Fundamentals of Fluid Mechanics for Chemical and Biomedical Engineers
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Lecture 28
Flow between Two Concentric Cylinders

Hello. So, in the previous lecture, we looked the laminar fully developed flow in a cylindrical channel, which has circular cross-section and we obtained the equation for the velocity profile in such a arrangement by simplifying the mass and momentum conservation equations in the cylindrical coordinates. Today, in this lecture, we will look at another problem which is flow between two concentric cylinders.

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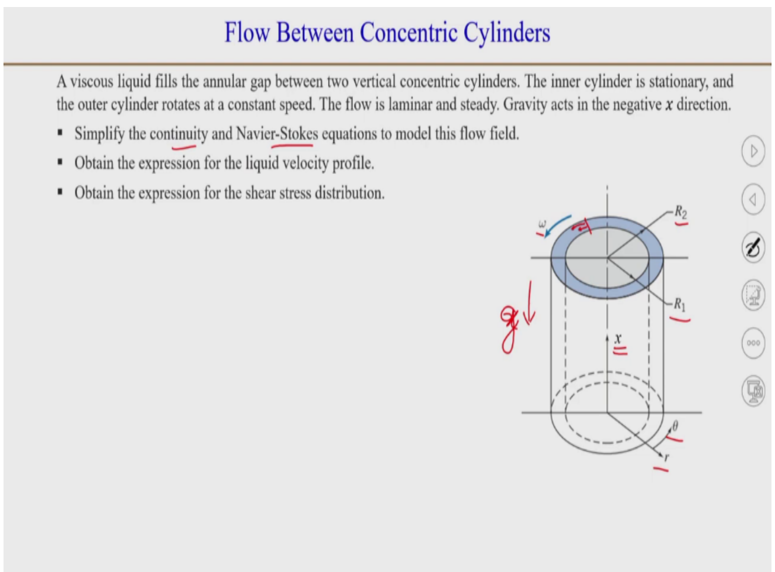

Now when we look at concentric cylinders, that means these cylinders, the two cylinders are, they have same axis and if you look at from the top you will see the cross-section as two concentric circles, the liquid is filled between the two cylinders as you can see by the blue color here. Now, you can have different arrangements.

For example, you may have, if the arrangement is vertical you may have the flow because of gravity driven in this cylindrical cavity or in the annular gap between the two cylinders. So, that is also you can consider that a case of a falling liquid film between, falling liquid film on a cylinder.

It might be inside the cylinder or outside the cylinder, and you will need to treat the outer boundary as the free surface or inner boundary as the free surface depending on where the film is.

But what we are going to look at in this lecture is that the flow is driven by the motion of one of the cylinders. So, in this case the outer cylinder rotates with an angular velocity of ω and because of the shearing motion, because of the shear caused by this cylinder the fluid starts rotating. So, it is a shear driven flow, if you remember when we talked about flow between two parallel plates and the upper plate moving with a velocity u, so again the flow was driven by shear in that case.

So, the gravity is acting in the negative x direction, you can see the arrangement, the coordinate arrangement that r and θ coordinates here and the axial coordinate is taken as x coordinate. The radius of the inner cylinder is R1 and the radius of the outer cylinder is R2 and we need to simplify the mass and momentum conservation equations, find the velocity profile.

So, in this case the velocity profile because the outer cylinder is moving so there will be a velocity in the angular direction here and there will be no motion along the vertical direction because there is nothing to drive the flow along that direction or we can neglect the motion along the vertical direction. So, this is actually gravity, gravity acts in the negative x direction, so we can neglect the motion in the vertical direction. And there will be as we will see that there will be no radial component of velocity.

