Fundamentals of Fluid Mechanics for Chemical and Biomedical Engineers
Professor Raghvendra Gupta
Department of Chemical Engineering
Indian Institute of Technology, Guwahati
Lecture 27
Fully Developed Flow in a Circular Channel

In this lecture, we will discuss fully developed flow which is laminar in a circular channel or a cylindrical channel. Now a circular channel or cylinder is a very frequent occurrence in our day-to-day life in chemical engineering applications where we encounter a number of pipes whose cross-section is circular or in biomedical applications, for example, the blood vessels, arteries, arterioles, capillaries, all of them are of circular cross-section.

Some of them may be flexible. But as a first approximation, we can use the laminar fully developed flow relationship to find out the pressure drop, for example, or for a given pressure drop the flow rate etcetera. So, we will look at this very important topic from chemical and biomedical engineering perspective as well as in general.

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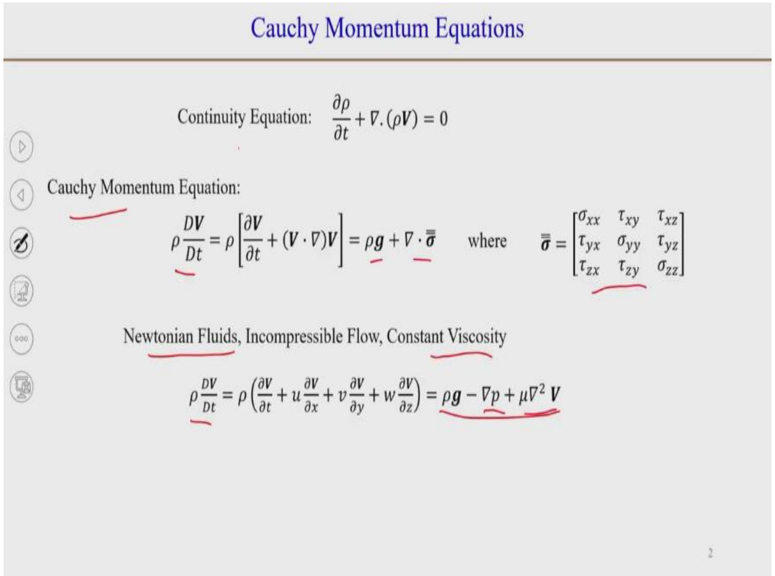

So, let us remind ourselves first the momentum and mass conservation equations. So, this is continuity equation and this is Cauchy momentum equation where we have a ρ, a substantial derivative of velocity, so D/Dt of velocity vector, that will be equal to gravity force, which is body force + divergence of stress tensor, which is a second-order tensor having 9 components as listed here.

And when we substitute it for Newtonian fluids and assume the fluid to be incompressible as well as having constant viscosity, it simplifies in the vector form like this, where we have the stress components getting dissolved × pressure which is the dominant normal stress, generally viscous normal stresses are negligible and from the viscous shear stress we get $\mu\, \partial^2 V$. So, $\partial^2 V$, actually will have viscous normal stress as well.

So, these are the conservation equations in the vector form, but we know that when we have a circular channel or a cylindrical channel of circular cross-section, it is convenient to use a cylindrical coordinate system for such a case, so what we will do, we will expand these system of equations of continuity as well as Navier-Stokes, which are momentum conservation equations in the cylindrical coordinates and then apply the assumptions and obtain the relationships for velocity profile etcetera there.

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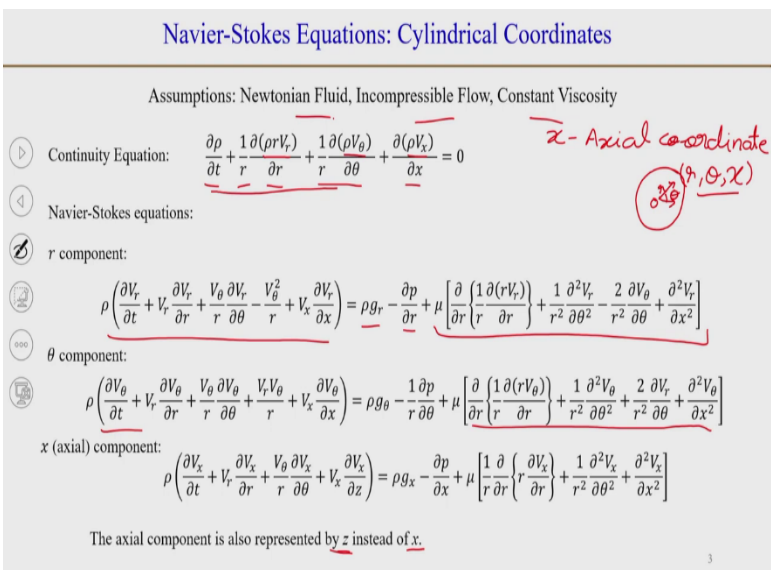

So, assuming all the, all these that the flow is Newtonian, it is incompressible and the viscosity is constant, we can expand the continuity and momentum equation in the cylindrical coordinates. We have already seen when we discussed the continuity equation that how this comes about from a vector equation or mass conservation in the vector form, how we can find out the mass conservation in terms of cylindrical coordinates.

