Fundamentals of Fluid Mechanics for Chemical and Biomedical Engineers
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Lecture 29
Lubrication Approximation

Up until now we have been looking at different solutions of Navier-Stokes equation where the flow was fully developed. When the flow was fully developed then we saw that the flow was one dimensional that means only one velocity component is non-zero all the other velocity components it might be in the cylindrical coordinates, or it might be in the Cartesian coordinates.

All other velocity components were zero, only one component was non-zero, so the flow was one dimensional. And this was achieved because we assumed that the flow is fully developed, so that there is the gradient along the x direction $\partial v/\partial x = 0$ sorry $\partial u/\partial x$ predominantly 0, because other velocity components we were able to achieve to be zero.

So, that way the nonlinear term in the Navier-Stokes equation where we had $u\, \partial u/\partial x$ or $v\, \partial u/\partial x$ such terms where we have multiplication of velocities which are non-linear terms, so such terms were neglected or they went to 0 and we were able to convert these partial differential equations × ordinary differential equations and which are more importantly in the ordinary differential equations and we could solve them analytically.

By solving them analytically, I mean that we could integrate them easily. Now we are going to look at a case where the flow is not necessarily one dimensional. We have a second component of velocity which is non-zero, but this component of velocity as we will see is very when you compare with the other component of velocity.

So, we will consider two-dimensional flows and in such two-dimensional flows you will see that if the flow is happening predominantly along the x direction that there is some y component of velocity, but that component of velocity, y component of velocity is very as compared to the main component of velocity, which is u, so v/u will be very, very less than 1.

And this is called lubrication approximation, so the lubrication approximation as the name suggests, it came because the first use of this approximation was done to solve problems in ball bearings where the lubrication etcetera come × picture. So, that is where the name came from. But the approximation is very useful in a number of problems, in biological flows as well as in thin fin flows etcetera.

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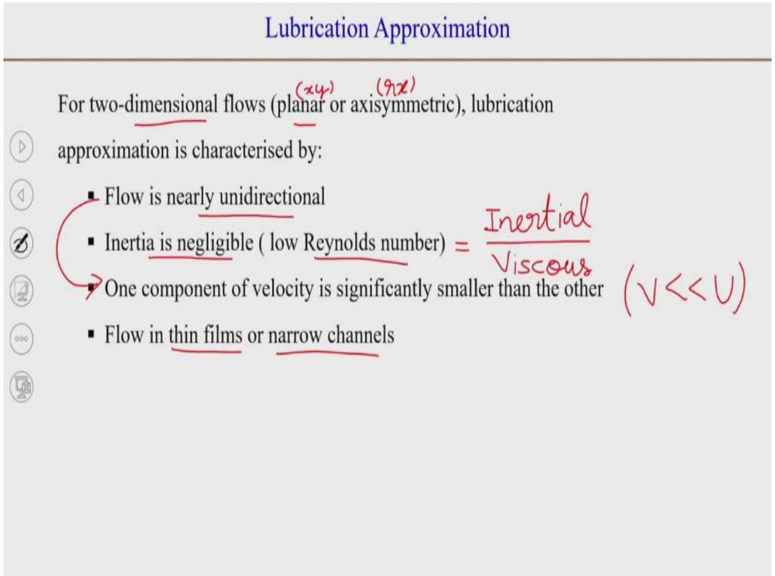

So, let us look at this. So, we will look at this lubrication approximation because we are looking at flow which is two-dimensional, so it can be two-dimensional flow and it can be in Cartesian coordinate, then we call it a planar, so then in that case it will be xy coordinate and axisymmetric, so in the axisymmetric case we will have a flow to be in r and x coordinates.

And it can be characterized that the flow is nearly unidirectional, so there is some other component of velocity, but that will be very low and we will see that with this is applicable we will be able to use this approximation only when we can neglect the inertia that means the Reynolds number, which is ratio of inertia or inertial forces /viscous forces.

So, when Reynolds number is low and inertia is negligible, that is also a condition for lubrication flow and a corollary of this is that flow is nearly unidirectional which means that one component of velocity is significantly smaller than the other, so let us say, if the flow is happening in the x direction predominantly than the y component of velocity v is significantly smaller than u. And as I said the flow is, or you will find such application of lubrication approximation in flow in thin films or flow narrow channels etcetera.