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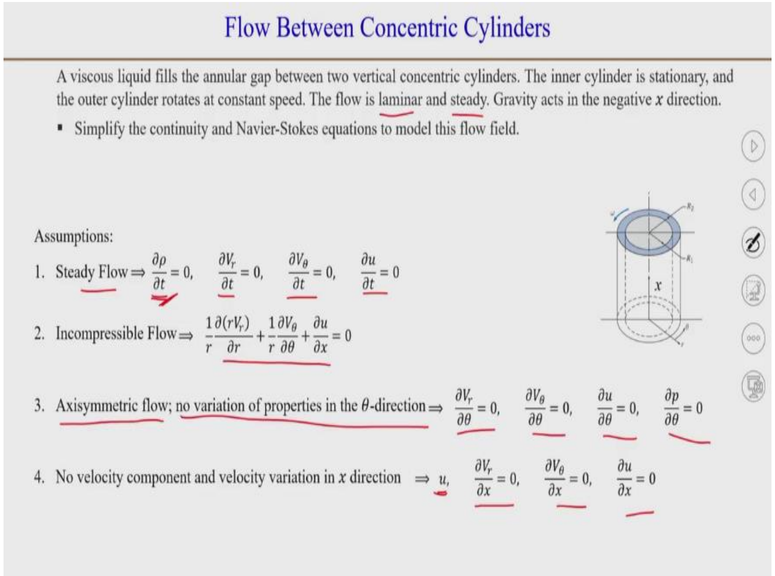

So, we can start by writing down the mass and momentum conservation equations in the cylindrical coordinates system and then let us write down the assumptions. So, the first assumption that we will assume that the flow is steady of course, we will also assume that the flow is laminar. So, all such cases which we are solving, finding the analytical solution for they are in all such cases the flow is laminar. The flow steady, then the flow is incompressible and the flow axisymmetric so which means that there is no variation of properties in the θ direction.

Remember when we talked about flow in a circular channel, then we looked at the θ component of velocity or V_θ was also 0, but here, V_θ will be non-zero and the axisymmetric flow simply means that the flow is symmetric about the axis. So, V_θ will be same at every θ. So, that means $\partial V_\theta / \partial \theta$ will be 0, and the gradients of other velocity components along the θ direction, which is $\partial/\partial\theta$ will be 0.

And there are no velocity component and velocity variations in the x direction. So, if we write down these assumptions in terms of equations, the first assumption of steady flow will simply mean that $\partial/\partial t$ for all the variables, all the velocity components and rho, they are 0. Then compressible flow will mean that in the continuity equation, $\partial\rho / \partial t$ will be 0 and ρ is a constant, so your continuity equation will come in this form, which is in vector form, $\nabla.\vec{V}$.

The axisymmetric flow will mean here that ∂/∂θ for all the velocity component as well as pressure will be 0, because it is not pressure drive flow, though the flow is happening along the θ direction, but this is because of the shearing motion. And we will neglect the velocity component in the x direction even if there would have been any flow along the x direction, so we will neglect and there are no gradients along the x direction, so ∂/ ∂x of Vr, of V_θ and u which is the axial velocity component are they are 0.

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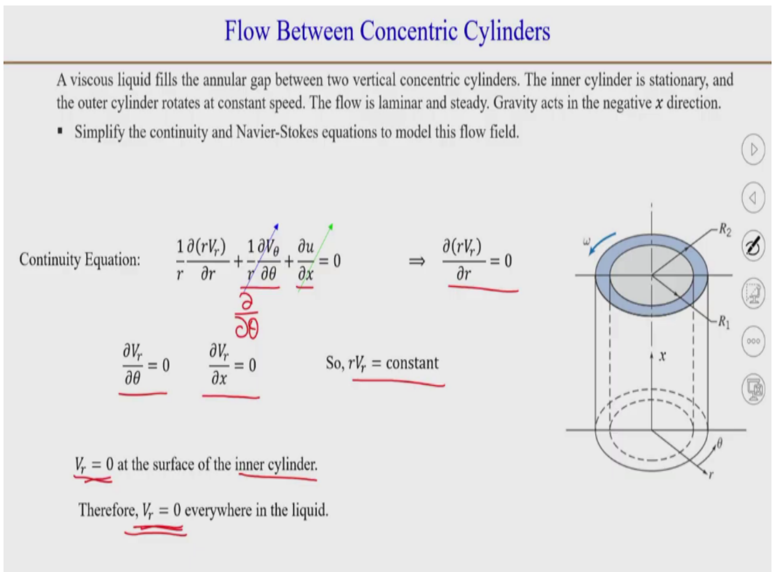

Now let us start with simplifying continuity equation. So, when we look at the continuity equation, these two terms will be 0, this term will be 0 because the flow is axisymmetric and ∂/∂θ of V_θ will be 0. This is 0 because u is 0 or you can also say that ∂/∂x of u is 0. So, that will simply mean that ∂/∂r of rVr = 0, which when integrated will give us that Vr = c /r, we can also take × account the fact here that the derivative of Vr with respect to θ is 0 because the gradients along the θ direction are 0 and ∂/∂x of Vr is 0 because gradients along the x direction are 0.

