Fundamental of Fluid Mechanics for Chemical and Biomedical Engineers

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Lecture 25

Flow between Two Parallel Plates

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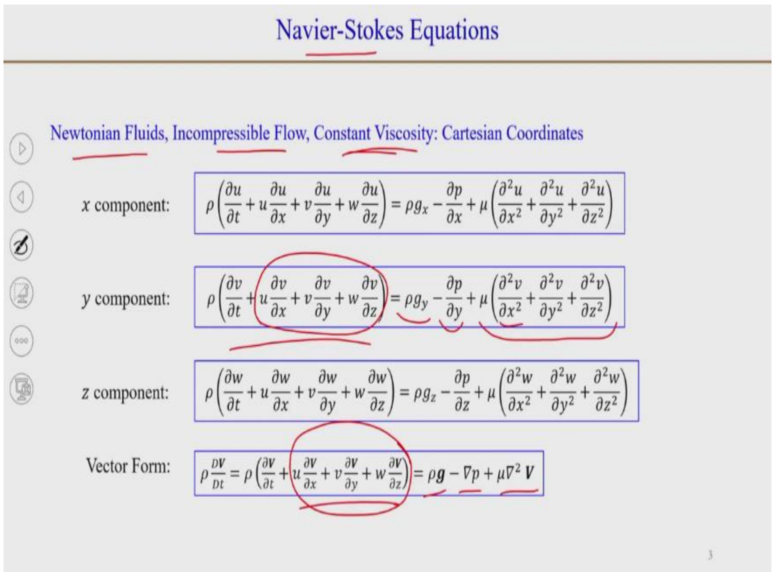

In the previous lecture, we discussed about the derivation of momentum conservation equations in the differential form, which we call Navier-Stokes equation for a Newtonian fluid. So, in this lecture, we are going to discuss a few examples and how to use the assumptions or simplifications to simplify the problems and find analytical solutions.

Of course, as I said earlier, that these equations are nonlinear and it is not possible to find the solution of these nonlinear partial differential equations for each and every case, and there are only certain instances where one can find the solution of these equations. So, let us just look at the equations. So, these are the x, y, z component which have stresses in there.

So, this is the Navier-Stokes or the Cauchy momentum equation, so to say in the general form where it is not necessarily that the fluid is Newtonian or non-Newtonian. So, on the left hand side we have acceleration, the first term is of course local acceleration, which is derivative of the velocity with respect to time and then convective acceleration, which is V.∇ of the velocity component and then gravity term and the stress derivative.

So, here the stresses will have normal stresses as we can see here in all the three x, y, z component equations and as well as shear stresses τ here. And we can also write this in the vector form, the acceleration, local and the convective acceleration, the gravity term and the divergence of second-order stress tensor where the stress tensor is in this form.

So, when we substitute the constitutive equations, constitutive equations means the relationship between stress and rate of strain for a Newtonian fluid in these equations and assume the viscosity to be constant. Then these equations assume the form what we call of the famous Navier-Stokes equations. So, the assumptions here are that the fluids are Newtonian, the flow is incompressible and the viscosity is constant.

In that case, the left-hand side is still same, that is ρ multiplied by the acceleration term. On the right-hand side, we have a gravity term or pressure gradient term and viscous stresses. So, you will have a viscous normal stress as well as viscous shear stresses here and the viscous shear stress will generally be larger in magnitude as compared to viscous normal stress, which is negligible in most of the cases.

You can write this in the vector form, again the viscous term, pressure term, gravity term, the accelerations. So, now we look at some of the special cases of Navier-Stokes equations. Of course, the fluid is Newtonian, incompressible at constant viscosity. But an additional assumption is that the flow is creeping flow. So, the inertial term is negligible.

So, if you go back to our equations, this is the term which represents the inertia. So, the inertial term in either of these equations, this is what we call inertial term, ρu^2, if we write inertia. So, this inertial term will be negligible if the Reynolds number is low, if the viscous terms dominate or if the flow is viscosity dominated and this will be true for slow flow or creeping flows.

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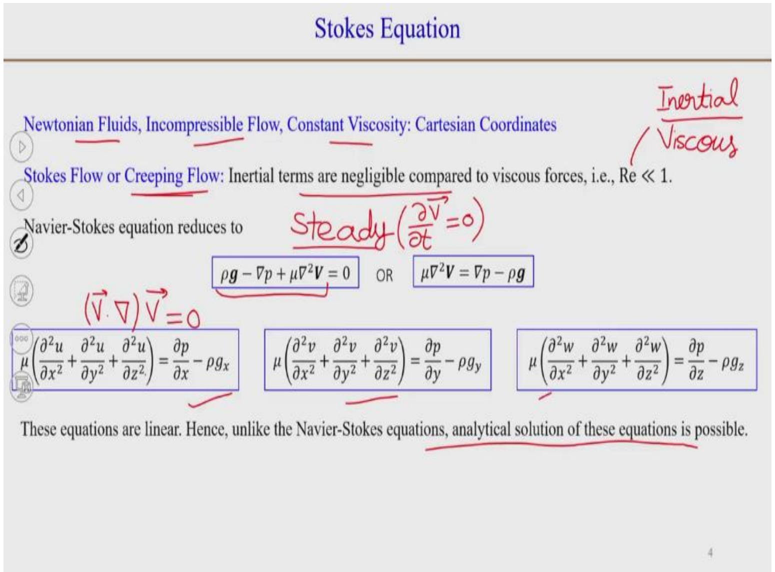

So, in these cases, inertial terms can be neglected. Reynolds number, you know as you know that ratio of inertial and viscous forces. So, Reynolds number is ratio of inertial and viscous forces. When Reynolds number is less than 1, then inertia is negligible when compared with the viscous forces. So, inertial term is negligible.

When the inertial term is negligible and the flow is steady, so we can also assume the flow to be steady, and when the flow is steady and inertia is negligible, then the left-hand side becomes 0 and we will have these terms. Now it was the inertial term which was V.∇ operated on the vector V which is the nonlinear term. And this is also the inertial term. So, when the flow is steady, ∂V/∂t is 0.

