Fundamental of Fluid Mechanics for Chemical and Biomedical Engineers
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Lecture 24
Navier Stokes Equations_ Derivation

In this lecture we are going to discuss momentum conservation, where we look at the differential analysis. So, when we started with the course, we looked or we discussed three different approaches to analyze the fluid flow problems.

The dimensional analysis, which we have already talked about, then macroscopic balances or integral analysis, where we were able to find out the average quantities or the quantities for one dimension analysis where we could assume the flow to be one dimensional or the forces on a surface, the total force on a surface, the integral pressure on a surface and so on.

Now, if we want to look into the detail, if we want to find out the force caused by a fluid on a sphere or the velocity field in this room, for example, at every point, then we need to do the differential analysis. So, in the previous module or in the previous week we have already built up the background for differential analysis where we looked at the fluid kinematics. So, we looked at the expressions for angular deformation.

We looked at the expressions for fluid acceleration and we could also derive the momentum, sorry, mass conservation or continuity equation in the differential form. Now, in this lecture, we will look at or we will derive the momentum conservation equation in the differential form and these equations when expressed in terms of stresses, we call these equations as Cauchy Momentum Equations.

And when we replace these stresses with the relations applicable for a Newtonian fluid. So, remember Newtonian fluids are the fluids where shear stress is directly proportional to the strain rate or we call shear rate or rate of angular deformation. So, for Newtonian fluids when we replace the stresses by the expressions applicable for Newtonian fluid, we, what we get is Navier stokes equations.

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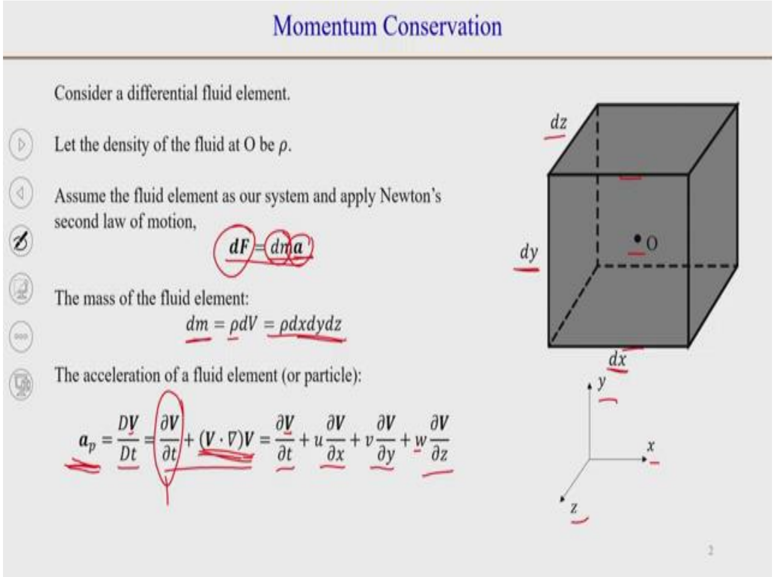

So, let us consider a differential fluid element and it is a cuboid in shape having a size or having the differential fluid element sides of dx, dy and dz along x, y and z directions respectively. The center of this fluid element is at point O, which is at a distance of dx/2 from both the faces in x direction, at a distance of dy/2 from the both the faces in the top and bottom directions and at a distance of dz/2 from both the faces along z direction.

So, we will assume, as we have done in the past, we will assume the fluid properties at point O, so the fluid property at point O, the density is ρ and we will apply. So, if we consider this cuboid fluid element as our system and apply Newton's law of motion on it, because that is from where our momentum conservation equation comes, so we will apply our Newton's second law of motion dF = dm into a.

So, our task is basically to find out these three terms dF, dm and a, and substitute these, and what we will eventually get is the conservation or momentum conservation equation. So, let us look at first that part dm, which looks simpler, simplest. So dm =, because this side has, this fluid element has the sides dxdy and dz, so the volume will be dxdydz and when we multiply it with the density the term dm will become ρ dxdydz.

So, we have the mass of this fluid element. Now, the next thing we can use is a, which is the acceleration of this fluid element. So, acceleration of a fluid element from the previous chapter where we studied fluid kinetics, we can write this acceleration of fluid element as capital D/capital Dt, which is substantial derivative in, if we write when we expand it ∂/∂t of vector V + V.∇ del operated on vector V.

So, the first term is, if we remember first term is the local acceleration or partial derivative with respect to time, whereas the next term, it represents the convective acceleration which is because of the bulk motion of the fluid, so how the momentum or how the velocity is being transported because of the bulk fluid motion.