So, that means Vr is not a function of θ and x and we can use this as an ordinary derivative also. So, a rVr will be a constant. Now r Vr is a constant, so Vr will be constant /r. Now we can use the boundary condition at the surface of the inner cylinder, which is stationary. So, because of the no-slip boundary condition there Vr = 0.

So, that mean if Vr = 0 at the inner cylinder, then Vr needs to be 0 everywhere in the in the liquid in the entire annular space. So, that means Vr = 0, we had that V axial component of velocity which is u equal to 0 and Vr = 0, so we are left with only one component of velocity which is non-zero and in this case this is V_θ.

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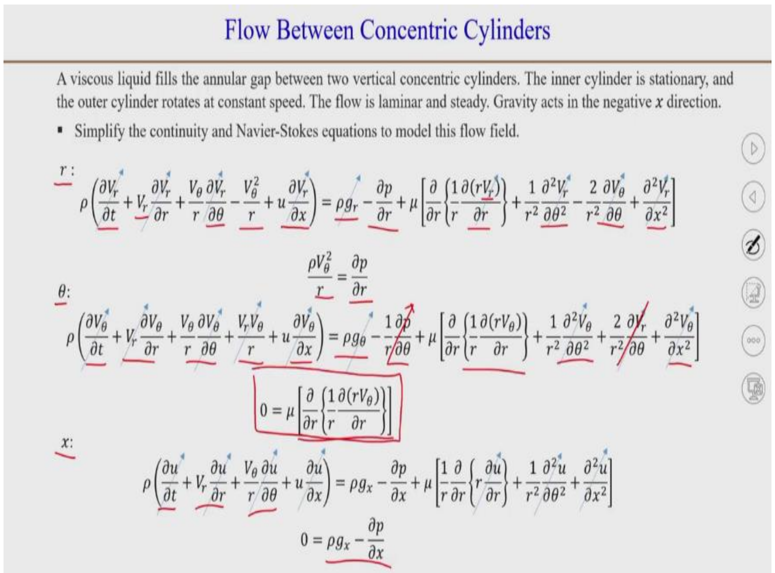

So, now we will try to simplify the equations of momentum conservation or Navier-stokes equations in the cylindrical coordinates. So, first we look at the momentum conservation equation in r direction and the first term will be 0 because the flow is steady. The second term will be 0 because Vr = 0. The next term will be 0 because we have ∂/∂θ as 0 for all the variables. And then next term, this term is 0 because ∂/∂x of Vr is 0.

There is no component of gravity, gravity acts in the negative x direction. So, there is no component of gravity along the radial direction, so this term is 0, and this term is 0 because Vr = 0. Next term is 0 because ∂/∂θ and second derivatives will be 0, this term will also be 0 because ∂/∂θ is 0, and the last term will be 0 because Vr = 0 or ∂2 /∂x2 will be 0.

So, what we have from this is the two terms which are left, which we are left with, so that is ∂p /∂r = ρV_θ^2 /r, so there will be a pressure gradient because V_θ will be non-zero. So, we will have a pressure gradient along the radial direction, which is, if you look at, it resembles the centrifugal force. Now let us look at the momentum conservation equation along the θ direction. The first term will be 0 because the flow is steady.

The next term will be 0 because Vr = 0, then ∂/∂θ = 0, Vr = 0, so this term will be 0 and ∂/∂x = 0 for all the velocity components, so this term will be 0. No component of gravity along the θ direction then the derivative along the θ direction and second derivatives are 0, so ∂2 /∂θ is 0, so this and this term.