So, we have this term. Now if this = 0 then the nonlinearity from this equation is gone and we can rearrange it in this form. So, because the system of equations have become now linear so we can find analytical solution of Stokes equation. So, you will see that lot of effort has gone in last 100 years or more in finding a solution of Stokes equation for different cases. When you expand it in the Cartesian coordinate form you get the three equations right.

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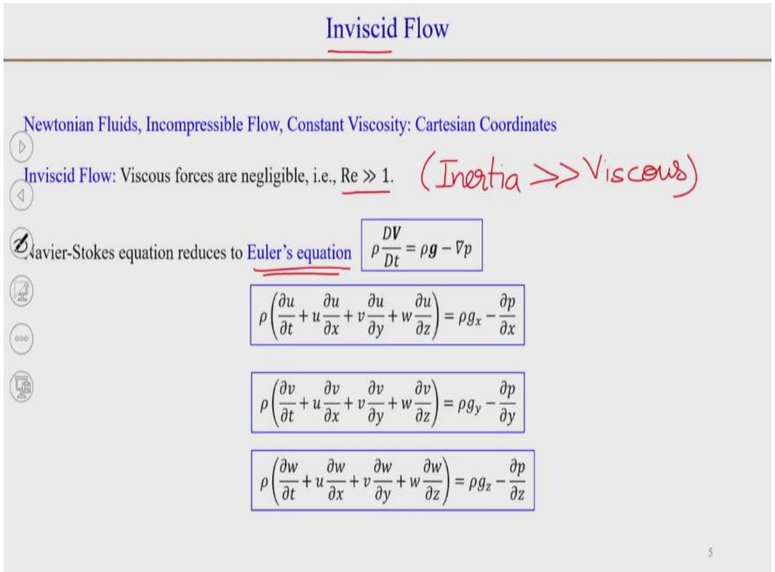

So, other approximation is on the other side when Reynolds number is very large than, very, very large than 1, that means the flow is dominated by inertia and very, very large than the viscous forces. So, in that case, we can assume the flow to be inviscid, and if that is the case, then we will not have viscous term or we can neglect the viscous term.

So, such equation is called the Euler's equation, whereas when we neglected the inertial term, the equation, the Navier-Stokes equations are called Stokes equation. So, at Reynolds number very, very less than 1 in that limit, we call the equations Stokes equation. Reynolds number very, very large than 1 and we can neglect the viscous term, then this equation is called Euler's equation.

So, we can write it in the expanded form and we will look at the inviscid flows and some of its cases in the next module.

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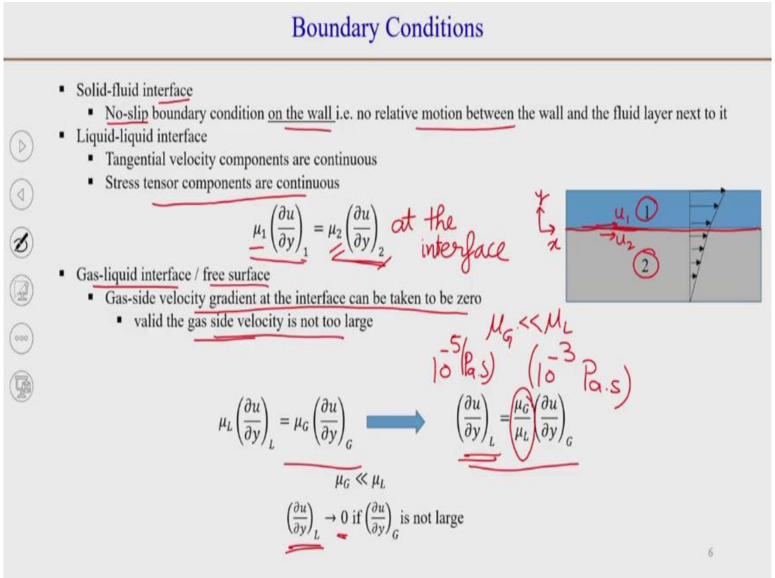

So, now we have the set of governing equations or set of partial differential equations, which describe the fluid flow phenomena, and to solve these equations we also need the boundary conditions. So, at the boundaries, they can be of different type. If the boundary is let us say a solid fluid interface, then as we saw earlier that at a gas solid or gas liquid interface we can use the no-slip boundary condition on a solid wall which simply means that there is no relative motion between the solid wall and the fluid layer next to it.

So, the tangential velocity component will be equal to the tangential velocity of the wall. And the normal velocity component will also be equal to the velocity of the wall. So, the velocity of the fluid adjacent to or the velocity of the fluid layer adjacent to the wall will equal to will be equal to the velocity of the solid wall. Now if there is a liquid-liquid interface, then in that case, let us say you have two liquids, liquid 1 and liquid 2.

And in such case what will happen, you will have at the interface, so this is the interface between the two liquids, these immiscible liquids, of course. So, the tangential velocity will be continuous, so you can say the velocity of fluid 1, if this we describe as x and y coordinates, so velocity of fluid 1, u_1 and u_2, if this is the direction as x, so u_1 will be equal to u_2 at the interface, and the stress components will also be continuous.

So, the velocity at the interface will be continuous as well as the tangential velocity will be continuous at the interface and the stress tensor components will also be continuous at the interface. So, the shear stress will be continuous as well as the normal stresses will be continuous. If we consider the interfacial tension, then at the normal interface, you will also need to incorporate the jump in pressure caused by a surface tension, but such problems, we will not have in this course at least.

So, at the interface, when we have a liquid-liquid surface, or liquid-liquid interface, then we will have velocity continuous as well as the continuity of shear stress. Now at a gas-liquid interface, so when we have a gas liquid interface, this is also called a free surface, for example, a flow in a drain, the liquid flows and at the top there is gas layer on it. So, such surfaces are called free surfaces flow in a river, flow in a canal.