So, that is the acceleration in terms of V where V vector is the velocity field. When we expand them in Cartesian coordinate then the first term, of course, will remain same, ∂/∂t of vector V + u ∂/∂x of vector V + v ∂/∂y of vector V + w ∂/∂z of vector V. So, we have dm of this fluid element and the acceleration of a, of this fluid element for the velocity field vector V.

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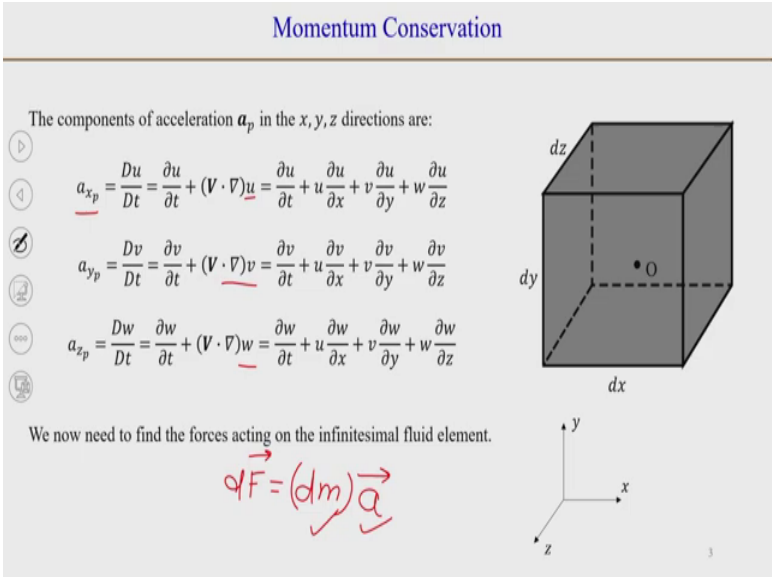

Now, the components we can further write that if we expand this velocity vector V in terms of u, v and w, then we can write the x component of vector, y component of vector and z component of acceleration vector. So, we have dm and a. Now, the next task is that we have already found in dF = dm into acceleration vector, so we have found dm and we found the expression for a in terms of the velocity vector in terms of dxdydz which are the sides of this fluid element and the density at the center ρ.

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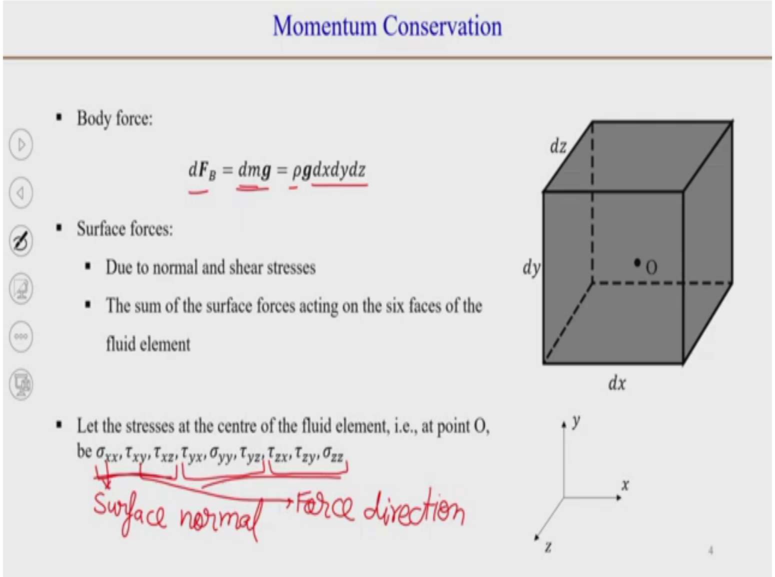

Now, let us find out the forces. So, the forces can be to kind of forces. The fluid element can experience body forces as well as surface forces, so let us look at body forces. And in this course, we are considering the body force that we are considering is the gravitational force only, so this makes things simpler that the gravitational force or the body force in this case will be that dm into gravity or gravitational acceleration due to gravity in the vector form.

So, when you expand dm, again you can replace dm with ρ dxdydz into g vector, so that is our body force. The next task is to find the surface forces and the surface forces that will come from the normal and shear stresses that are being applied on the control surface of this fluid element or in, on the, on all the surfaces of this fluid element.

So, this fluid element which is a cuboid element, we have six faces, two left and right top and bottom and front and back. So, we will write down the expressions for the normal and shear stresses on all the surfaces one by one and then combine them together. And as we know that there are at a particular points in this case at the center of this fluid element at point O we will have a nine component of stresses because stress is a second order tensor.