They will be 0 and the last term is 0 because there are no derivatives along the x direction or ∂2 /∂x2 is 0. So, now from this we will be left with the viscous term, and there is no pressure variation along the θ direction, so ∂p /∂θ will also be 0, so we were left with only viscous term which = 0. And from this we will be able to find the velocity.

Now, the last momentum conservation equation which is along the x direction, so the first term is 0 because flow is steady. The next term is 0 because Vr = 0 and the third term here is 0 because u is 0, actually in this equation most of the terms will be 0 because u = 0 so we can say all of these terms will be 0. And there is that ρgx - ∂p /∂x, so you will have a hydrostatic pressure gradient along the vertical direction.

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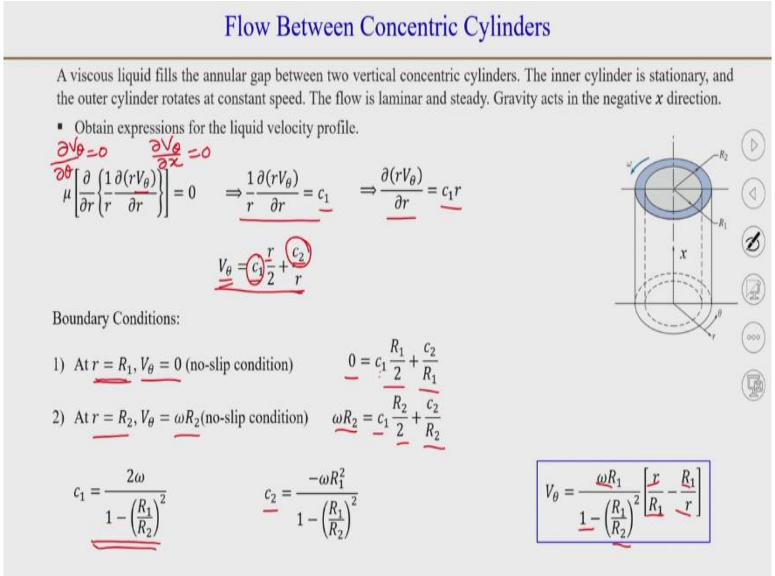

Now we will take this term and integrate it, so that we can find the velocity profile. So, we can integrate this term and we know that V_θ is not a function of θ because ∂V /∂θ = 0 and we also know that $\partial V_\theta / \partial x$ is equal 0, so V_θ is only a function of r, so we can as well use ordinary derivative in place of partial derivative.

So, when you integrate it, you will get 1 /r, $\partial/\partial r$ of rV_θ that = c1. Now when you integrate or when you simplify it, you will get $\partial/\partial r$ of rV_θ = c1r, you can integrate it further, so you will get rV_θ = $c1r^2$ /2 + another constant of integration which is c2.

And then you can find V_θ, so that will be c1 r /2 + c2 /r. So, that is the velocity profile in the annular space between the two cylinders and you can see that this is a function of r. There are two constants c1 and c2 which we need to find. So, we need two boundary conditions here and we have two walls, on the inner wall, at r = R1, V_θ = 0 because the cylinder is stationary.

The outer wall where r = R2, $V_\theta = \omega$ R2, so we can use both the boundary conditions there. So, when we use the first boundary condition that at the inner cylinder r = R1, V_θ = 0, so we will get 0 = c1, R1 /2 + c2 /R1. And the second boundary condition will give us that at r = R2, $V_\theta = \omega$ R2, that will be equal to c1 R2 /2 + c2 /R2.

Now we can simplify these equations and can find the values of c1 and c2, so you can multiply the first equation by R1 and second equation by R2 and you can eliminate c2 then once you eliminate c2 from that you will get c1, you can subtract the equation, the second equation from the equation and you will get the value of c1. Once you get c1, from this you can say that c2 = - c1 × R^2 /2, so c2 will be c1 × a R^2 /2.

So, 2 ω × - 1 - R^2 /2, so you can use, you can find the equation after substituting the values of c1 and c2 in this equation. So, you get the velocity profile which is $V_\theta = \omega$ R1 / 1 – (R1 /R2 $)^2$ × r /R1 - R1 /r. So, this is a function of r.