So, the first remains same, ∂, ∂/∂ t of ρ + 1/r ∂/∂ r of ρ r Vr, so you may note here that it is ρ r Vr and the next term 1/r, ∂/∂ θ of ρ V_θ and + ∂/∂ x ρ Vx. So, note here that what we have used here is x is the axial coordinate. So, when we have a cylindrical coordinate system, we will have at an arbitrary distance at an angle θ at a distance r from origin, you will have r θ and let us say z.

So, many times we use z of or you can also use x because there is no x, y otherwise here. So, r θ and x coordinate are axial coordinate are cylindrical coordinate system that we are going to use here. So, this is continuity equation. Then Navier-Stokes equation for r component you will have left-hand side, which is the fluid acceleration and gravity term, pressure gradient term and the system expanded here.

We will not derive these, you can find such form of equations in standard textbooks and using the same procedure that we derive the equation for a Cartesian coordinate system, these equations can be derived, but they are slightly more cumbersome to find. This is the equation for θ component, especially this is the bit which looks very different and the axial component.

So, let us look at how we can, using these equations, how we can simplify them for a fully developed flow in a circular channel. Here we have used for axial coordinate x in many textbooks or in many places you will also find this in terms of z. So, it might be r θ x coordinate or r θ z coordinate, x or z are axial coordinate.

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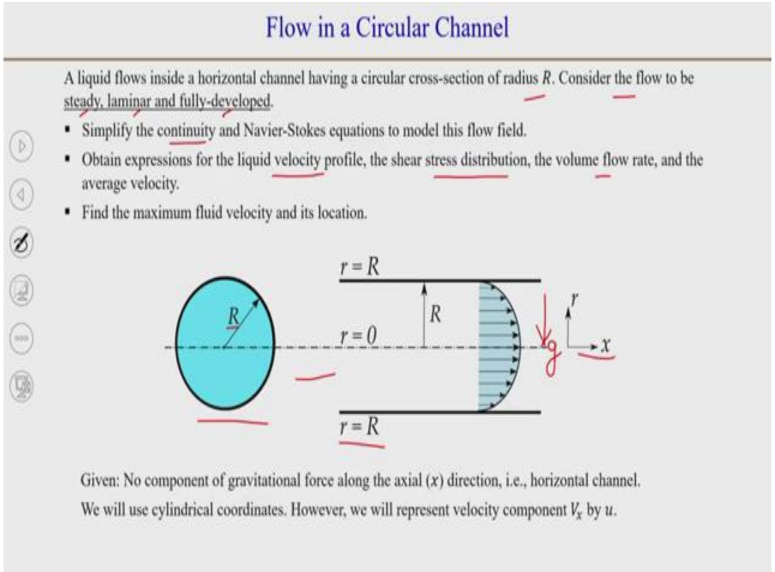

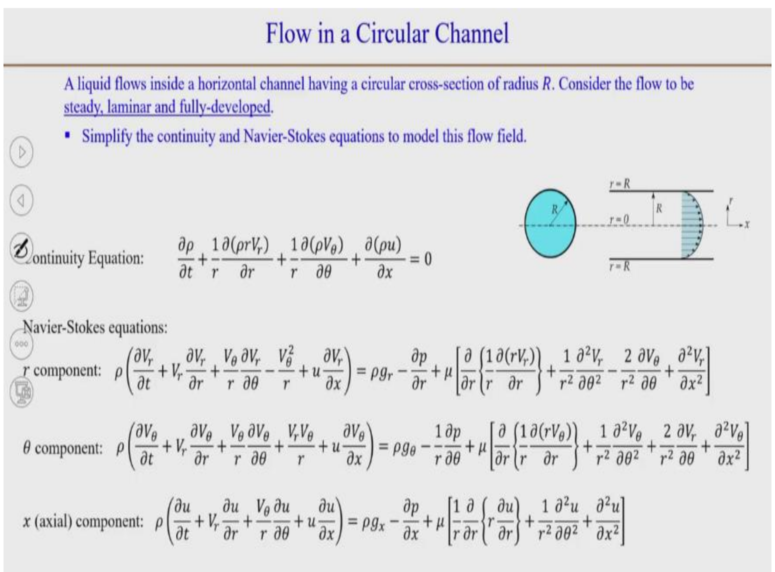

So, we can define the problem in such a manner that a liquid flows inside a horizontal channel having a circular cross-section which has radius R and we can assume the flow to be steady. So, it is time independent, the flow is laminar and flow is fully developed. So, first, we will simplify the continuity and momentum conservation equations.

So, find out that what are the simplified equations from the three equations in r θ z coordinates or r θ x coordinates and the continuity equation. And then once we have simplified we will obtain the velocity profile of the liquid. We will obtain the shear stress distribution, the volume flow rate and average velocity. From that we will obtain what is called the Hagen–Poiseuille equation which relates pressure drop with the flow rate and we will also find the maximum fluid velocity and where does it occur in the channel.

So, this is a schematic diagram where we see the cross-section and the radius of the channel is R and we take at R = 0 we can keep the origin and the radius is R so wall is at r = R and the flow happens along the axial direction. Now the flow is in x direction and the pipe is placed horizontally so the gravity will act in the vertically downward direction and it will have components in and r and θ directions.

In the equations, in the previous slide we used Vx as the x component of velocity, here we will use in place of x simply u as x component of velocity to make things easier for typing. So, let us write down the continuity and momentum conservation equations, like same equations that we saw in the previous slide, so we will skip them through and let us start with the assumptions that we will have for a fully developed steady laminar flow in a channel.