And this comes about because on each side we will have at a particular area we will have three components of stresses which will be acting on a particular surface, it will have three components of stresses along x, y and z direction, so that is what we have. σ_{xx}, τ_{xy}, τ_{xz} and τ_{yx}, σ_{yy} τ_{yz} τ_{zx}, τ_{zy}, σ_{zz}.

So, if you remember σ_{xx} is the normal stress on the x surface. So, the first subscript represents the surface or the plane on which the force acts, so that is the direction of surface normal; whereas the second subscript represents the force direction. We discussed this already when we discussed about a stress field. And this stress is positive, if both of them are positive.

If the area normal is positive and the force in, and the direction of force is also positive then the stress will be positive, if both of them are negative then also the stress will be positive, but if one of them is positive and another is negative. So, if the surface normal is in the negative y direction and the force is in positive y direction then it will be negative.

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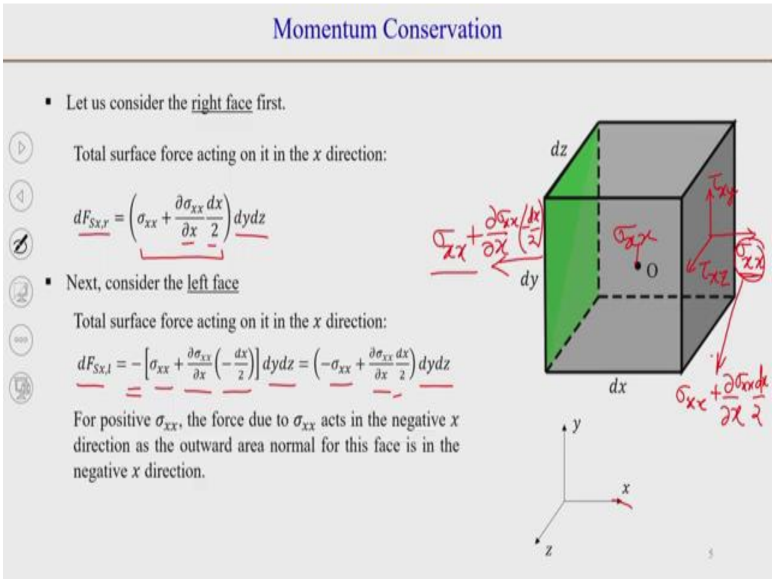

So, with this let us start looking at each faces and try to find the surface force on all the six faces. So, we will start with the right face, which is along the positive x direction here. So, on this face, if we draw the stresses there will be three stress components here and one is normal stress and two shear stresses. This is a x plane, so the first subscript will be x for each case and second subscript will depend on the direction in which the force is acting.

So, this is σ_{xx} which is normal stress, τ_{xy} acting along y direction and τ_{xz} acting along the z direction. Now, we are going to look at because we will do the analysis or we will combine the forces for, on the cube on all the faces which act along the x direction. So in each case, for each face we will be concerned with the force that acts along the x direction.

So, on the right face the force that acts along x direction is σ_{xx} and that is the stress, but remember the stress at point O is σ_{xx}, so from Taylor series expansion the stress here will be

σ_{xx} + ∂/∂x of σ_{xx} into dx/2. And the area of this surface will be dydz. So, if we write down the total force acting on this, on the right surface the total force acting will be the stress which is σ_{xx} + ∂/∂x off σ_{xx} into dx/2 into dydz.

Similarly, next, we will consider the left face. So, on the left face the force that we will be looking at σ_{xx} or the normal stress on this first because that is what will be acting in the x direction, so that will be σ_{xx}, but now it is in minus_{xy} to minus dx/2 distance from point O, so we will write ∂/∂x of σ_{xx} into minus dx/2 so we can say, and the area normal will be in the negative x direction.

So, if we are considering the stress to be positive then the force will also be in the negative direction. So that is why we have written the force on left face = minus, the area of course will be dydz and the stress will be σ_{xx} + ∂/∂x σ_{xx} into minus dx/2 or we can simplify it, so minus σ_{xx} + ∂/∂x of σ_{xx} into dx/2 into dydz.

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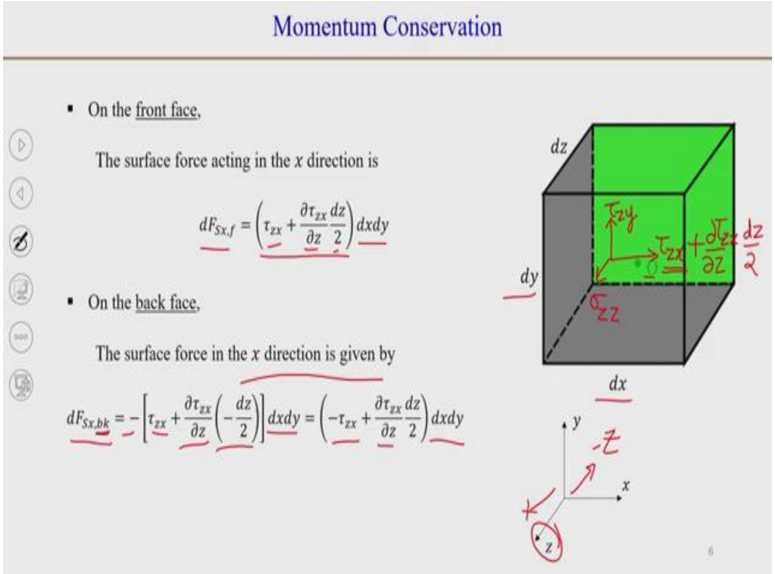