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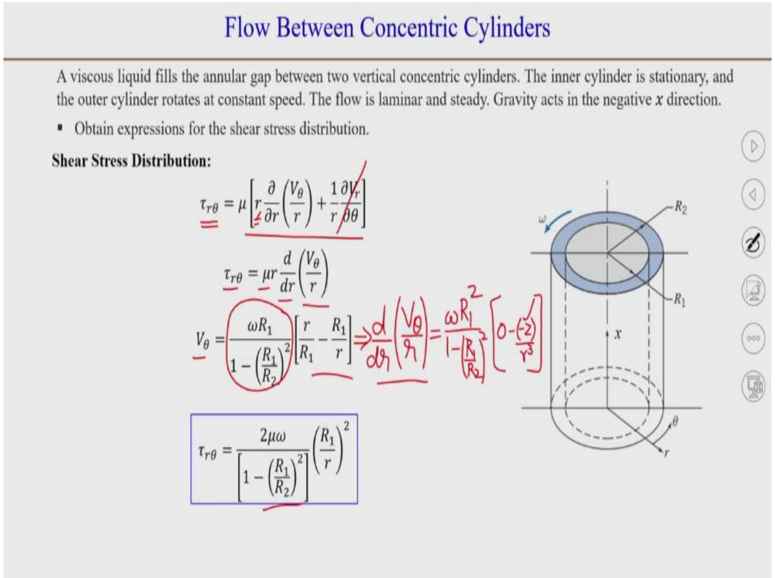

Now the next task is to find the shear stress distribution. So, to find the shear stress distribution because the flow is again along one direction only so you will have only one component or say among the 6 shear stresses you will have only two shear stresses τ r θ or τ θ r which is non-zero, so that is the general expression for τ r θ in the cylindrical coordinates.

And because ∂/∂θ is 0 or Vr is 0, so this term will be 0 and you will have τ r θ = μ × r, d /dr V_θ /r, you can use ordinary derivative because V_θ is a function of r only. So, you can use the expression for V_θ to find the derivative, so when you find derivative of V_θ /r with respect to r, then a you will need to multiply this equation by r, all this is constant.

So, you will have this constant say ω R1 / 1 – (R1 /R2)^2 × the first term in bracket will become a constant because you divide by r so it will become 1 /R1, so that will be 0 -, the second term will be R1 / r^2, so you will have to differentiate r^{-2}, when you do that so you will get - 2 × r^{-3} or you can write /r^3.

So, the derivative will be of V_θ /r, will be this expression. Now you have to multiply this with r so one of the r will be cancelled here and you will get τ r θ = μ times 2 which comes from here 2 × ω /1 – (R1 /R2)^2 × R1^2 by r, so we had a R1 here.

So, this should have been $R1^2$, so you will have $R1^2 /r^2$, because one of the r will get cancelled when you multiply by this r. So, that is the expression for τ r θ or the shear stress in between the two cylinders as a function of r.

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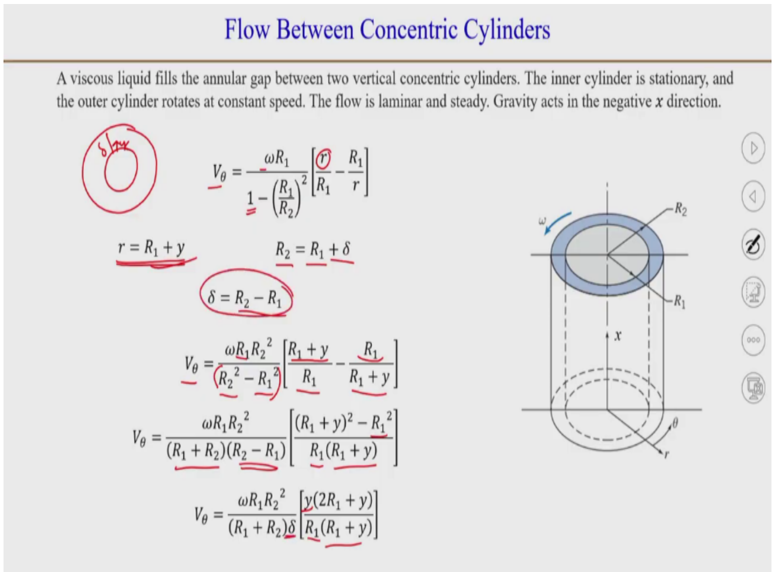

Now we will just look at the expression here, which is for the velocity and try to simplify it with some assumptions. So, if we say that r = R1 + y, so let us say you have the two concentric cylinders and you assume that the distance from the inner cylinder is y and the distance between the two cylinders is δ, so you can write r = r = R1 + y and R2 = R1 + δ, so δ is a constant here.