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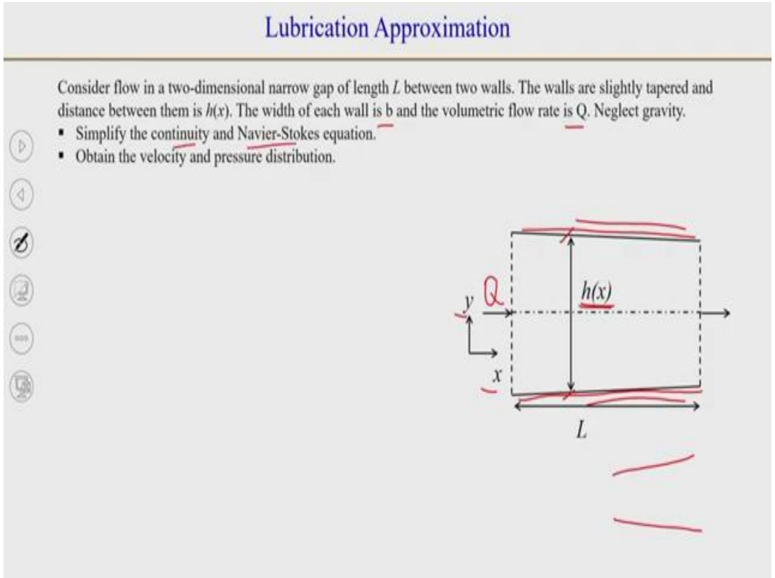

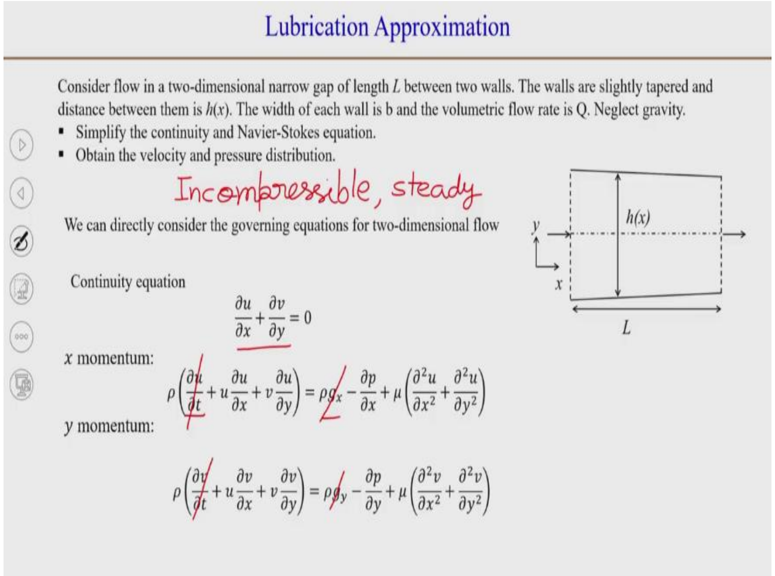

So, we will take up a example where we consider a two-dimensional flow in x and y coordinates, so a there are two walls which are not parallel to each other. So, this gap you can see that the distance between the two walls here is a function of axial coordinate or not the axial coordinate but the x coordinate, which is the direction in which the flow is happening and you can say that the flow rate here is Q, that has been given.

The width of these walls normal to the screen is b, and we will not consider gravity to make our life simple here and as you can see that these lines, they are slightly tapered. It is in this case the flow is say these walls are converging, but you may as well have the flow between the walls which are slightly diverging and approximation will be valid there also.

So, as we have been doing earlier, what we will do, we will start with the governing equations which are mass and momentum conservation equations or continuity and Navier-Stokes equations. And then we will obtain the velocity and pressure distribution. We will also assume here that the Reynolds number is low.

So, we will start with the writing down the governing equations for a two-dimensional flow in Cartesian coordinate and we will assume that the flow is incompressible. So, we will start with writing governing equations when it is two-dimensional flow and flow incompressible then our continuity equation will simply reduce to $\partial u/\partial x$, + $\partial v/\partial y = 0$.

The x momentum equation, because the flow steady, so this term can be removed and we will neglect gravity so this term can also be removed and I have already removed the terms which contained w or $\partial/\partial z$. So, this is, with these terms removed, we will have our x momentum conservation equation and then similarly y momentum conservation equation. The unsteady term is neglected and the gravity term also can be removed.