So, next we look at the momentum conservation for the front face, which is the positive z surface, so the area of this surface will be dxdy and the area vector will be pointing out in the positive z direction. If we draw the stresses acting on this surface again, we will have σ and two τ one normal stress and two shear stresses. The first subscript in each case will be z because that is what the direction of surface normal is, and depending on the directions.

So, the normal stress will be acting in the z direction, this shear stress is acting along the x direction and this along the y direction. We are concerned with the forces acting along the x direction because we want to write down the total force on this fluid element along the x direction. So, we know the area and we know the stress.

The stress at this point will be τ_{xz} + or at this surface because it is at a distance of dz/2 from point O so the stress at the surface at the front will be $\tau_{zx} + \partial/\partial z$ of τ_{zx} into dz/2. So that is what we have here, the force on the front face = the area multiplied by the stress $\tau_{zx} + \partial/\partial z$ of τ_{zx} into dz/2.

Now, we will consider the face on the back. Now again, we will have three stresses on the other side and the force because the stress is positive so the force on this will be acting in the negative direction. The area vector is acting in the negative direction, the area vector on the back face will be in the minus z direction.

So, on the face and the back on the back the x direction force will be force on the back because we will have another subscript to define or to represent back we have used the subscript bk here. So, this = minus $\tau_{zx} + \partial/\partial z$ of τ_{zx} multiplied by minus dz/2 because the surface at the back is at a distance of minus dz/2 or at a distance of dz/2 in the negative direction from point O.

The area is dxdy again so we can simplify, we can take the minus sign inside the bracket so this will give us minus $\tau_{zx} + \partial/\partial z$ of τ_{zx} into dz/2, this whole multiplied by dxdy. So, we have done the exercise or we have obtained the forces acting on the four surfaces along the surfaces or the planes along the x direction planes along the z direction.

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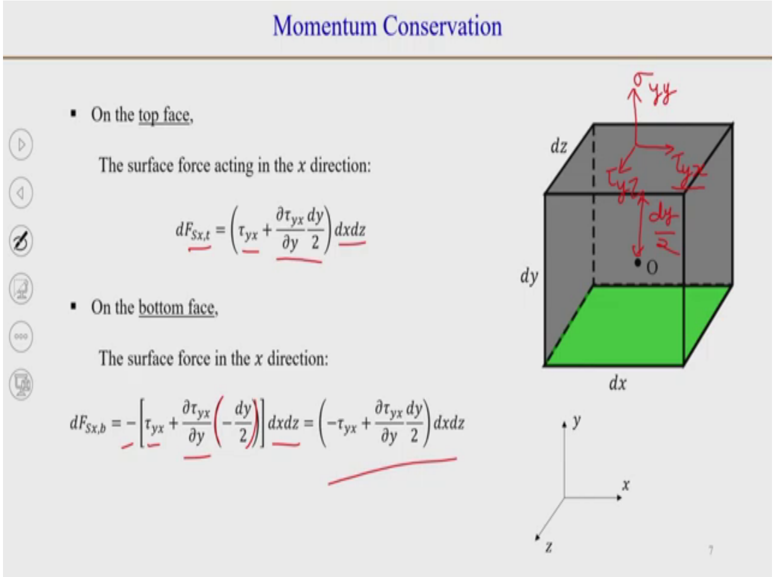

We can do that exercise now for the surfaces on y directions so the surface on the top and surface at the bottom. If we look at the surface on the top, the force is acting on it or the stress is acting on this surface or the area vector is in positive direction, so we can write there will be one normal stress two shear stresses, and this is a y surface, so the first subscript will be y σ y now, σ_{yy}, τ_{yz} , τ_{yx}.

We are concerned with the force along the x direction. So, this will be, the force will be area dxdz multiplied by τ_{yx} + ∂/∂y of τ_{yx} into dy/2 because that is the distance dy/2 from point O of the top surface, so the top surface has a distance of dy/2 from point O. Now, on the bottom face, we will do the same exercise, but then area vector will be pointing in the negative direction.