Now in such cases, you can neglect the gas side velocity gradient at the interface. So, when you have two fluids adjacent to each other because of the continuity of shear stress, you can write a μ_1 $\partial u/\partial y$ in fluid 1 and μ_2 $\partial u/\partial y$ in fluid 2 at the interface, which basically represents the continuity of shear stress at the interface, if the normal, this is written if the normal to this interface is pointing in the y direction and interfaces is along the x direction.

Now generally the viscosity of gases, so if we write down this equation for gas and liquid, we can write it in this form and the viscosity of gas is significantly lower than the viscosity of liquid. The two orders of magnitude lower generally, the viscosity of gas is two orders of magnitude lower than the viscosity of liquid. So, viscosity of gas is of the order of 10^{-5} Pascal-seconds, whereas viscosity of liquids at room temperature, for example of water is of the order 10^{-3} Pascal-seconds.

Now if this is the case, then if the gas side velocity is not too large, then we can write down the, if the gradient of velocity in the gas side is not too large then we can rearrange our equation in this form, and because of this ratio being very small we can neglect the gradients in the liquid side at the interface. So, $\partial u/\partial L$ at the interface will be negligible and that boundary condition can be used at a gas-liquid interface.

So, at a gas-liquid interface, we can assume the velocity gradient at the interface in the liquid side to be 0 given that gas side velocity is not very large.

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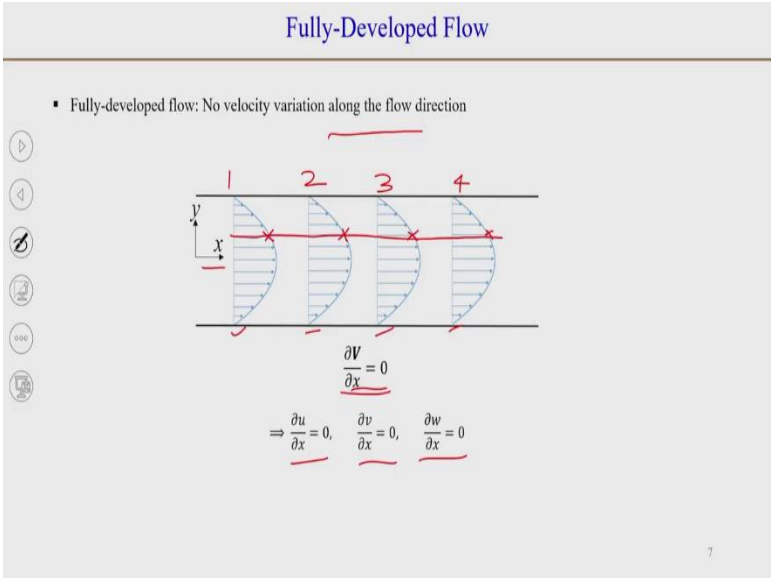

Now another thing that we need to look at or that we will encounter again and again is what is called fully developed flow. So, a fully developed flow refers to that there are no gradients in the velocity along the direction of flow. So, if you have flow in a pipe as shown here and the flow is happening or the liquid is flowing along the x direction then fully developed flow will mean that $\partial V/\partial x = 0$, and this we can see from here that the velocity profile is invariant as we move along the x direction.

So, if you turn these locations at location 1, 2, 3 and 4, and if we take any transverse location, so if we measure the velocity at these points is they are all same. So, $\partial V/\partial x$, if we write then the velocity at this point divided by $\Delta x = 0$. So, that means this is going to be 0. Now if we write it in the component form, it will be $\partial u/\partial x$, $\partial v/\partial x$ and $\partial w/\partial x = 0$.

Normally in such cases when the flow is fully developed the flow is unidirectional. So, only one component of flow will be there, only u will be non-0, v and w in these cases are 0.

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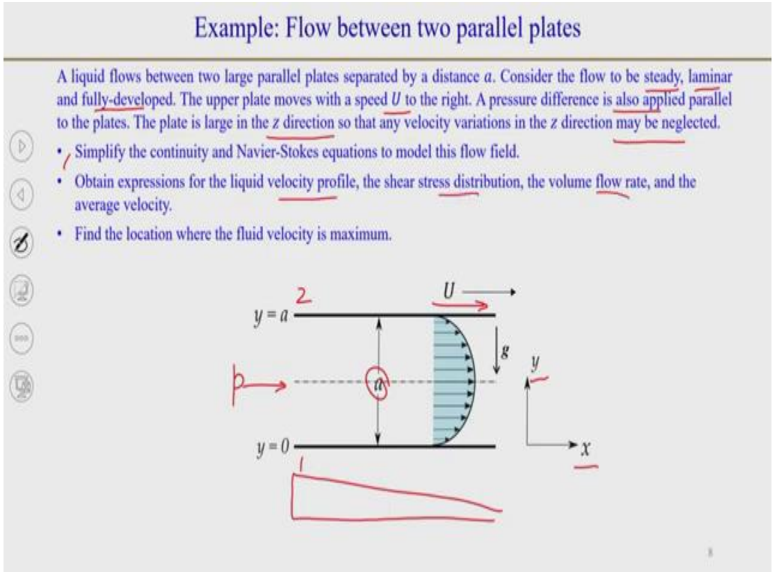

So, let us look at an example now. What is given here that a liquid flows between two large parallel plates, so this is plate 1 and play 2, and these plates are separated by a distance a. We consider the flow to be steady that means ∂/∂t for all the things are 0, for all the variables is 0, flow is laminar, and the flow is fully developed.

The upper plate moves with a speed u to the right. So, the upper plate is being moved with a force and the velocity of this upper plate is constant which is given as capital U and there is also a pressure difference. So, the flow, if we look at the pressure here, so the pressure is causing or pressure will cause the flow. So, there is a pressure gradient present in the flow.