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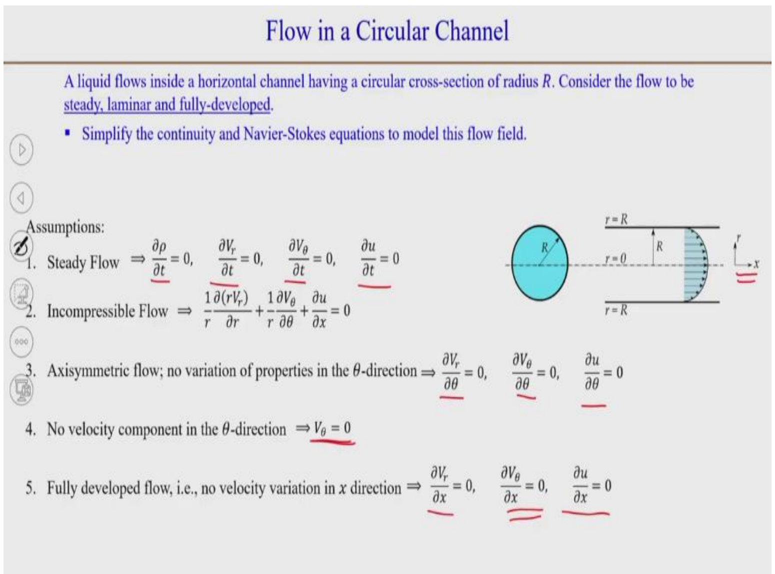

So, first assumption that the flow is steady, that means that all the time derivatives will be 0, so in the continuity equation we will have ∂/∂ t of ρ is 0, ∂/∂ t of Vr, V_θ and u, which is axial component of velocity, all of them are 0, then flow is incompressible, so ρ is constant and our continuity equation will become simplified.

We can take ρ out of the derivatives and because of steady or incompressible flow ∂/∂ t term will anyway become 0. Then flow is axisymmetric, so axisymmetric means that the flow is symmetric about the axis, so there is no flow along the θ direction, and there are no gradients along the θ direction, so ∂/∂ θ term or ∂^2/∂ θ 2 term will be 0 as well as V_θ will also be 0. So, we can write that all the velocity components, their derivative with respect to θ will be 0 and there will be no velocity component in the θ direction.

Then flow is fully developed, so flow is fully developed, that means the velocity profile along the x direction remains independent of x coordinate that is the direction in which the flow happening. So, that means the derivative of velocity components along the x direction are 0 or with respect to x direction. So, ∂/∂ x of Vr = 0, ∂/∂ x of V_θ = 0 and ∂/∂ x of u = 0. This will become in significant ∂/∂ x of V_θ = 0, because we already know that V_θ = 0.

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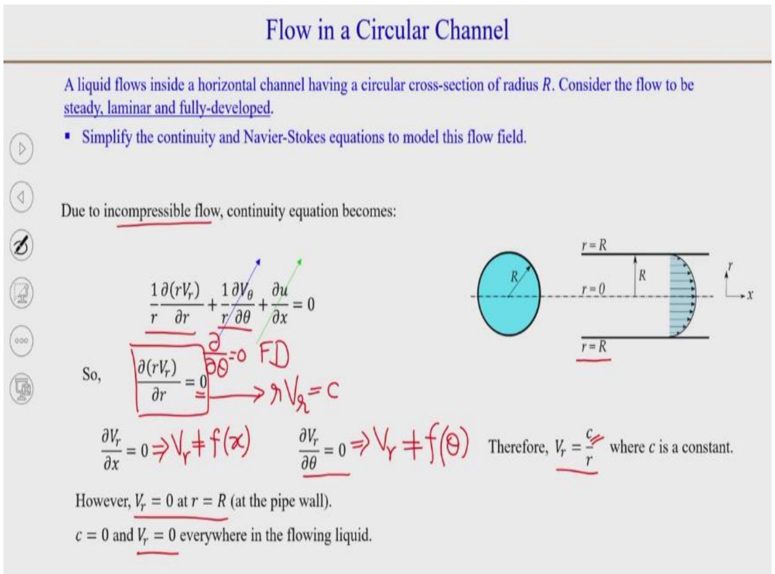

So, with listing down all these assumptions, now we can start simplifying the equations. So, we will begin with continuity equation which we have already written in the form for an incompressible flow. So, in this if we see all the derivative with respect to θ, they are 0, so this term becomes 0 and flow is fully developed, so the derivative with respect to x are 0, so both of these terms are 0 and we will end up with this term only 1/r ∂/∂ r of rVr, that = 0.

Now. if we look at Vr, ∂/∂ x of Vr = 0 because the flow is fully developed and the gradients along the θ direction is 0, so ∂/∂ θ of Vr = 0. So, this suggest that Vr is not a function of x and from here we can conclude that Vr is also not a function of θ or it is independent of θ.

So, if we look at this equation now and we can integrate it to find that rVr is a constant, so when we integrate this equation we get rVr = constant. Let us say this constant is c, so we can write down that Vr = constant/r, so Vr = c/r. Now as we have done earlier for some of the problems that we solved in Cartesian coordinate system.

For example, flow of a falling liquid film or flow between two parallel plates that we saw that for the transverse component for the velocity component for the y component of velocity, velocity at the wall = 0. Similarly, we can use Vr = 0 at r = R. So, that means because Vr = 0 at r = R that means c has to be 0, because at r is equal R, Vr will be c/ R and if it is 0 that means constant c = 0.