Now we can substitute these, or you can also have a δ = R2 - R1, so when you substitute this in the expression for velocity you will get V_θ = ω R1 as it is, now we can simplify this so we can write this $R2^2 - R1^2$, the terms in the bracket we will replace r /R1 + y here, so this will become R1 + y /R1 - R1 /R1 + y.

Now we can write this R2 + R1 × R2 - R1, so that is what we have done here. Then we can simplify the term in the brackets. So, when you take LCM it will be R1 × R1 + y of the first term become $R1 + y^2 - R1^2$. So, you can simplify the term in the bracket again. So, this will remain R1 × R1 + y, when you write this it will be R1 + y - R1 which will be y only and you will have R1 + y + R1 so that will be 2 R1 + y.

Now you can also replace that R2 - R1 = δ here. So, we have replaced the R2 - R1 with δ.

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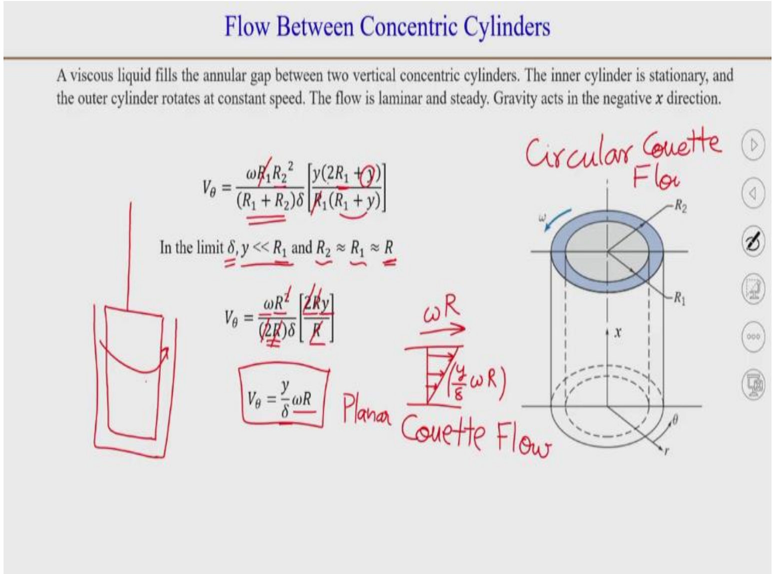

And if we assume that the distance between the cylinders or the space between the cylinders is very thin or the film between the cylinders is very thin the gap between the cylinders, which is δ is very small when you compare with R1, so in that case what will happen that you can also assume that R2 and R1, they are approximately equal. So, you can write them equal to R.

Now if we substitute this approximation in here, so what you will get you can write R1 + R2 in the denominator here as 2R, this R1 and this R1 will get cancelled, you will have ω and $R2^2$, you can write as R^2, because y is small when you compare with R1, so you can neglect this y and this expression will be 2 R1 or 2R, so 2R multiplied by y /R1 + y, you can write as R1 or R here.

So, you can write this as, now R and R will cancel out. So, what you will be left with 2 will cancel out and what you will be left with as y /δ × ω R. So, y /δ is, so if you look at this, the flow between two parallel plates and the upper plate moving with a velocity ω R, so the linear velocity profile you will have a y /δ ω R.

So, in the limit when the gap between the cylinders is very small when compared with the radius of the cylinder, then you can very well assume the flow to be of, flow between or as the flow between two parallel plates. So, you can think of, that means what does this suggest, approximation suggest that you have a very big cylinder and at the top of cylinder or in between the cylinders, the efect of curvature will be very small and you will have this expression to be valid.