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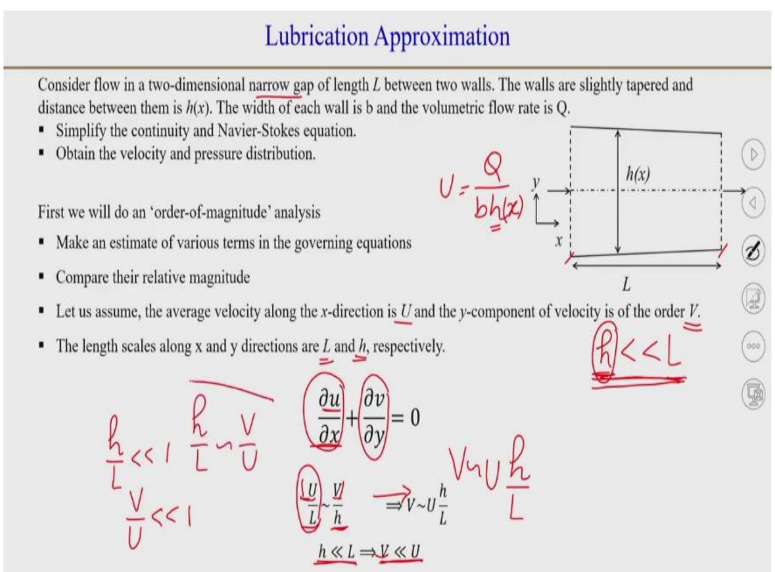

Now what we will do is, we will do an order of magnitude analysis. So, by looking at the relevant scales, in terms of length scale or in terms of velocity scales for the corresponding variables, we will try to find out the relative magnitude of different terms and then we will try to neglect the terms which are negligible with respect to other terms. So, we will start with the continuity equations.

So, we can assume that the velocity along the x direction, the flow rate is Q and Q/bh, you can say that that =velocity U. Now remember here that this h is varying slightly along the x direction, so h is a function of x and the distance between say inlet and outlet of this channel is L, so the length of this channel is L, and we can assume the length scale along the x direction is L and along the y direction is h.

We will also assume that this narrow gap, so h is less than L, that is one of the conditions, because the flow is narrow, so the flow is in, sorry, not because the width of the channel is significantly smaller than the length, so h is very, very less than L here, that is what we are going to consider. Now, we will also consider that let us assume that the y component of velocity is the order of y component of velocity is V.

So, with these assumptions we can write down this term, magnitude of this term, $\partial u/\partial x$, so this we can write ∂u as U, and ∂x, we can write as L, so $\partial u/\partial x$ we can say that its magnitude is of the order of U/L, and the magnitude of $\partial v/\partial y$ term, we have assumed that v or y component of velocity is it can be scaled as or it can be of the order of V and the distance along the y direction we can take as, h, which is the distance between the two plates.

Now, we know U which is average velocity and h and L are the geometric dimensions, but we don't really know what is U, so do not really know what is V, so we can see that V =Uh/L. And we have assumed here that h is very, very less than L, so that means from here we can see that V will be very, very less than U and so from this we can say that h if an h is less, less than L then V is very, very less than U.

So, with the assumption that the one dimension of our system which is h, the transverse direction is significantly smaller than the other dimensions which is the direction along which the flow is happening, then we can say here that the component of velocity V will be less than the component of velocity U. So, this is the first thing that we obtain from our approximation, so we can now say that h/L is less than 1 and V/U is also less than 1 and we can also say that h/L and V/U, they are of same order.

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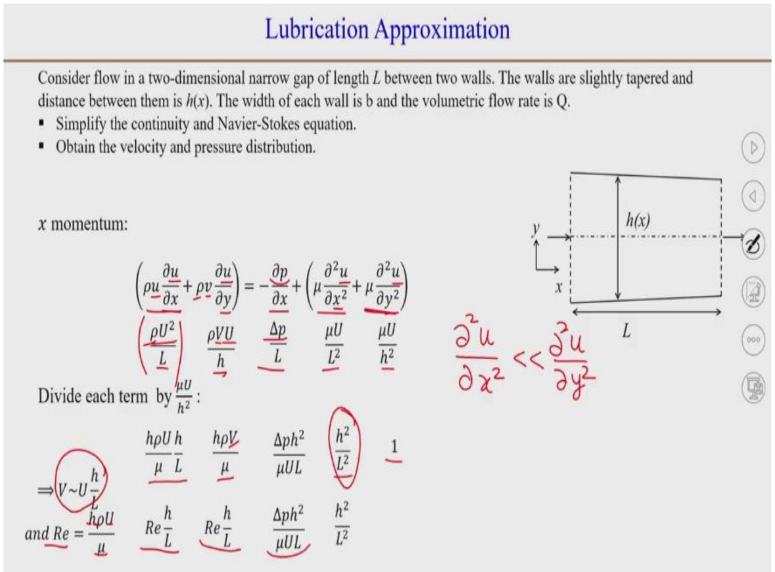

So, this is what we learned from the continuity equation, let us is now look at the momentum equation. So, we write down the momentum equation without unsteady and gravity term which we neglected. Now the first term if we look at, the first term will be of the order of ρ, for u, we can write U here and a for another u, so it will become U^2. And for x we will have L, so the first term, the order of first term is ρ U^2 / L.