So, the force will also be in the negative direction, so that the stress is positive on this surface. So, the force we can write minus dx by dx into dz which is the area of this surface multiplied by τ_{yx} + ∂/∂y of τ_{yx} multiplied by minus dy/2, which is the distance of this surface from point O. And we can simplify it or we can take the minus sign inside. So now, we have written down the forces acting along the x direction on the six surfaces of this infinitesimal cuboid element.

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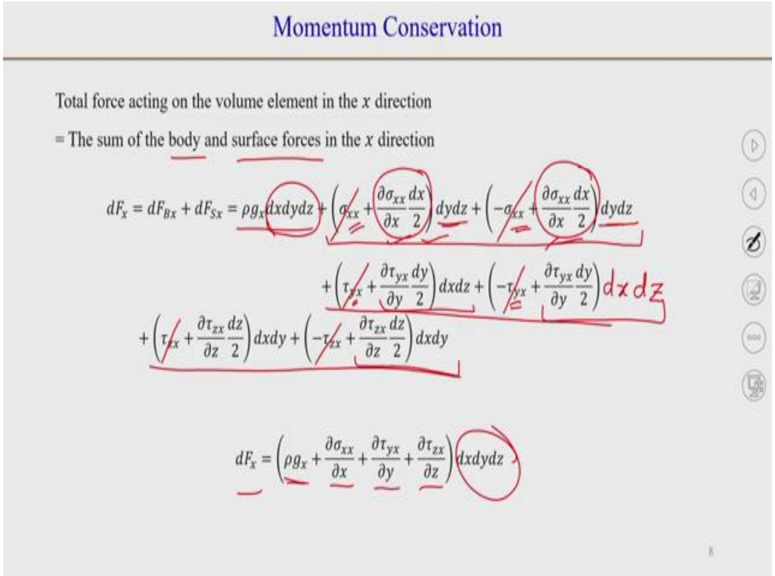

So, we can combine the all the body and surface forces, which act along the x direction and we can write this down. So this is the body force and the force σ_{xx}. So this is the force on the two x surfaces, the force on the two y faces and force on the two z faces, if we look at this the σ_{xx} and minus σ_{xx}, they will be cancelled out because both of them are multiplied by dydz.

Similarly, τ_{yx} and minus τ_{yx} they will be canceled out sorry there is a typo here, we can write dxdz. Then the same thing can be done here also τ_{zx} and minus τ_{zx} they will cancel out. So when we combine, if we look at these two terms, the second and third term here dydz is common and both of these terms both the terms are $\partial/\partial x$ of σ_{xx} into dx/2, so when we combine this will become $\partial/\partial x$ of σ_{xx} into dx.

So, adding these two terms will give us $\partial/\partial x$ of σ_{xx} into dxdydz and the same will be true for other terms. So, from here we will get, adding these two terms we will get $\partial/\partial y$ of τ_{yx} into dxdydz into ∂ by the last term will be $\partial/\partial z$ of τ_{zx} into dxdydz. So the force along the x direction will be $\rho\ g_x$ because we take dxdydz outside the bracket. So, $\rho\ g_x + \partial/\partial x$ of $\sigma_{xx} + \partial/\partial y$ of $\tau_{yx} + \partial/\partial z$ of τ_{zx}, so that is the force along the x direction.

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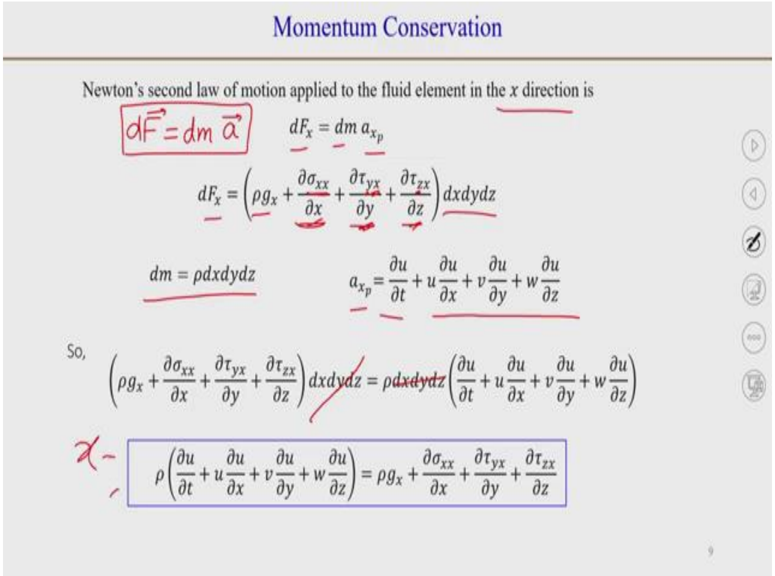

So, now we have dF =, dm into acceleration for a fluid, fluid element. If we take the component of all this, if we write the equation in Cartesian coordinate system and take the x component then this will be dF x = dm into acceleration along the x direction. And we can, we know all these terms, so dF we have just found out that dFx or the force along the x direction = dxdydz multiplied by in bracket $\rho\ g_x + \partial/\partial x$ of σ_{xx} $\partial/\partial y$ of τ_{yx} $\partial/\partial z$ of τ_{zx}.