So, now we know that Vr = 0, so Vr = 0, V_θ = 0, that means flow is one dimensional, we have only one component of velocity which is non-zero, which is along the axial direction or u is non-zero. So, this is what we have obtained by simplifying the continuity equation.

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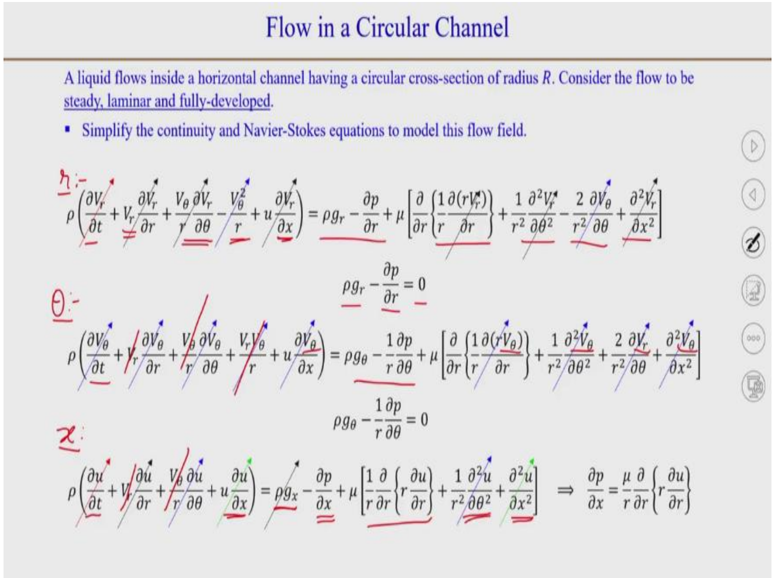

Let us is look at the momentum conservation equations. We start with the momentum conservation equation along the r direction. So, we know that the flow is steady, so this term is going to be 0.

Now we know that Vr = 0, so this term will be 0. We know that derivative with respect to θ is 0, or v_θ is 0, so this term is 0, and then we know that ∂/∂ x for the velocity components are 0, so this term is also 0, Vr = 0, so we can write this term to be = 0, Vr = 0, so we can write this term to be equal to 0 and Vr = 0, so this term = 0.

Now V_θ is also 0, so this term, all the terms which have V_θ are also 0. So, we will simply obtain from this that ρ gr - ∂/∂ r of p = 0, so that is the simplified form of momentum conservation equation along the r direction. Now we write the momentum conservation equation along the θ direction and see how we can simplify it. So, we will start with that the flow is study.

So, this term is 0, then we can use the fact that Vr = 0 and V_θ = 0, so this term is 0, v_θ = 0, so this term is equal 0, so Vr_θ are 0, so this term is 0, and V_θ = 0, so this term is equal 0, V_θ is 0, so this is also 0, this are 0 because V_θ = 0, this term is 0, because Vr = 0 and this term is 0 because V_θ is 0.

So, again, we have two terms, one because of gravity and the other because of pressure, so we can write ρ g_θ - 1/r, ∂/∂ θ of p = 0. So, that is the simplified form that we obtained from a θ momentum equation or momentum equation along the θ direction. Then we can write the momentum equation along the axial or x direction and we can start using the assumptions.

So, the flow is steady, this term is zero because Vr = 0, so this term will become 0 and V_θ = 0, so this term will also become 0, flow is fully developed, so this term will become 0, and we do not have any component of gravity along the axial direction, because we have assumed the flow to be horizontal, so this is ρ gx will also be 0, because gx = 0.

And again, the derivative of u with respect to θ will be 0, so this term is 0, and because of the fully developed flow derivative of u with respect to x are 0, so this term is 0. So, now are left with the one term in the pressure gradient in the form of pressure gradient - ∂/∂ x of p and another viscous term 1/r ∂/∂ r of r ∂/∂ r of u. So, we are left with these three simplified equations.

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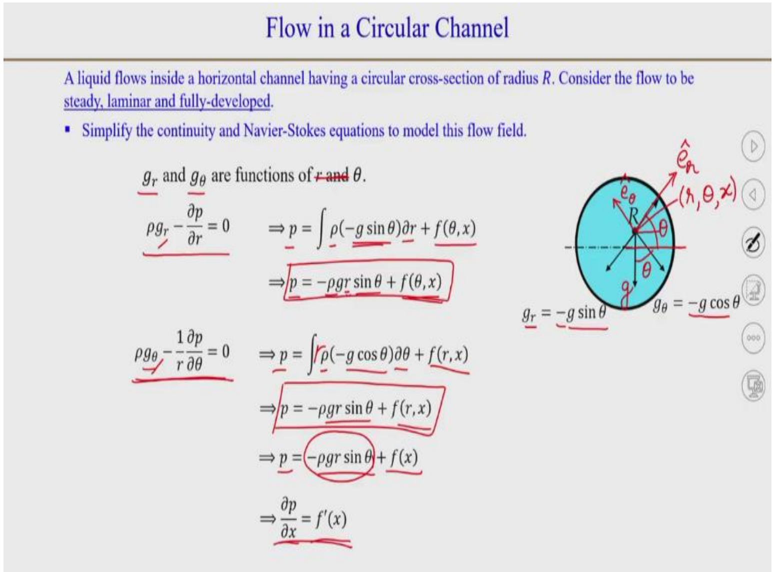

Now let us look at the two equations from r and θ coordinates, r and θ coordinates, so we get from r momentum equation, ρ gr - ∂/∂ r of p = 0, from θ momentum equation we got ρ g_θ - 1/r, ∂/∂ θ of p = 0. Now if we look at gr and g_θ because the flow is horizontal, so the will be function of actually θ not of r.