The second term we have a ρ here, for v we will use V and there is another u, so U and for y we will have h. So, ρ VU/h, then let us say the pressure is of the order of Δ p and for Δ x we will have L, so Δ p/L here and this term will be μ, U/L^2, so for u and for x we have L and x is^2, so L^2, so μ U/L^2 and next term will be μ U/y^2, so we will h^2.

Now we can try to divide each term, μ U/h^2, because of this is a thin film and we will have a viscous term dominating here. So, we will see that how other terms compare with the viscous term where we are considering the gradient along the y direction. So, if we divide the first time by μ U/h^2, or we multiply by h^2/μ U, then then this term we will get from the first term we will get ρU / μ × h/L.

The next term we will get ρ × V × h / μ U and U will cancel out. The next term will have Δ p/L multiplied h^2 / μ U, and then μ U will cancel out here, so h^2/L^2 and this term will be 1. Now if you look at this term, we see that h/L is less than 1, and among the two viscous terms here, so we can see by looking at just this itself that ∂ 2u/∂ x2 is significantly less than ∂ 2u/∂ y2.

So, among the two viscous terms only ∂ 2u/∂ y2 term will be significant. We can replace this V/Uh/L here, and we can also define the Reynolds number for the flow here. So, h ρ U/μ, which is where we use h as the characteristic length scale and U as the characteristic velocity scale, so we can see that the first term in the inertial term is Re h/L and second term also will come out to be Re h/L.

So, we can see that both the inertia terms are of the same order, ρ U ∂ u/∂ x, ρ V ∂ u/∂ y, then we have a pressure term and two viscous terms.

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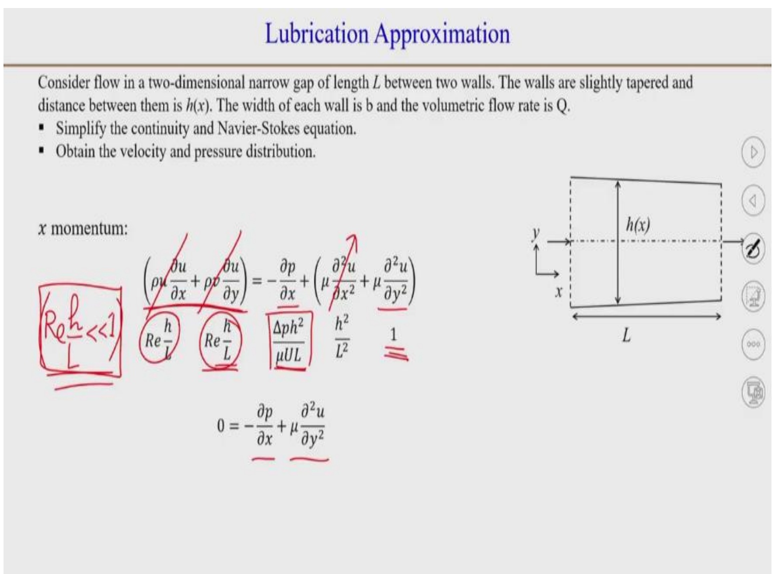

So, these are the relative magnitudes, now we can see here that we can neglect this term when we compared with this, now h/L is and Reynolds number is also small, so we can say that Re h/L is significantly less than 1, so one another assumption we assume, we have assumed here before that h/L is less than 1.

Another assumption we are going to make here that Re × h/L is less than 1, so when we do that then we can neglect these two terms with respect to 1, so these two terms are also negligible. So, this means because this is the inertial term, this is the fluid inertia. So, we can neglect the inertial terms with the assumption that Re × h/L is less than 1.

Now we are then left with two terms, pressure gradient term and one viscous term. So, this simply says, if you remember the same equation, this is the same equation that we obtained for flow between two parallel plates. Now here what we have, we still have that the flow is two-dimensional, so there is some v component of velocity here and p ∂ p/∂ x, now we need to see that there we had the pressure gradient was independent of other component.

So, ∂ p/∂ y was 0 and we need to see and assess here what happens to a ∂ p/∂ y or is pressure dependent on y or not. But this term suggests that the pressure gradient, because this is a pressure driven flow, the flow is driven by a pressure gradient and it is balancing. So, the pressure gradient which is the energy that is being lost because of the viscous losses, pressure gradient is providing the energy to overcome those losses.