You can see here that these terms are coming $\partial/\partial x$ of σ_{xx} $\partial/\partial y$ of τ_{yx} so it corresponds to the force on which the stress acts, similarly $\partial/\partial z$ of τ_{zx}. So, the derivative is taken with respect to the force on which the or with respect to the plane on which the stress acts. And dm = ρ dxdydz, the acceleration is $\partial/\partial t$ of u + u $\partial/\partial x$ of u + v $\partial/\partial y$ of u + w $\partial/\partial z$ of u.

So we can substitute all this and we will get the expression for the momentum conservation along the x direction. So, on substituting we get this and dxdydz can cancel out. So, we will get the momentum conservation equation, which is also called Cauchy momentum conservation or Cauchy momentum equations, which gives the momentum conservation along the x direction.

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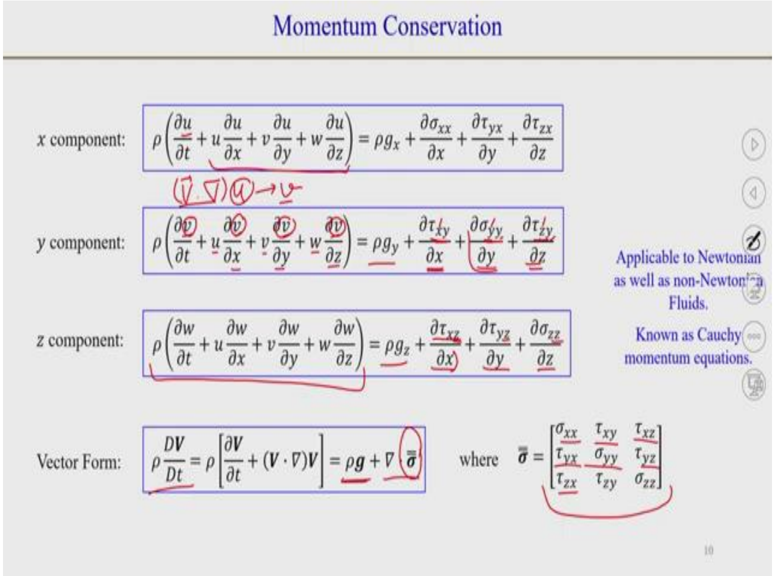

So, this is x component of momentum conservation equation and on the similar lines we can write the y component of momentum conservation equation. So, we can replace u component or the x component of velocity with y component of velocity here. We have already seen how do we write the acceleration. So, because this is vector V.∇ of u these are the three terms.

So, we can write from there that this will be replaced by, the u will be replaced by v when we write the x component, it will be u when we write the y component, this will be v, but this will remain same. So, we did this u ∂/∂x, v ∂/∂y + w ∂/∂z will remain same this will be v throughout. So, that is for the left hand side.

On the right hand side you will have ρ g_y + ∂/∂x of τ_{xy}, ∂/∂y of σ_{yy}, ∂/∂z of τ_{yz}, where all the stresses τ_{xy}, the stress acting on the x face on the x face along the y direction, stress acting normal stress on the y face along the y direction, stress acting on the z face along the y direction and the differentiation will be ∂/∂x because it is acting on the x face, ∂/∂y acting on the y face, ∂/∂z acting on the z face.

Similarly, we can write the equation for z component. So, again the v here will be replaced by w, so that is dm a per unit volume ρ gz here and ∂/∂x τ_{xz}, τ_{yz} and σ_{zz}, so these are the stresses and their differentiation with respect to the plane normal on which they. So, ∂/∂x of τ_{xz}, ∂/∂y of τ_{yz}, ∂/∂z of σ_{zz}, so these are called Cauchy momentum equations.

We can write down in the vector form. So, substantial derivative of velocity vector V multiplied by ρ that = ρ g + ∇ operated on the second order stress tensor, where second order stress tensor will have nine components and these are the nine components σ_{xx}, τ_{xy}, τ_{xz}, σ_{yx}, σ_{yy}, τ_{yx}, σ_{yy}, τ_{yz} and so on.