So, if we look at, we take a point here which has coordinates, so we can say that the coordinates of this point r, r θ and x, so at this point, the angle will be θ and the gravity acts in vertically downward direction. So, we can, because the angle between the horizontal line and this line is θ, so vertical line and normal to it this angle will be θ, and this will be my direction of unit vector along θ direction and this will be direction of unit vector along radial direction.

So, the gravity, when we find out gravity component along the θ direction, the magnitude of this component will be g cos θ and it will be in the negative θ direction, so that is why we have a - sign there. Now similarly for, for the component of gravity along the radial direction, the magnitude will be g sin θ and it is in the negative direction.

So, we will have - g sin θ. Now a with this we can substitute the values of gr and g θ here, so if you substitute the value of gr and integrate, we will get p =, or we will get ∂ p/∂ r = ρ gr and when we integrate we will get ρ gr, so in place of gr we can substitute - g sin θ and ∂ r and when we integrate, we will get a function which is independent of r, but it can be a function of θ and x, so on integrating this equation, we will get p = d sin θ is independent of r.

So, we can write this, ρ g sin θ with - there and on integrating we will get r, + a function of θ and x in general. Now we can integrate this equation, so when we integrate, we will get p = ρ and in place of g_θ we can substitute - g cos θ, ∂ θ, + because we are integrating with respect to θ, so the constant can be a function of r and x in general.

Now when we integrate this, we have missed a r here, so there should have been a r, so that p = - ρ gr sin θ + f of r, x, so if we compare the two expressions that we have obtained for p that is - ρ gr sin θ + f of r, x, - ρ of gr sin θ + f of θ x, so from this we can say that the function will be a function of x only, it will not be a function of θ and r, so the pressure will be function of - ρ gr sin θ + f of x.

Now if we differentiate this with respect to x, pressure with respect to x, then we will get ∂ p/∂ x = f′ x and the thing to note here is that this term is not a function of x, so its derivative with respect to x will become 0, and this relationship which tells us that partial derivative of p with respect to x is only a function of x, this we will use in the subsequent analysis.

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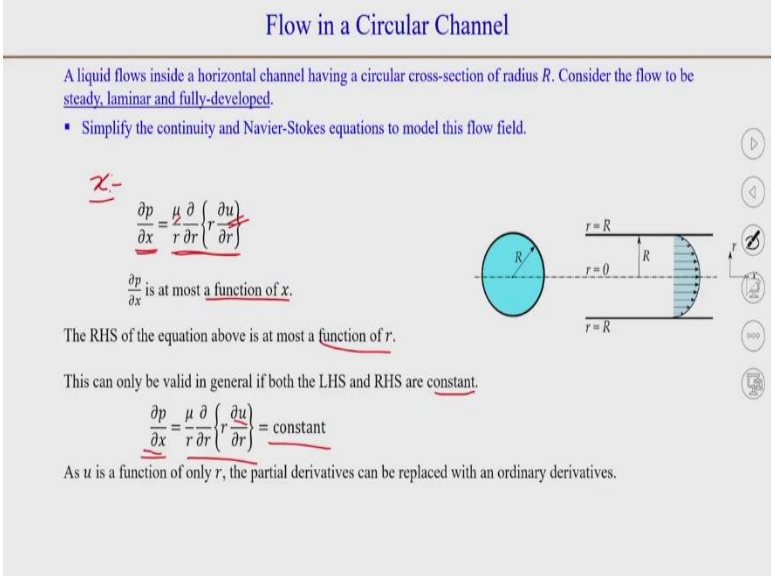

So, we have written this the simplified form of the momentum equation along the x coordinate. Now this we obtained that - ∂ p/∂ x + μ/r × ∂/∂ r of r ∂ u/∂ r and if we rearrange it, we will get this form. Now as we saw that the left-hand side which is ∂ p/∂ x is a function of x at most, it is not a function of r and θ. Now the right-hand side is the derivative μ is a constant and this is a derivative of u multiplied by r and so on.

So, u is not a function of θ because we saw that ∂ u/∂ θ = 0 because the ∂/∂ θ for all the velocity components is 0, u is also not a function of x, because ∂ u/∂ x = 0 because of fully developed flow, so u can be a function of r only. So, if that is the case we say that the left-hand side is a function of at most of x and right-hand side can be a function of at most r, that means both of them are going to be equal, and they are constant.

So, they are neither function of x nor of r, but they will be constant. So, we can write this that ∂ p/∂ x are equal to the right-hand side that will be equal to a constant. Now because u is only a function of r, it is not a function of θ and x, so we can write these partial derivatives in terms of ordinary derivatives. Similarly, for ∂ p/∂ x, we can write it as dp/dx.