So, it is the balance of pressure gradient and the viscous forces that will come × picture here and these two terms will be of the same order. That means of $\Delta p\, h^2 / \mu U L$ and 1, so you can see that this term or you can say that this term is also of the order of 1.

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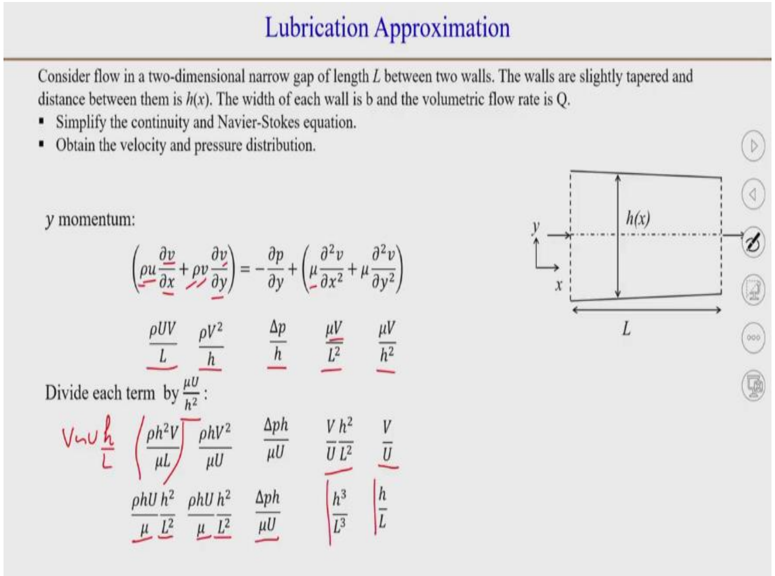

Now let us look at y momentum equation, so in the y momentum equation, the first term we can write ρ U V/L. The next term we can write ρ V × V, so V^2 /h. Then the next term we can simply write Δ p/h μ × V/L^2 and μ × V/h^2. So, then again we can divide these terms by μ U/h^2 and see that how these terms compare with the viscous terms here.

So, we have the first term, ρ × h^2 /μ U, U will cancel out and V so ρ h^2 V /μ L, then the next we will have ρ × h, one h cancel out × V^2 /μ U. Next term Δ p/h, multiplied by h^2/μ U, so one h will cancel out and you will have Δ p h/μ U, and the next term will be μ, μ will cancel out, so we will have V/U × h^2/L^2.

And the last term μ, U, μ will cancel out and h^2 will cancel out, so we will have a V/U, so we can see here, we can replace V with the Uh/L, so when you do it in the first term so you will get ρ h, U/μ × h^2/L^2. In the second term, so you will get ρ as U/μ × h^2/L^2. Then the next term, Δ p h/μ U, it remains same, because there is no V here, V/U we can replace by h/L, so h^3/L^3 and similarly h/L here.

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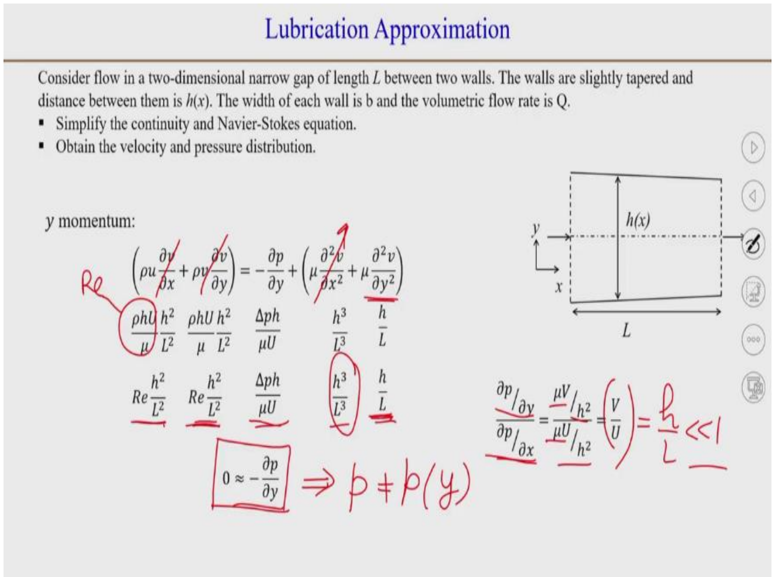

Now, so we can write this as Reynolds number =ρ h U/μ, so the first term will be Re h^2/L^2. Next term, Re h^2/L^2 and we have pressure gradient term and h^3/L^3 and h/L. So, we can see here that this term is negligible with respect to this term, h^3/L^3 will be significantly smaller than h/L.