And this is applicable to Newtonian as well as non-Newtonian fluids because we have written the forces or the stresses in terms of stresses only, we have not replaced the, these terms with the relationship for the Newtonian fluids. So, these expressions are valid for Newtonian as well as non-Newtonian fluids. Now, we will see what will happen if we replace these stresses by the relationship between shear stress or stress and rate of strain for Newtonian fluid. These equations are known as Cauchy momentum equations.

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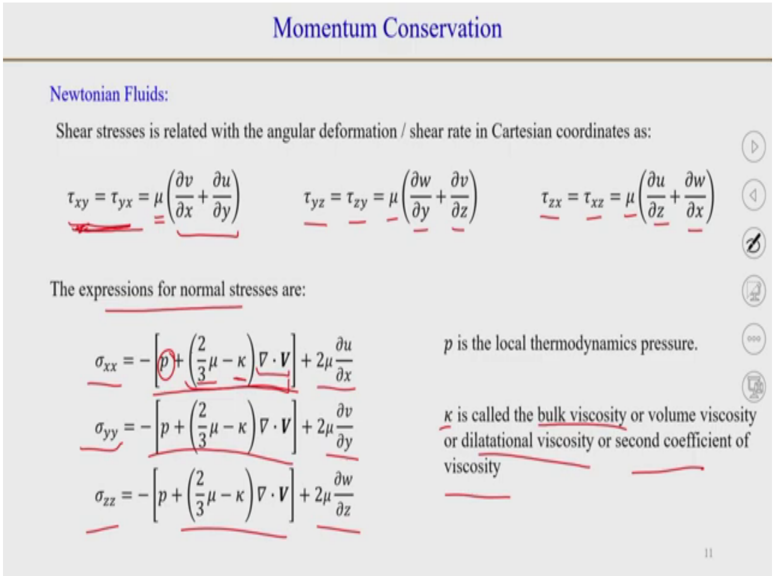

Now, for Newtonian fluid, we know that the shear stress is proportional to the rate of strain or angular deformation or shear rate, and we saw in the general form that on an xy plane when a shear force is applied τ_{xy} that will be equal to μ into the rate of strain or the angular rate of angular deformation in the xy plane and that ∂/∂x of y + ∂/∂y of u, that we saw when we looked at the kinematics and found rate of angular deformation of a fluid element.

Similarly, for a yz plane, so τ_{yz} will be equal to τ_{zy}, that will be equal to μ into ∂/∂y of w + ∂/∂z of v and for a zx plane that will be equal to $\tau_{zx} = \tau_{xz}$ and equal to μ into ∂/∂z of u + ∂/∂x of w. So, these are the expressions for the shear rates or sorry shear stresses in terms of shear rates and you have covered here six stresses or six component of the stress.

Now, we are left with the three normal stresses. So, without going into their derivations, we will be or I will be giving you the expressions for the normal stress which are in the form. So, if you look at σ_{xx}, the viscous normal stress is 2 μ ∂/∂x of u, σ_{yy} 2 μ ∂/∂y of v and σ_{zz} = 2 μ ∂/∂z of w. Now, we have this term in each case.

So, p here is the local thermodynamic pressure, which we can obtain from a equation state for a known molecular mass and temperature. So, if you use ideal gas p is, ρ = PM/RT you can use the relationship to find the thermodynamic pressure. Now, this is the term which is coming because of the stretching or compression of the fluid element that is why we have ∇.V term here.

If you remember we talked about the linear deformation of the fluid element ∇.V , which comes out to be, which will be multiplied by 2/3 μ minus κ and κ is called the bulk viscosity or the volume viscosity or because it represents dilation, so it is a dilatational viscosity or the second coefficient of viscosity. So, it has a number of names. But the good thing is that we are looking at the incompressible fluids, so this term is going to go away very soon, because ∇.V will be equal to 0 for an incompressible fluid.

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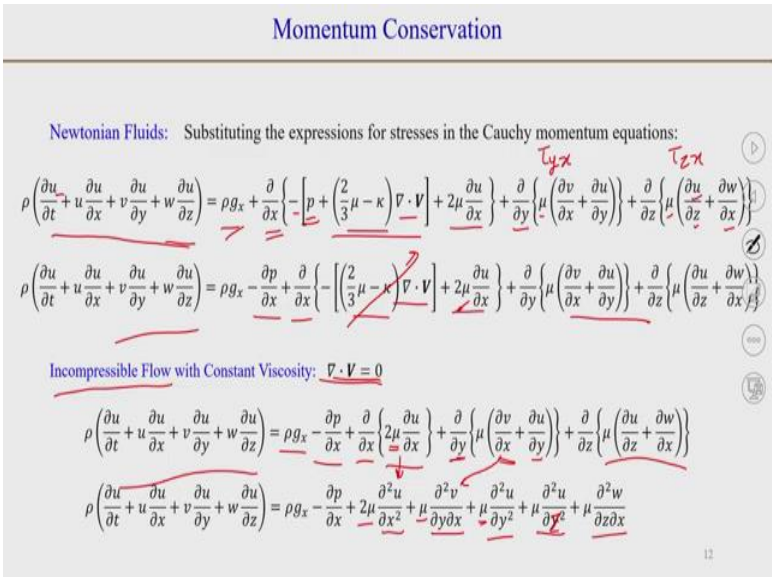