So, in the viscometer, when one have the viscometer arrangement generally what happens that there is a cylinder, a outer cylinder and in this outer cylinder the liquid is filled and the inner cylinder is introduced and it is hanging by a wire here. Now one can rotate this cylinder using a motor arrangement or one, so if can rotate at a specific angular velocity, so you can have a ω R fixed and because of that there flow between the two cylinders.

So, this arrangement is slightly different from the problem we just discussed, in that the inner cylinder moves her and the outer cylinder is fixed, but that is a small change in the boundary conditions and we can obtain the expression for this case where the outer cylinder is fixed and inner cylinder is moving.

So, such arrangement quite often as a viscometer and you remember that this arrangement where the flow was shear driven between two parallel plates, we used to call it as Couette flow, so this arrangement is called when there are two infinitely parallel or two parallel infinitely long and infinitely wide plates, we call it planar Couette flow, where is when it is the flow between two infinitely long cylinders and one of the cylinders is moving or they may be moving, they might be moving, both of them might be rotating and their angular velocity are different.

You can find such a general case also, but then the boundary conditions will be that at the inner cylinder you will have r = R1, V_θ = ω R1 and the ω R1 and at the outer cylinder you will have V_θ = ω 2, R2, so you will be able to find general expression. The flow, this such kind of flow is generally known as circular Couette flow.

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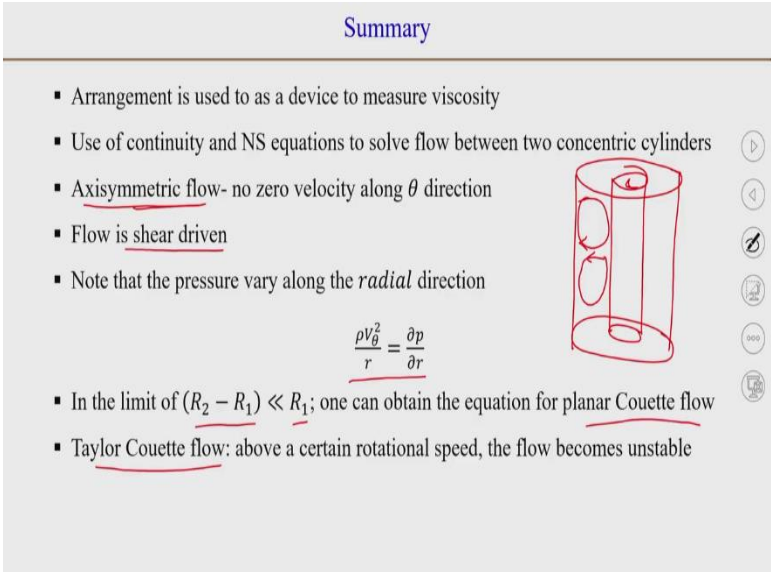

So, if we summarize what we have discussed today, we discussed a co-annular arrangement of circular cylinders where the cylinders are coaxial and found the velocity profile when the outer cylinder is rotating and have used or simplified that mass and momentum conservation equations, used the boundary conditions for such case and we noted here in this case that even though the flow is axisymmetric, but V_θ is non-zero.

So, a $\partial p / \partial \theta$ was 0 and all the three velocity components, their derivative along the θ direction will be 0, that is what the axisymmetric flow mean. And in this case there was no pressure gradient to drive the flow. The flow was driven by the shear, the pressure vary along the radial directions, so we obtained this relationship and we can substitute the value of V_θ here and also calculate what is the pressure gradient along the radial direction.

And we have also seen that in the limit when the distance or when the separation distance between the two cylinders is very small as compared to the radius of the cylinder, then we obtain the equation for a planar Couette flow. The velocity profile is same that in case of a planar Couette flow. Now when the cylinders are moved with a very high rotational speed, then what happens that above a certain speed.

Let us say if the inner cylinder is being rotated with a speed, so above certain speed there are vortices forming, the flow becomes unstable and there are vortices forming and these vortices are known as Taylor vortices after a Taylor, after whom this cylindrical arrangement is also named as Taylor Couette flow, so looked at the instability in such cylindrical arrangement. So, we will stop here. Thank you.