Similarly, the inertial terms Re h^2/L^2 and Re h^2/L^2, they will be significantly less, then so these terms can be negligible. Now we will again have ∂ p/∂ y and μ ∂ 2v/∂ y2. These terms which balance each other because other terms are neglected. But let us see, because remember that the viscous term in the x momentum equation was of the order of 1.

So, we will just compare the pressure gradient terms and we can see that the pressure gradient, the y, $\partial p/\partial y$ and $\partial p/\partial x$, when we the compare these two terms, so $\partial p/\partial y$, because the pressure gradient in both the cases we can say that the pressure gradient is of the order of this viscous term. So, we can say that $\mu V/h^2$ and $\mu U/h^2$ and that will be V/U, which is h/L and less than 1.

So, that means that this term here, they are very or the pressure gradient is negligible. So, from here we can say when you compare the $\partial p/\partial y$, it is compared to $\partial p/\partial x$, so it is quantity and we can say that $\partial p/\partial y$ is negligible which will give us that p is constant with respect to y. So, that means the pressure does not vary in the transverse direction.

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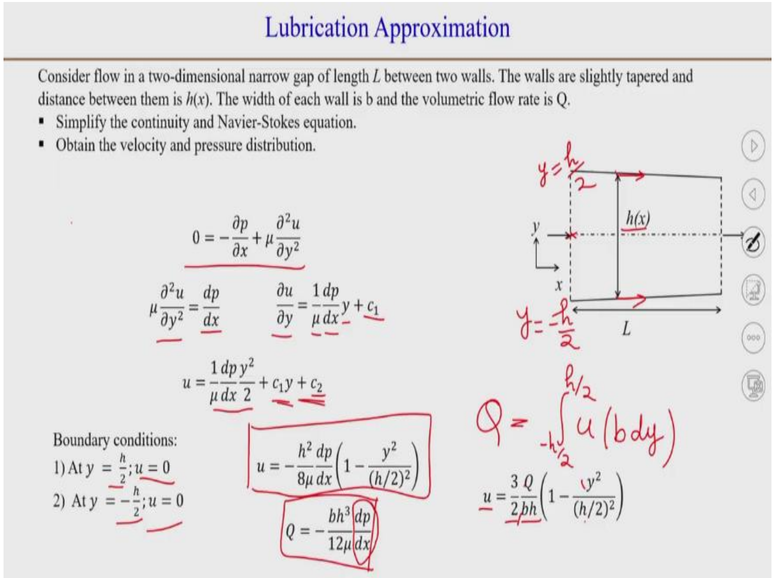

Now, we can a integrate this equation, which is the simplified x momentum equation, we can write because we found it is two-dimensional flow and we saw that p is independent of y, so we can write this $\mu \, \partial 2u/\partial y2 = dp/dx$ and we can integrate, so when we integrate first time we will get $\partial u/\partial y = 1/\mu$ or $dp/dx \times y$ + constant of integration and another integration and we will get $1/\mu \; dp/dx \; y^2/2 + c1 \, y + c2$, and now our task is to find these two boundary conditions.

So, for the boundary condition of the tangential velocity at the two walls which is let us say our origin is here at the axis, so the top wall or at any place, the top wall is y is h/2 and bottom wall we can say $y = -h/2$. Now at the walls when you talk about tangential velocity there will be some u and v components, but both the components will be 0 because the tangential velocity is 0, you can resolve it you can take its components as well as the velocity normal to it because this is non-porous walls.

So, velocity normal to it is also 0, so you can take that the u component of velocity is 0 at $y = + h/2$, $y = - h/2$ and from that we can find the constants c1 and c2. So, c1 will be 0 and we can find c2 which will be equal to, $c2 = - 1/\mu \; dp/dx \times h/2^2 /2$, so once we replace the constant c2 by its value, then you will get the velocity profile.

Once you get the velocity profile, we can find the flow rate Q =integral - h/2 to h/2, u × bdy, where b is the plate with normal to the screen. And after the integration, we will get the flow rate which

=- $bh^3/12\ \mu \times dp/dx$. So, we can replace or we can find out dp/dx from here, because we can replace this and Q is known quantity, so we can write velocity profile in terms of Q.

So, u will be 3/2 Q/bh and 1 - y^2 /h/2^2. So, remember here that Q is a constant plate with b or wall with b, normal to the skin is also constant, y of course is the coordinate in the transverse direction and h is the plate width at any place, so h is a function of x here. So, we have obtained u component of velocity.

