that the force that is caused by the pressure difference across the sphere is called firm drag and the pressure difference that is caused by the viscous forces is called friction or viscous drag.

So, the viscous, the contribution from a pressure here is 2 π μ RV or one-third of total, and contribution of viscous drag is two-third of total. So, this is where we have the viscous stresses are most dominant because the Reynolds number is very low. So, viscous stress is dominant here, where there the contribution of viscous stress is two-third of the total drag. As we will see later on that as the Reynolds number increases, the contribution of viscous drag will become, in terms of fractions will become smaller and smaller.

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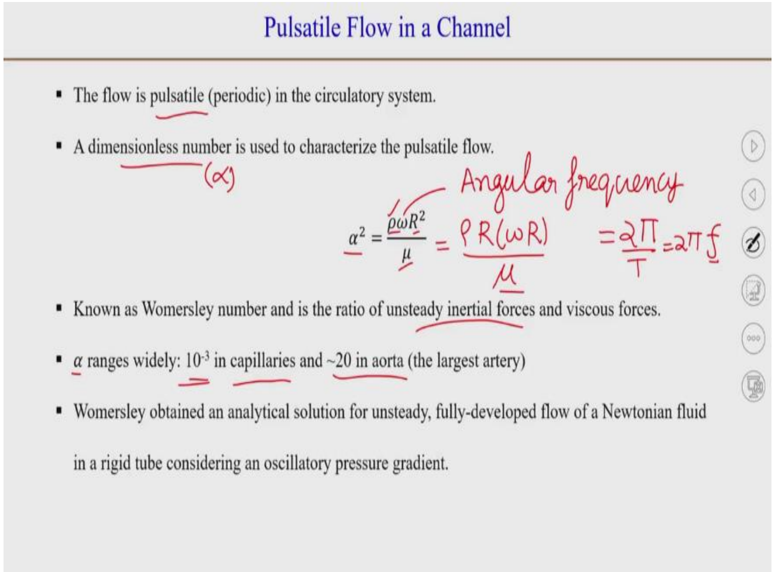

So, that was about Stokes in brief, then when we look at say flow in a cardiovascular system or in a circulatory system, so what is different from what we have studied, they are pipes or they are channels, but the wall is flexible and the flow is pulsatile. We know that in a normal human being the heart beats about 72 times or the pulse rate is about 72 per minute. That means the heart pumps the blood in the body 72 times, so in each cycle, there are 72 cycles of blood flow in 1 minute. So, the flow is pulsatile.

You can say that the flow is almost periodic if it is rhythmic, then the flow is periodic. Now all we have studied until now, we have considered flow to be steady. But what we talk about say in cardiac flow is the flow is pulsatile, even with steady flow we could obtain a certain information, for example, the Poiseuille relationship which is the relationship between the pressure drop and flow rate for a laminar fully developed flow, for the steady flow for a Newtonian fluid was derived Poiseuille, who was a physiologist, and he wanted to understand the flow of blood in vascular vessels.

So, there is a lot of physics or there is lot of information that could be understood just by using the steady-state relationship. But if you want to relax that assumption and want to understand more about what is happening or more accurately we want to look at then we can understand the pulsatile flow in the circulatory system. If we also consider the flexible tube, then we will need to understand or we will need to take × account the effect of pressure on the elastic walls or off the vascular vessels such as arteries or arterioles.

Now if the walls flexible then because of pressure there will be some deformation of the walls and you need to understand this deformation, again calculate the velocity field and then the deformation or we need to solve the equations for the deformation of the solid walls and the flow equations simultaneously. So, anyway, what we are going to discuss just now is the pulsatile flow in rigid tubes.

So, a dimensionless number is used to characterize this pulsatile flow and this number is called Womersley number. So, the Womersley which is denoted, denoted as α^2 and generally the Womersley number is called as α, so this $\rho\ \omega\ R^2/\mu$ where ρ is of course the density of the blood or density of the fluid, ω is angular frequency, which =2 π / t, or time period or 2 π f, so which is frequency and R is radius of the channel and μ is viscosity.

So, you notice here, if you can see that this is basically a kind of Reynolds number, which is ρ R × ω R / μ. So, ω × R is a velocity scale and ρ R × velocity scale divided μ, you have a Reynolds number here. So, it is the ratio of unsteady inertial force, and the viscous forces.

Now if one want to find the solution for it, you can look × some books where the solution for pulsatile flow in the rigid tubes has been found and it is called Womersley solution. So, what is done here or what Womersley did is because any pressure signal using Fourier series, it can be decomposed × sins and cosins.

So, he obtained that he considered if the pressure gradient which drives the flow is pulsatile or it is sinusoidal or it has a cosin, so he considered a sinusoidal pressure gradient term and he also assumed all the assumptions for a fully developed flow as we make in Hagen–Poiseuille, so for example the flow is incompressible, flow is fully developed and there the flow is considered to be Newtonian and the viscosity is constant, all those assumptions and of course the flow is axisymmetric and the channel is cylindrical or the tube is cylindrical.