The plate is large in the z direction, so as you can see here the plane in the in this screen is xy plane and the flow or the z coordinate is normal to the screen. So, it is given that the plates are large in the z direction, so there is no velocity variation in the z direction or we can neglect the velocity variations that are there in the z direction.

So, first we will simplify the mass and momentum conservation equations and then we will obtain the expressions for the velocity profile, the shear stress distribution, flow rate and average velocity. And then we also need to find the location where the fluid velocity is maximum. So, we will start with listing down all the assumptions that are involved in this problem.

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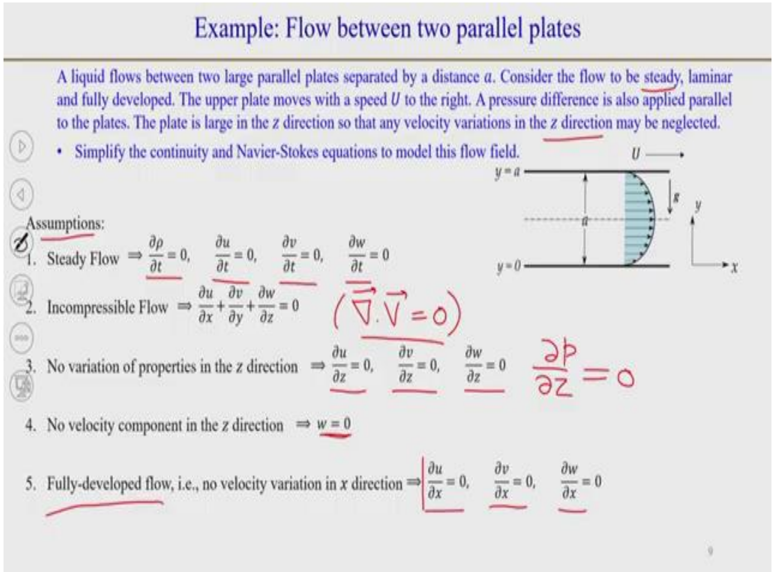

The first assumption we see is that the flow is steady. It is given, so when the flow is steady that means ∂/∂t for the variables is 0. So, in the continuity equation, we will have ∂ρ/∂t = 0, ∂u/∂t in the momentum equation, ∂v/∂t and ∂w/∂t, the rate of change of the velocity component with time is 0. Then the next assumption is that the flow is incompressible. So, for the incompressible flow, we have ∇.V = 0, the continuity equation in this form.

So, our continuity equation will reduce in this form. Then there is no variation of properties in the z direction as is suggested here. So, ∂u/∂z = 0, ∂v/∂z = 0, ∂w/∂z = 0 and ∂/∂z of pressure also we will see that it will come out to be 0. Now the next assumption is that there is no velocity component in the z direction, because there is no flow along the z direction.

The flow is happening only in the xy plane. So, the z component of velocity w = 0. And the last assumption is that the flow is assumed to be fully developed. So, that means because the flow happens along the x direction and as a result ∂u/∂x, ∂v/∂x and ∂w/∂x of w, all of them are 0. So, let us now write down the equations and try to see that what are the terms that will become 0 because of these assumptions.

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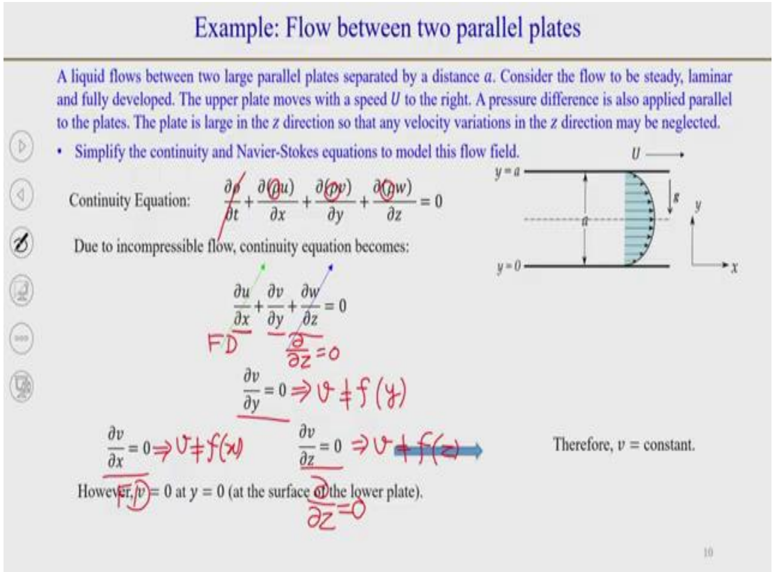

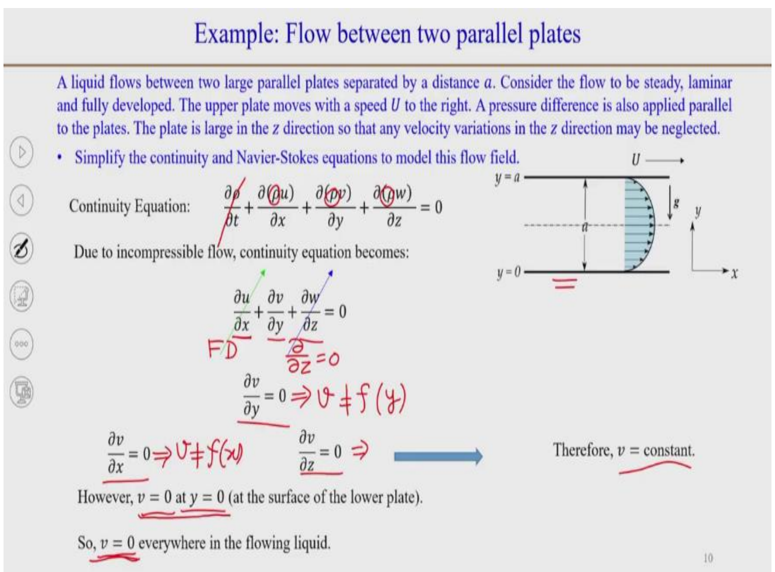

So, first we will start with the continuity equation, and because of the steady flow this term will become 0, because the flow is incompressible so we can take ρ out of it and we will have our continuity equation reducing to ∂u/∂x, + ∂v/∂y, + ∂w/∂z that becoming equal to 0. So, the first term ∂u/∂x is equal 0 because the flow is fully developed. Then the last term, ∂w/∂z = 0 because we know that ∂/∂z for all the velocity components is 0.