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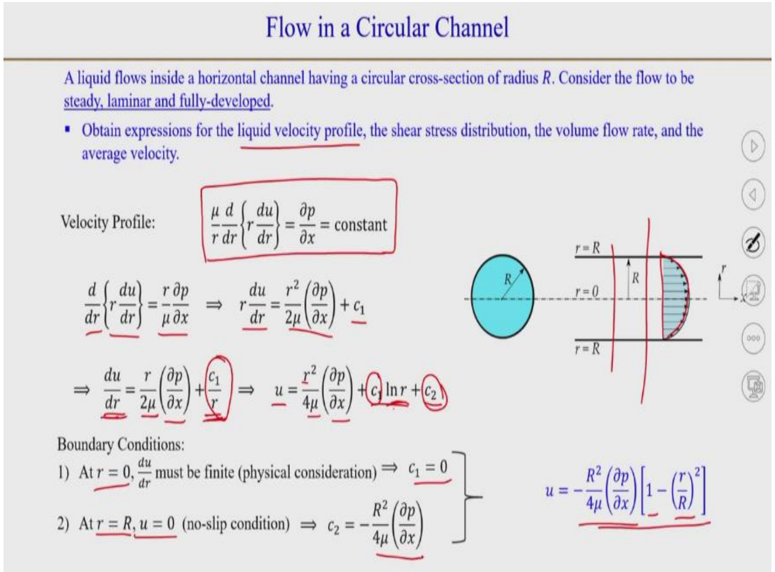

So, now we have simplified using all the simplified equations, the continuity equation, the momentum, all the three momentum equations, this is the equation that we obtained. Now we can integrate it, and see what we get. So, when we integrate this equation, we will get or we can rearrange it, so r will go this side and μ will go in the denominator, we will get d/dr of r du/dr = r/μ, dp/dx or ∂ p/∂ x.

Now, on integrating, we will $r^2/2\mu$ ∂ p/∂ x + a integration constant, let us say this constant is c1 and that will be to r du/dr. We can integrate it further, but before we will to divide it by r, so we can get du/dr = $r^2/2\mu$, 2μ will become r/2μ, when we divide it by r × ∂ p/∂ x, + c1/r.

Now when we integrate it further, then we will get r, on integrating r, we will get $r^2/2\mu$, so $r^2/2$ and there is already 2μ, so it will become $r^2/4\mu$ × ∂/∂ x of p, + when we integrate 1/r, we will get ln r, so c1 ln r + another constant of integration, this we call c2. Now our next task is to find out these two constants, c1 and c2 and to find these we need two boundary conditions.

So, we have the first boundary condition will be that at r = 0, if you look this equation at r = 0, du/dr will become infinite if c1 is non-zero, right? So, this for du/dr to remain finite, we will need to have c1 = 0 or you can look at from here that for this to remain physical, ln r to remain definite or for u to remain finite at r = 0, c1 has to be 0.

And another boundary condition which is at the wall, so at the wall we have r = R, which is the channel wall at this, because of no-slip boundary condition u = 0 or axial component of velocity is 0. So, from the first consideration we get c1 first constant of integration = 0. And from the second consideration, we will get u = 0 at r = R, so we will get $R^2/4\mu \times \partial$ p/∂ + c2 = 0 because this term has become 0 now, because c1 is 0.

So, we will get c2 = - $R^2/4\mu$ ∂ p ∂ x and we substitute these two, so we can write down this - $R^2/4\mu$ ∂ p/∂ x × 1 - r^2/R^2. So, that is the velocity profile in the liquid. At any section in this pipe, we will find that the equation of this velocity profile is represented by this equation, you noticed it that this is the equation of a parabola. So, that is why it is called that the fully developed laminar incompressible flow in a circular channel is parabolic, where velocity profile is parabolic.

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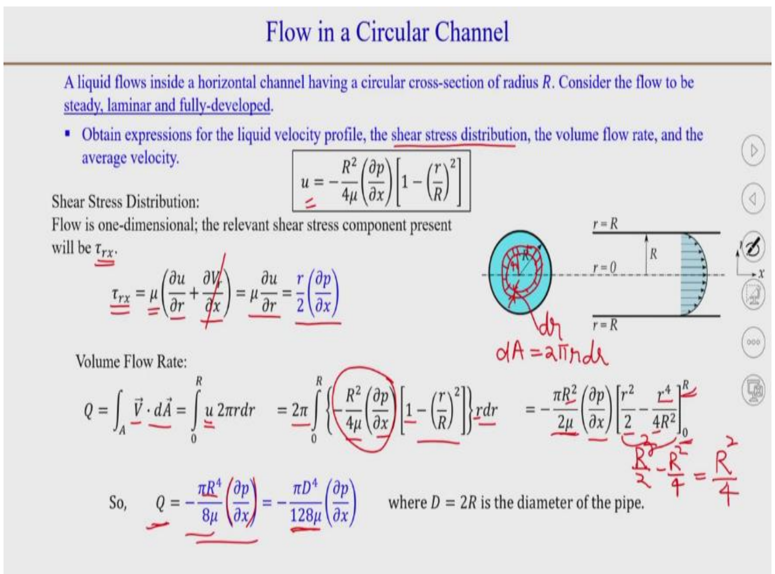

Now the next thing is that we need to find the shear stress distribution and we know that the flow is one dimensional so only one component of shear stress will be present, which will be τ rx. So, τ rx, if we write down the expression for it for a Newtonian fluid in terms of rate strain, so that will be μ × ∂ u/∂ r + ∂/∂ x of Vr and ∂/∂ x is 0 or Vr is 0.

So this term is 0, so we will have this simplified to be that τ rx which is the shear stress distribution that = μ × ∂/∂ r of u and when we write down this, the derivative of it, we found just in the previous slide that = R/2μ ∂/∂ x of p. So, we get τ rx = r/2 ∂ p/∂ x.