So, using those assumptions he obtained the solution which includes bessel function etcetera. So, if you look at this Womersley number, the value of Womersley is very when the radius is small, so 10^{-3}, which is, the capillaries in our body and of the order of 20 in aorta which is the largest artery, which take the oxygenated blood from heart to all the entire body which is called systemic circulation. Now so the value of α ranges between these two values of about 5 order of magnitude change.

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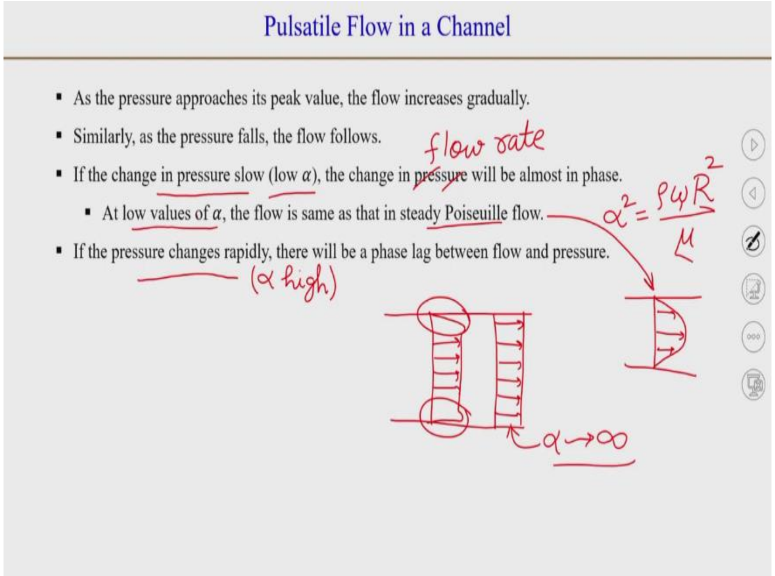

Now he obtained a solution as I said, and what he learned or what we can say qualitatively about the flow is that because if the pressure gradient is sinusoidal then the pressure will reach to a peak value or the pressure difference will reach to a peak value and then come down and go up again. So, when the pressure difference is large, the flow will tend to become also large and when the pressure difference is decreasing, then the flow will also decrease gradually.

Now the question comes that how will that happen? Will there be a lag between the two on will there be or they will be occurring simultaneously? So, that will happen if your change in pressure is slow, so if α, or let us write α^2 which is $\rho\ \omega\ R^2 / \mu$. So, if α is small, that may be because ω is low or maybe R is low, which is the case in capillaries or μ is high, because if we talk about blood then ρ and μ are going to be fixed.

It is for ω and R is changing, R will be changing when we move from aorta, largest artery to capillaries which are the smallest vascular vessels, and ω is the frequency that will vary depending on the conditions or depending on the exercise states etcetera. So, if for either of these reasons if α is low, then the pressure will change slowly, the frequency is low then the flow will respond almost in phase, so if the change in pressure, then that should be change in flow rate.

So, they are almost in phase. And in fact, when the values, value of α is small then the flow will be like a Poiseuille flow and you will have a parabolic velocity profile as we obtained in steady Poiseuille flow. So, the velocity profile is parabolic in such a case. On the other hand, if there is a rapid change in pressure, so that means α is high, either frequency is high or the radius is high then there will be a phased lag in the flow and pressure, because the flow will not be able to respond to the change in pressure immediately.

So, there will be a change in the, there will be a phase lag between flow and pressure and the velocity profile will look more like a flat profile. So, you will have some gradients near the wall where viscous forces are dominant. But in the center you will have uniform flow, in fact for α tending to infinity, you will have an inviscid kind of flow which have listed in the subsequent lectures so that will be when the velocity profile will look like a plug flow when α is approaching infinity.

So, that is about Womersley solution, but again, when we talk about Womersley solution, we have the assumption of fully developed flow, so that we have assumed here that in the entire vessel the flow profile is going to remain same. So, that is again an approximation. But the good thing is that this can provide us an analytical solution and we can find or we can have a lot of insight × the flow using this.

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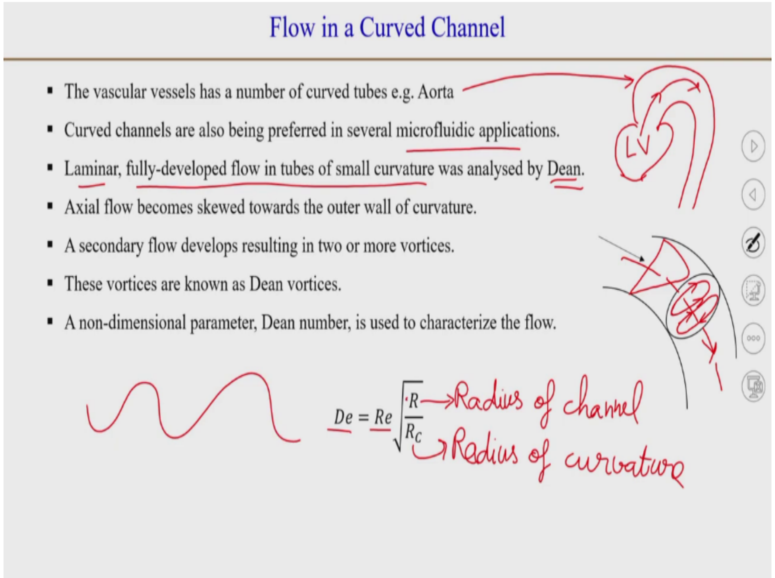