So, that will give us the ∂v/∂y equal to zero. And we already know that ∂v/∂x = 0 because flow is fully developed and ∂v/∂z = 0 because ∂/∂z of all the velocity components is 0. So, from this we know that v is not a function of x and v is also not a function of z. And this tells us that v is not a function of y either, so that means v is a constant.

So, now we know that v is a constant and other thing we know that because of the no-slip boundary condition the velocity at y = 0 at the bottom plate which is stationary. The velocity is 0, because the bottom plate is stationary, its velocity is 0 and because of the no-slip boundary condition the velocity components will be 0 there.

So, at y = 0 v = 0, so that means we have a v = 0, the constant that we talked about here. This constant will turn out to be 0. So, the velocity y component of velocity v is 0 everywhere in the fluid. So, that is the conclusion that we could derive by applying the assumptions in the continuity equation.

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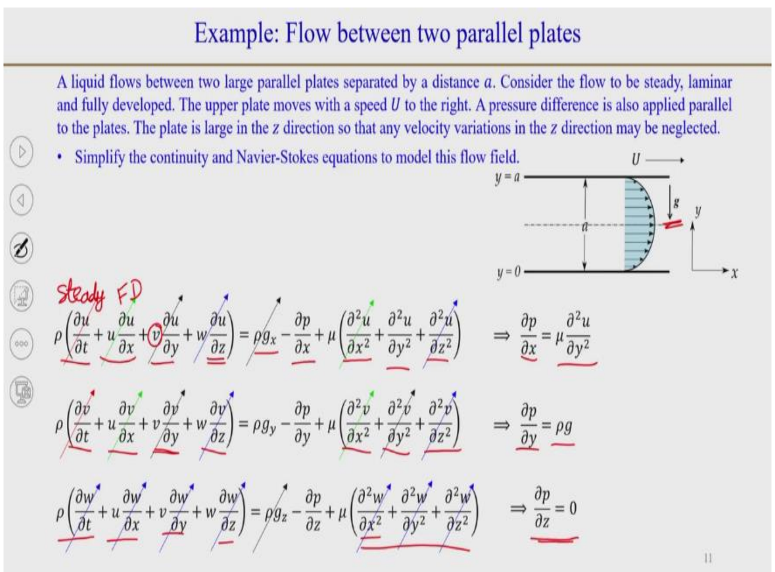

Now, let us look at the momentum equations. So, we write down the momentum equation in the x direction. And the first term is 0 because the flow is steady. Now this term here u ∂u/∂x = 0 because the flow is fully developed, so ∂u/∂x = 0. And ∂u/∂x = 0. So, its derivative will also be 0, so this term is also 0.

Next the derivative of velocity components with respect to z is 0, so ∂u/∂z = 0 as well as the second derivative of u with respect to z is 0. Now we just figured out that v = 0, so this term will also be 0. And as we can see the gravity acts, the arrangement is horizontal, so the gravity acts in the negative y direction, so there is no component of gravity along the x direction so we can also say that $\rho g_x = \rho g_x = 0$.

So, we end up with only two terms here, $- \partial p/\partial x + \mu \, \partial^2/\partial y^2$ or $u = 0$ or we can write it in the other form, so $\partial p/\partial x = \mu \, \partial^2 u/\partial y^2$. So, let us look at the momentum conservation equation in the y direction. The flow is steady. So, this term, the first term goes. The next term because $\partial v/\partial x$ is 0 because the flow is fully developed.

Then there are no gradients in the z direction for any velocity component, so $\partial v/\partial z = 0$, $\partial^2 v/\partial x^2$ is 0 because the flow is fully developed, no gradients along the z direction, so this term is also 0. Now there are, because $v = 0$ and we also figured out that $\partial v/\partial y = 0$, so from either of these arguments this term will become 0, and because $\partial v/\partial y$ is 0, so its derivative with respect to y will also be 0.

So, $\partial^2 v/\partial y^2$ is also 0. Now from this we find out that $\partial p/\partial y = \rho \, g$, that means because of the hydrostatic pressure there will be pressure gradient along the y direction. Now we come to the third equation, which is the momentum equation in the z direction and we can see because $w = 0$, there is no velocity component along the z direction. So, $w = 0$ and all terms which involve w, they are going to be 0, $\rho g_z = 0$ because no gravity component along the z direction.

So, that will give us $\partial p/\partial z = 0$, so this we could derive from here or we could have simply seen that the velocity components are 0, the gradients of velocity components u, v, w were 0 in the z direction. So, from that we got this relationship. Now we have simplified the momentum conservation equations for x, y, and z directions.

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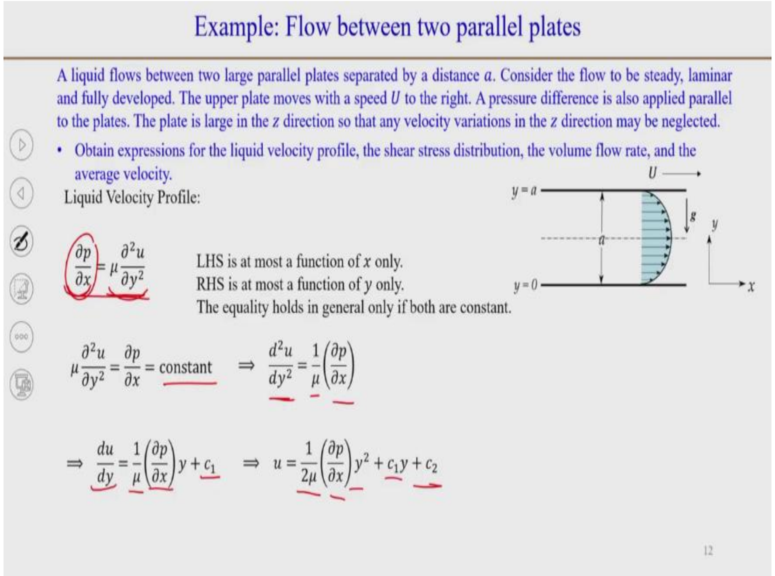

And the next objective is to find the velocity profile so we can simplify this equation. Now this is a partial differential equation. The left-hand side is a derivative with respect to x, whereas the right-hand side has derivatives with respect to y. Now we know that $\partial p/\partial z = 0$, so that means p is not a function of z. And $\partial p/\partial y$ we found out that it is a constant.