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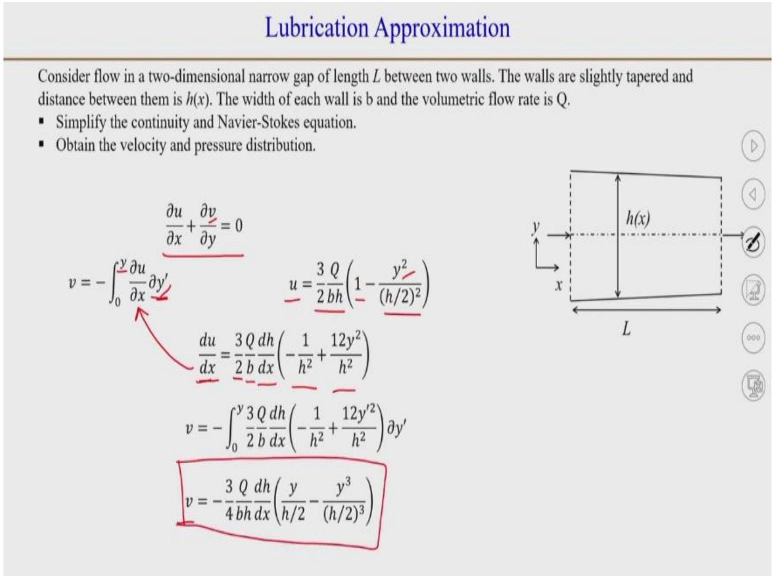

Now there is some component of velocity v also which is the velocity along the y direction. So, using the continuity equation here we can find out the y component of velocity, so we can write this equation and rearrange so ∂ v/∂ y =- ∂ u/∂ x and we can integrate from 0 to y ∂ y, so y is nothing but a dummy variable so that we can differentiate between the limit y and the variable y inside the integral sign.

And we have obtained this u =3/2 Q/bh, 1 - y^2 /h/2^2. Now we can evaluate du/dx, so here Q/b is constant with respect to x, but h is a function of x, so we can evaluate this du/dx and it will come out to be 3/2, Q/b, dh/dx - $1/h^2$ + 12 y^2/h^2, I request you to verify it for yourself and see that if there is any mistake in the calculations.

So, once you obtain du/dx, you can replace it here and we can obtain v by integrating it from 0 to y, so we can replace and then do the integral, so once the integral is done you will be able to obtain

v. So, that is v component or y component of velocity obtained, now we have obtained u component and v component for the velocity field. So, we have obtained the velocity distribution. And next is to obtain the pressure distribution.

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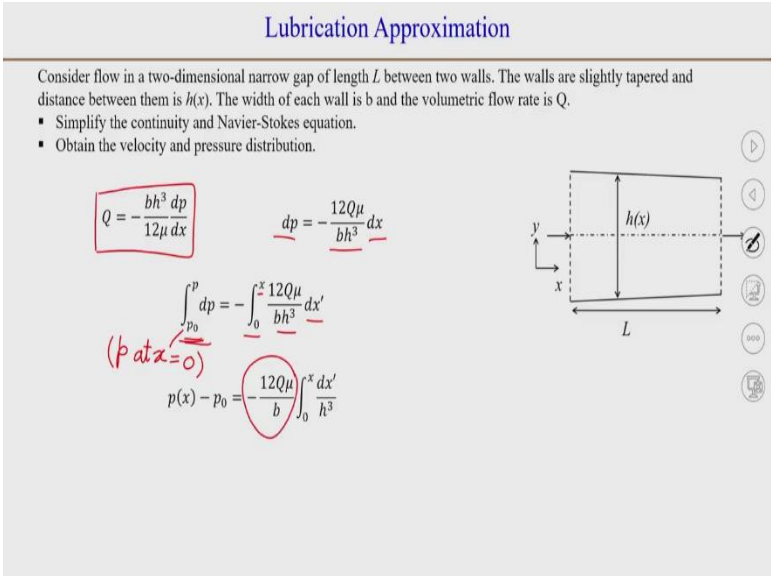

We have already obtained pressure gradient in terms of flow rate or flow rate in terms of pressure gradient, we can rearrange it to write dp =- 12Qμ /bh^3 dx and h is a function of x, so we can to find dp we can integrate pressure so we can say that if the pressure at x =0, so p0 is pressure at x =0 and we can integrate it from 0 to x, so at a distance x, remember that pressure does not vary along the transverse direction or that variation is negligible, so we can write 12μ Q /bh^3 dx‘.