The next flow that we are going to consider is flow in curved channels. So, this is a very common occurrence, flow in curved channel is a very common occurrence in our vascular system, so for example human aorta which bring the blood from heart, so this is let us say left ventricle. And aorta bring blood from heart, so you can say that there is a change in direction here.

So, the artery or this aorta is curved, and you have number of bifurcations in arteries or in other places you have or in many other places you have curved channels or the curved vascular vessels, so the question is, will there be a flow in curved channels similar to what we observed in a straight channel or are there going to be any differences?

So, about a century ago, Dean studied analytically laminar fully developed flow in tubes of small curvature. And what he observed is that when you have a flow in curved channels as in shown here, then the velocity profile is not symmetric and the velocity profile becomes a bit skewed towards the outer curvature. So, the flow is moving towards the outer wall and then it is brought back together.

So, apart from the flow in the stream-wise direction or in this direction here, if you take a cross-section you will see some secondary flow as well. So, if you cut the tube and look at the flow or if the flow is happening in this normal to my palm, and then you will see some the circulations here and this is called secondary flow and as you can see they are vortices. So, these vortices are name as Dean vortices after, after the gentleman.

And the flow can be characterized as strength of these vortices, the occurrence of these vortices etcetera can be characterized by a non-dimensional number again, which is defined as Dean number and it is Reynolds number multiplied by root of R which is radius of the channel or the tube, and Rc is the radius of curvature. So, smaller the radius of curvature the larger the Dean number or you can say the smaller, the larger the ratio R/Rc, so either R is small of Rc is large, the Dean number will be large.

Now this will also be a function of Reynolds number. So, that means the flow where, the conditions where the flow is laminar, say microfluidic applications where the channels are of the order of few millimeters or less, then in such cases the flow is generally laminar and the problem with laminar flow is that there is no inherent mixing. As a chemical engineer, I would like to have for different applications for the reactions etcetera I need mixing of reactants etcetera.

So, a lot of research in microfluidics goes × developing micro-mixers, and one of the strategy there is to use curved channels, so for example somebody might use say serpentine channels, so the channels follow this path and you will have in this cross-section there is a lot of mixing happening and this mixing is used for, I mean, that is a passive mixer, so the Dean flow is used significantly for passive mixing applications.

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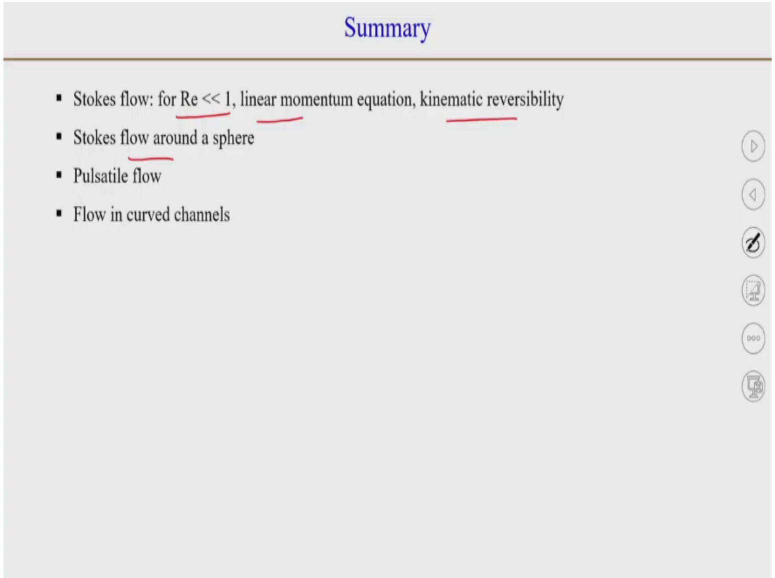

So, in summary what we have looked today, we have looked at Stokes flow which is valid for Reynolds number significantly less than 1, the momentum equation becomes linear there and as a result we have kinematic reversibility in the flow, that means the flow from left to right or right to left will look or appear to be same.

We also looked at Stokes flow around a sphere, the distribution of velocity, pressure, the expressions for them and how does it vary. Then we looked at briefly pulsatile flow and we should remember what is Womersley number and then we looked at flow in curved channels, where we looked at the Dean number and we learned that there is secondary flow if there is curvature in the channel then there is secondary flow in the channel. So, we will stop here. Thank you.