So, it is also not a function of y, so p can be a function of x only or $\partial p/\partial x$ can be a function of x only. So, the left-hand side is a function of x only and the right-hand side from the similar arguments, we know that $\partial u/\partial x = 0$ because flow is fully developed and $\partial u/\partial z = 0$ because there are no gradients along the z direction so u will be function of y only. So, this term will be a function of y only.

So, that means the left-hand side is a function of x, right-hand side is a function of y only and that will be possible only when both of these terms are equal to a constant term. So, let us say, or now we know that they are equal and this = a constant neither a function of x nor a function of y. So, we can write this down, now we can write this in terms of a total derivative because u is not a function of x and u is not a function of y.

So, $d^2u/dy^2 = 1/\mu\ dp/dx$, which is the gradient along the x direction of the pressure gradient along the x direction and we can integrate it. So, when we integrate, we will get $du/dy = 1/\mu$, or $\partial p/\partial x$ into $y + c_1$ a constant. Now we integrate it again, so when we integrate it again, we will get $1/2\ \mu$

$\partial p/\partial x$, or $y^2 + c_1 y + c_2$. Now we have another constant which is c_2 now, so our task is now to find out these two constants, c_1 and c_2 and for that we can use the boundary conditions on the bottom wall and the top wall.

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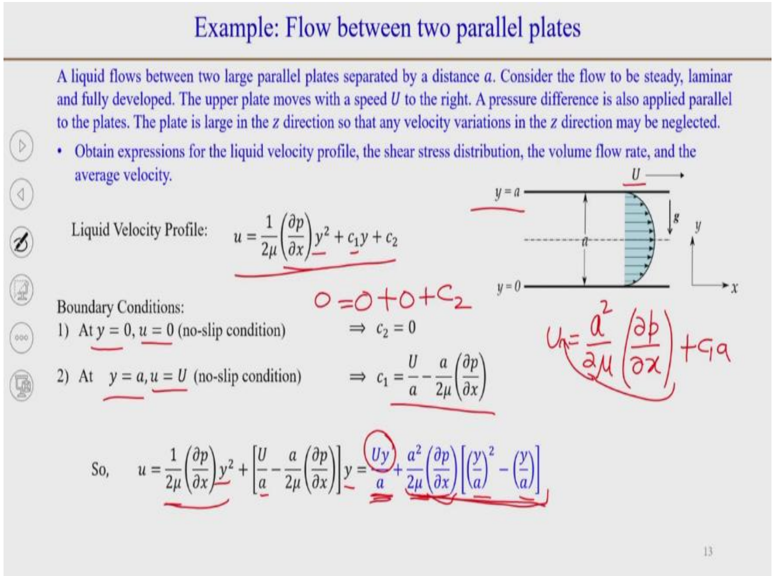

So, from the no-slip boundary condition on the bottom wall where y = 0 and the wall is fixed. So, the velocity will be u = 0 and we use it, so u will be 0 and that will be at y = 0 + 0 + c_2. So, we get c_2 = 0. Now at the top wall, which is at y = a, as we can see here, and it moves with a velocity u, so from no-slip boundary condition u = capital U here. Note that this is an example where we see that the wall is moving, so at the moving wall, the fluid velocity will be equal to the velocity of the moving wall and not 0.

So, when we use this, we will get c_1 = u/a - a/2 μ, $\partial p/\partial x$, that we can simply see from here that small u = capital U, 1/2 μ into a^2 when we replace y = a, $\partial p/\partial x$ + c_1 a. And we can bring c_1 on the, or we can bring it on the other side and so this it is subtracted and then divide by a, so we will get the value of constant c1, and we can replace the values of constant c_1 and c_2 in this equation.

So, we will get u = 1/2 μ ∂p ∂x, y^2, which remains same + c_1, so c_1 = u/a - a/2 μ $\partial p/\partial x$ into y and c_2 is 0. And we can rearrange this a bit, so we can see that the first term is Uy/a, by multiplying U/a with y + $a^2/2\mu$ So, $a^2/2$ μ into $\partial p/\partial x$ into y/a^2 - y/a.

So, $a^2/2\mu$ is, and $\partial p/\partial x$ is taken out of the bracket. So, this term will become y^2/a^2 because you will need to multiply and divide by a^2 and the other term will be - y/a, so you have the two components. The first term is a linear term and the second term is a quadratic term here in y.

So, you can see that if U = 0 then this term will go, the capital U = 0 then this term will go away. Whereas if $\partial p/\partial x$ = zero, then the second term will go away. So, these are the two contributions. The first term is the contribution in the velocity profile, which is because of the linear or because of the shearing by the upper plate whereas the second term, the quadratic term is because of the pressure gradient that is causing the flow.