Next we need to find the volumetric flow rate. So, the volumetric flow rate can be found when we integrate V dot dA over the cross-sectional area because the velocity varies along the r direction, so we can take a elemental area at a distance r from the origin and the thickness of this elemental area is let us say, dr, so the area of this strip will be dA = 2π rdr, so we can substitute the value of u from here, which is - $R^2/4\mu\ \partial/\partial$ x of p × 1 - r^2/R^2 × 2π rdr, so 2π is a constant. It can come out of integration multiplied rdr.

Now we can integrate all of this is constant because ∂ p/∂ x is independent of r, so we can take this out of the integral sign and this will become 2π × - $r^2/4\mu$, so when we simplify it, it will become - π $R^2/2\mu$ × ∂ p ∂ x and we need to integrate. So, this will, the first time when we integrate this multiplied by r, so when you integrate r you will get $r^2/2$.

The second term, r^2/R^2, R^2 is of course a constant when you multiply it by R, you will get r cube, so when you integrate r cube you will get r^4 / 4 and we will substitute the limit from 0 to R which is 0 is at r = 0 at the axis, R is radius at the wall.

So, when we substitute this, the first term will become $R^2/2$ and the second term will become $R^2/4$, so this will give you $R^2/2$, so you will get Q = -, sorry from here you will get $R^2/2$ - $R^2/4$, which will give you $R^2/4$. So, when you substitute this, you will get - π r^4 because r^2 × r^2 that will come r^4 × 2μ × 4 will become 8μ × ∂ p/∂ x, that is the relationship for the flow rate.

And it can be written in terms of diameters, so when you replace R = D/2, so R^4 will be D^4/ 16 and 16 × 8 will be 128. So, you will have Q = - π d^4/ 128μ × ∂ p/∂ x. So, that is the relationship for the volumetric flow rate. You can also change it to mass flow rate, so you can multiply this by density, Q by density.

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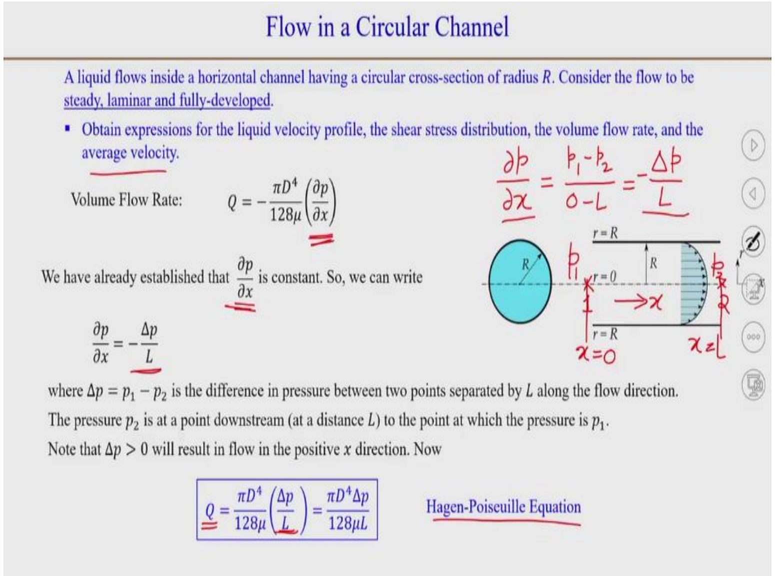

And now next task is to find out the average velocity which will be the volumetric flow rate divided by the channel cross-sectional area. So, before we do that, we can see here that ∂ p ∂ x is constant, when we saw the left-hand side of the partial differential equation and right-hand side are equal then we established that ∂ p/∂ x is a constant and we could have written it as a constant.

Now we write this as Δp/L, which is the pressure gradient. So, if you have two points, let us say point 1 and point 2 in a channel. And let us say pressure at point 1 is p1 and pressure at point 2 is p2, because we have positive x direction along this direction, so let us say at point 1 we have x = 0 and at point 2 we have x = L, then we can write ∂ p/∂ x or dp/dx, that will be equal to p1 - p2 equal to 0 - L.

So, if we say that p1 - p2 = Δp, then we will have - Δp/L. So, ∂ p/∂ x is - Δp/L, we know that the flow happens from higher pressure to lower pressure, so Δp will be a positive number and L is the length, so that will also be a positive number so the pressure gradient ∂ p/∂ x will be negative.

So, we need to remember that and we can write this ∂ p/∂ x simply in terms of - Δp/L and the relationship will change now the - sign will go away and you will have Q = π^4 / 128μ × Δp/L. And this equation is known as the famous Hagen–Poiseuille equation which relates Q with Δp/L or the pressure drop per unit length with the volumetric flow rate in a circular channel.