So, we will get p, x - p0 =- 12Qμ /b integral 0 to x, dx’/h^3. And if we know that how does h vary with x, we can replace here and find the variation of pressure with respect to x, all of these are constant.

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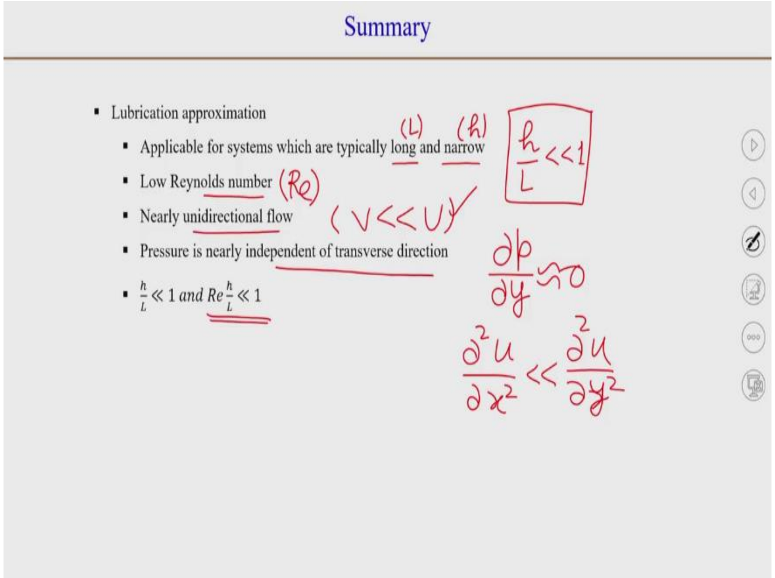

So, in summary, if we look at what we have done is we have looked that a flow where the flow is two-dimensional, but the one component of velocity is significantly smaller than the other. So, by making some assumptions, we have been able to identify or we have been able to find the velocity and pressure field, because once we have obtained the conservation equations then our goal to solve different problems is to find the solution of these system of equations and we will try to find that if such an approximation can fit × some particular solution.

For example, we can use this for flow in thin films, flow between two narrow channels or flow between even two circular cylinders, we can try to use such approximations, say there is a flow between so flow in a channel where the wall is porous, even though the walls are parallel to each other then there will be a y component of velocity and the same there we will not have h/L less than 0, but we can see if there is not the flow from the walls is lesser than the main flow, then we can also use or we will be able to find that v is less than u, or we can also use this approximation to solve the problems of flow on a porous wall.

So, this is a useful technique to solve different problems in a number of applications. So, lubrication approximation is applicable for systems typically which are long and narrow that means we can say h/L where h is the narrow dimension and L is the longer dimension. So, that is one approximation that we need to use and the Reynolds number is low.

So, low Reynolds number we can have, but the ultimate condition which we will need to see are based on which we have neglected the inertial term is that Re × h/L such terms should be negligible. That is what our limit for Reynolds number is, so if h/L is very small, let us say of the order of 10^{-3} 10^{-4} then we can use such approximation for Reynolds number of 100 or so.

Then as we have seen the flow is near unidirectional that means the v component of velocity is very, very less than u, and we saw that $\partial p/\partial y$ which is the pressure gradient in the transverse direction, that was significantly less than $\partial p/\partial x$, so we neglected it. We also saw here that $\partial 2v/\partial x2$ was significantly less than, $\partial 2$, sorry we should say, actually it is valid for both the velocity components, but it is more relevant for u.

So, I will just write u here, so $\partial 2u/\partial x2$ is significantly smaller than $\partial 2u/\partial y2$ and all of these things are the components that we need to remember to find some of these things, we start as an assumption and some of those results we find when we apply those assumptions. So, we assumed here that h/L is less than 1 and from that we got V is less than U and then we could neglect the inertial terms when we saw order of magnitude that Re h/L is less than 1, so we can neglect or we could neglect the initial terms.

We could neglect one of the viscous terms by this that $\partial 2u/\partial x2$ is less than $\partial 2u/\partial y2$ and we could also, when we compare $\partial p/\partial y$ and $\partial p/\partial x$, we saw that their ratio is of h/L, so we could say that $\partial p/\partial y$ is negligible. What we also learned in this lecture is the order of magnitude analysis.

So, when it is not possible to do everything analytically by using some common sense and by making approximations and finding out the relative magnitude of different terms in the equation, we will be able to simplify the equations and try to find analytical solutions. So, lubrication approximation is useful and apart from that, we have also learned here order of magnitude analysis, which we should try to practice and get a hang of it, because it comes handy in a number of cases. We will stop here. Thank you.