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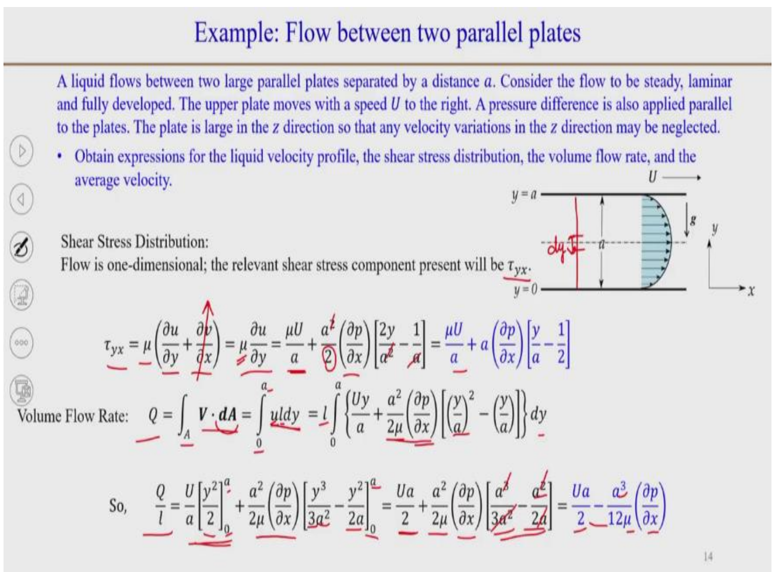

So, we can find out the shear stress distribution. Now, the flow is one dimensional flow happens only along the x direction. So, the only component of shear stress which will be non-zero will be τ_{yx} or τ_{xy} which of they will be equal, so we can write τ_{yx} for a Newtonian fluid because the flow is Newtonian so we can write $\tau_{yx} = \mu$ into $\partial u/\partial y + \partial v/\partial x$, but this term is 0, $\partial v/\partial x$ is 0 because the flow is fully developed.

So, you finally have this = μ into $\partial u/\partial y$, and when we differentiate with respect to y, so the first term will become, because it was linear, so it will become a μ multiplied because there is a multiplication by μ, so μ multiplied by u/a and $a^2/2$ $\partial p/\partial x$, y^2/a^2 so that when differentiated becomes 2y - a^2 and y/a will become 1/a.

So, we can further simplify it, μ u/a + 1 a will cancel out, throughout, so you can have a into ∂p/∂x, y/a - 1/2 where the 2 is multiplied inside the bracket. So, we have got an expression for τ_{yx}. Now next thing we find or we need to find is the volumetric flow rate. So, volumetric flow rate is V.dA for a differential area a, so if we integrate it over a cross-section, so because the flow is fully developed so you take at any cross-section the velocity profile is going to be same and we integrate over the cross-sectional area A.

The area will be, if you take a small length here, say dy and the distance normal to plate you can take it to be l, so the area will be ldy multiplied by the velocity at that location because the velocity is varying with respect to y. So, we will need to integrate and we will integrate it from 0 to a. So, integral uldy, we can replace the expression for u and l is a constant.

So, l will come out of the integration, l integral 0 to a, U/a into y + $a^2/2$ into ∂p/∂x multiplied by y/a^2 - y/a into dy and on simplification we can bring l on the other side. So, Q/l = we can integrate term by term. So, U/a, the first term will give U/a and when you integrate y, you will get $y^2/2$ and the integration limits from 0 to a.

The next term will have $a^2/2$, ∂p/∂x which is all constant with respect to y and when you integrate the terms inside the bracket, the first term will give you y^3 /3 and a^2 is there already, so $y^3/3a^2$, $3a^2$ - when you integrate y you will get $y^2/2$ and there is a in the denominator, so $y^2/2a$ integration limits from 0 to a, and when you substitute the limits from the first term you will get U/a into $a^2/2$. So, a and a will cancel out and you will get Ua/2.

The next term, you will get $a^2/2$ ∂p/∂x + on substituting the value of a you will get $a^3/3a^2$ - a^2 /2a. So, when you simplify the first term remains same Ua/2 - ∂p/∂x, we as it is, the term within the bracket is, so this will give you a/3 - a/2. So, a/3 - a/2 will be - 1/6. So, - 1/6 multiplied by 1/2, you will get - 1/12 into a^3/μ. So, that is flow rate per unit plate width.

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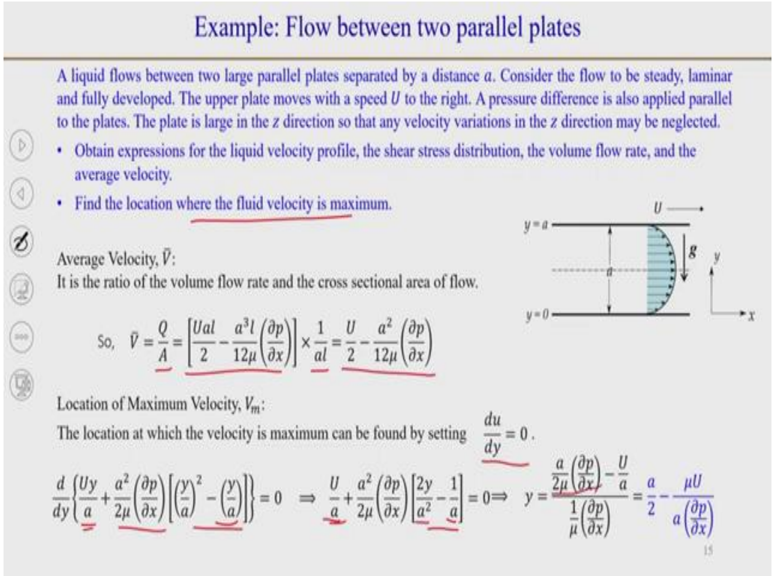

And next we need to find the average velocity. So, average velocity will be flow rate divided by the cross-sectional area of the flow. So, cross-sectional area will be a into l, so Q which we found just now divided by al and we can find the average velocity. Next, we also need to find the location of the maximum velocity. Now the location of maximum velocity we can substitute in the expression of velocity or du/dy where du/dy = 0.

So, we can find the differentiation of the velocity expression, which is Uy/a + this so we can differentiate it and the differentiation of first term will give you U/a + this term is constant with respect to y. So, $a^2/2$ μ ∂p/∂x and then differentiation of the terms in bracket will give you $2y/a^2$ and this term differentiated will give you 1/a.