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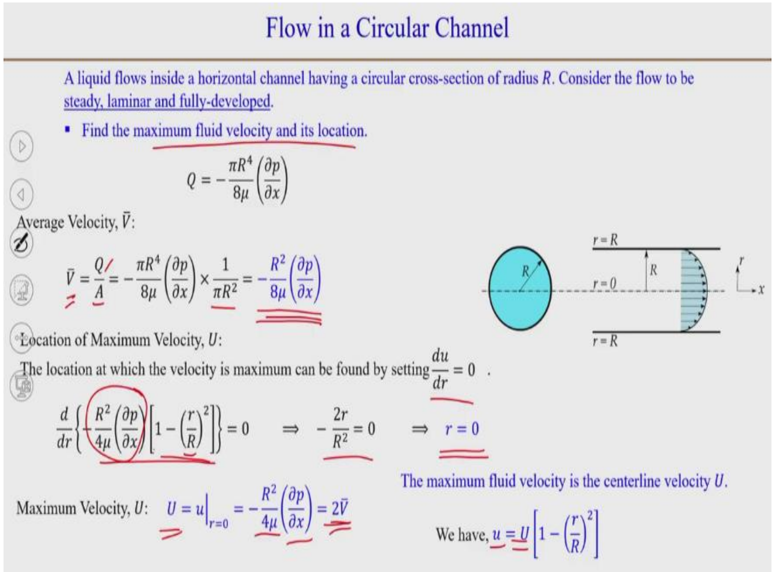

And next we need to find the average velocity. So, to find the average velocity we can divide the volumetric flow rate by the cross-sectional area which is πr^2 and we will get that $\bar{V}$ = - $R^2/8\mu$ $\partial p/\partial x$, so if you look at the expression for the velocity profile then we can write this = V average × 2, because we just so that V average equal to $R^2/8\mu \times \partial p/\partial x$.

So, we can write that u = 2 × 2 times of average velocity × 1 - r^2/R^2. So, that is an easier expression to remember. Now we also need to find the maximum fluid velocity and its location, so we can find this by writing the expression for du/dr and setting it equal to 0, so the expression of u, you can derive it with respect to r and find this = 0, and when you derive this is all constant.

So, the first term is also a 0 because 1 is a constant, so derivative of 1 with respect to r will be 0 and when you to differentiate this you will get - $1/r^2$ × differentiation of r^2 which will be 2r, so you will get - $2r/R^2$, that means the derivative is 0 at r = or du/dr is 0 and r = 0.

So, that means the velocity is maximum at r = 0, that you could have seen from here itself that if you substitute this term is going to be maximum, because this term is positive because - $\partial p/\partial x$ is positive. So, this term, the entire term in the bracket is going to be maximum when we have r = 0, so at r = 0 the velocity is maximum.

So, you could find the maximum velocity by either way, the location of maximum velocity and the expression of this will be - $r^2/4\mu$ $\partial p/\partial x$ which is also equal to you can see from here that = twice of average velocity, so the maximum velocity twice the average velocity. So, we can write that the velocity profile = U which is the maximum velocity × 1 - r^2/R^2. So, that is the velocity profile.

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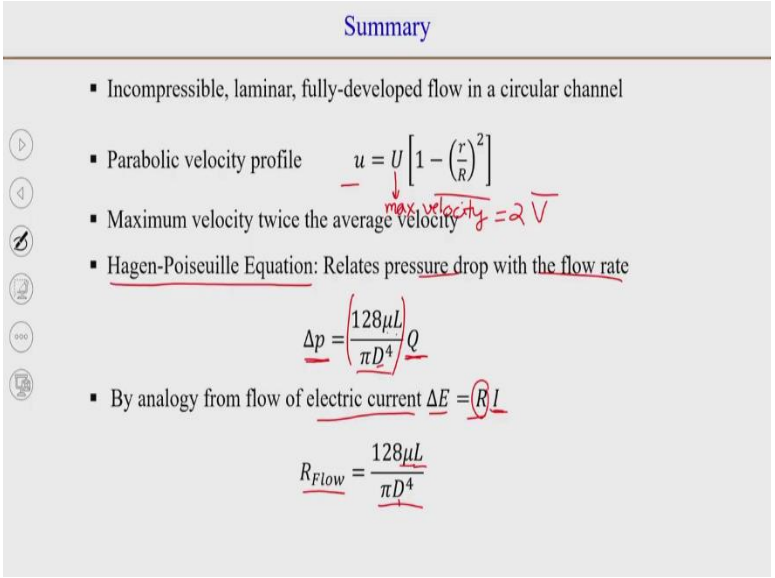

So, we have learnt quite a bit of things today. Let us just summarize that, that for incompressible laminar fully developed flow in a circular channel, the velocity profile we obtain is u = U × 1 - r/R, R^2/R^2 and this is maximum velocity, U is maximum velocity. And that is also equal to twice of average velocity.

Now we have derived the Hagen–Poiseuille equation which relates pressure drop with the flow rate and this is the expression. So, we have, we can rearrange it to find out pressure drop in terms of flow rate, which = Δp = 128 × μ × L /π, D^4Q, where μ is the fluid viscosity, L is the length of the pipe and D is the diameter of the pipe.

Now you can have it in analogy or analogue with the electric current relationship where we have the voltage difference = current times resistance and here the flow happens because of pressure difference. So, we have pressure difference = the flow of fluid or the flow rate of which is Q, here the flow rate of charge is the current, so we have a analogue here.

So, by analogy we can say that the flow resistance R can be represented by 128μ L /π D^4, and this is used quite a bit in human physiology to find out the relationship or find out the relationship for the flow resistance. So, this = $128\mu \, L / \pi \, D^4$. You can notice here that the resistance to flow is proportional to viscosity and it is proportional to length.

So longer the channel, more the resistance, higher the viscosity, more the resistance and you can see that it is inversely proportional to D^4, that means smaller the channel, smaller your D, smaller the channel size, more is the resistance to flow and it increases very rapidly because you have D^4. So, for the same flow rate if you look at the pressure drop will be high in the channels of small diameter. So, that is all we have for this lecture. Thank you.