So, when you simplify it, you will get from here, you can write $2y/a^2$ = U/a or - U/a + a/2 μ ∂p/∂x divided by this term which will be 1/μ ∂p/∂x. And on simplification, you will get a/2 - μ U/a ∂p/∂x.

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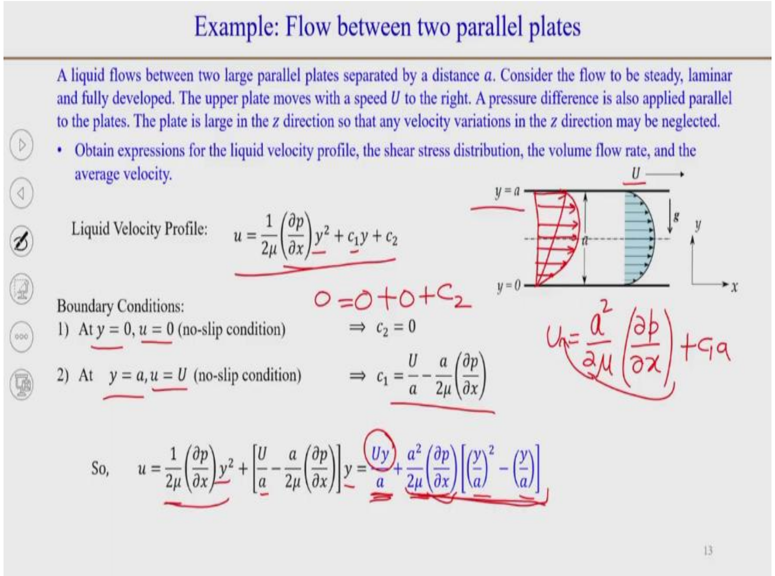

Now in this problem we had both, we were looking at the flow between two parallel plates and the upper plate was being driven by a certain velocity U and there was a pressure gradient which was driving the flow. So, there were two factors which were driving the flow and we saw in the expression also, while velocity profiles in these expressions or in these figures we have drawn like this, but actually when we see the equation and we plot it, the velocity profile will be somewhere sometime something like this.

And the location of maximum velocity will of course depend on the pressure gradient and U and their relative magnitudes. So, this combines the first term which is a linear term and the second term which is a parabolic term.

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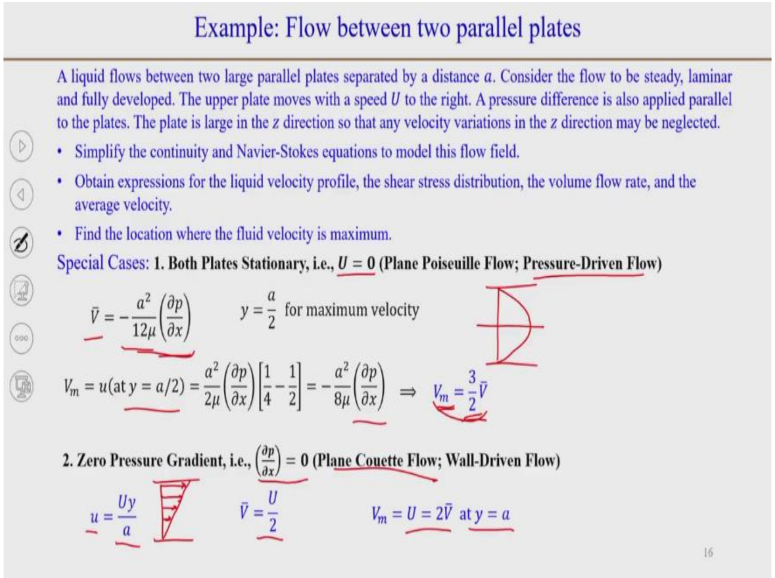

So, here we can look at the two special cases that when both the plates are stationary, so there is no shear driven flow, there is no flow such that because of the plate movement, because both the plates are stationary and the flow is only pressure-driven flow. So, in such a case, the velocity profile will be parabolic. So, it will be symmetric about the mid plane and the average velocity will be - $a^2/12\ \mu$ into $\partial p/\partial x$ and that will be maximum velocity.

We can find all that. And the important relation that we can observe from here is that when we have a pressure-driven flow between two plates which are parallel to each other, the average velocity will be 2 by third of maximum velocity or maximum velocity will be 3/2 or 1.5 times of average velocity. The velocity profile is parabolic in this case.

Now if it is 0 pressure gradient, so that means there is no pressure that drives the flow, the flow is only because of the shear that is being provided by the moving wall, so in such a case, we will have the velocity u = capital U y/a which is a linear velocity profile. So, the velocity profile in such a case will be a linear velocity profile, and it is also known as planar Couette flow. And the velocity at y = a is of course the velocity at the top or velocity at the top plate.

The mean velocity here is U/2 because it is a linear velocity profile and the maximum velocity = twice the mean velocity. So, in this example what we have been able to look into or what we saw here is flow between two parallel plates, two different cases, which we have combined together. The first case was that the flow is being driven by the shear provided by the top plate.

And the second example was that there is flow between two parallel plates and both the plates are stationary. We have combined them together that what if the upper plate is moving with a velocity U and there is a pressure gradient in the flow. So, we found the velocity profile is such a case and we could also find the shear stress, average velocity and so on.

And some of these results are very important to remember, for example, we need to remember that when we have wall-driven flow or planar Couette flow, the velocity profile is linear, when there is pressure driven flow between two parallel plates then the velocity profile is parabolic and the mean velocity or the maximum velocity is 3/2 times of the mean velocity.

We have learned in this lecture that what is fully developed flow, what are different boundary conditions at solid wall or gas liquid and liquid-liquid interfaces. So, we will stop here and in the next lecture, we will further look into a few more examples where we will simplify the terms and find out how to apply Navier-Stokes equation to some other problems. Thank you.