So, far the Newtonian fluid we can substitute all these terms, all the expressions for the stresses from there, from the previous slide. So, we have the left hand side dm into a divided by dxdydz this is what the left hand side is, it will remain as it = ρ g_x + this term was ∂/∂x of σ_{xx}, the normal stress along the x direction. So, σ_{xx} is replaced by minus of p + 2/3 μ minus κ ∇.V + viscous normal stress 2 μ ∂/∂x of u. The next term ∂/∂y of, because it is in the x direction, so we will have the forces along the x direction, so that would have been τ_{yx} and this would have been τ_{zx}. So, when we write ∂/∂y of τ_{yx} that will be equal to μ into ∂/∂x of v + ∂/∂y of u. Similarly, τ_{zx} will be equal to μ into ∂/∂z of u + ∂/∂x of w. We can further simplify this now.

So, the left hand side will remain same, the right hand side first term ρ g_x remain untouched. We can open this bracket and minus ∂ p/∂ x comes out. Then the next term will be ∂/∂x minus 2/3 μ minus κ ∇.V + 2/2 ∂/∂x of u and we can keep this other terms as it is. Now, we can use the two approximations that we can assume the flow because the flow is incompressible.

So, ∇.V will be 0, which will make this term to be 0 and the viscosity is constant, so it can come out from the derivatives. So, the left hand side is again same, the first two terms on the right hand side remain same, this term will become 0, then you will have ∂/∂x 2 μ, ∂/∂x of u and so on. Now, taking the viscosity out, so you will have this term changing to 2 μ, $\partial^2/\partial x^2$ of u + from the derivative of partial derivative with respect to y.

So, ∂/∂y of ∂ v/∂ x that will give you this term which is μ, μ is constant. So, $\partial^2/\partial y \partial x$ of v and then partial derivative is up to y of ∂/∂y of you, so that will give you μ, $\partial^2/\partial y^2$ of u and the last term μ can be taken out and this should have been μ into $\partial^2 u/\partial z^2$ + μ into $\partial^2 w/\partial z \partial x$. So, that is the simplified form of the equation.

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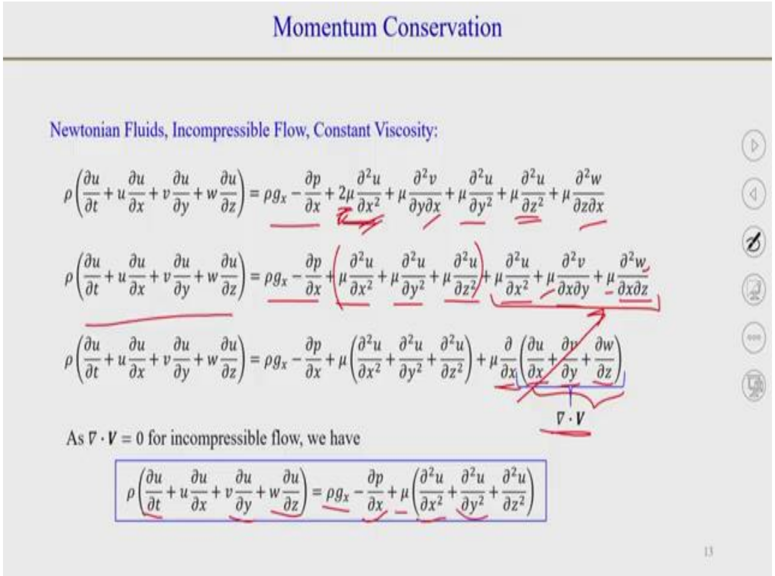

Now, we have this equation right here. So, now we can write this down further or simplify this further. So, left hand side remains same, the first two terms, they are same and we can expand this term or we can write this term twice in place of 2 μ we will write it μ $\partial^2/\partial x^2$ of u. So, first we write μ $\partial^2/\partial x^2$ of u because we want to combine all the second derivative or $\partial^2/\partial x^2$, $\partial^2/\partial y^2$ and $\partial^2/\partial z^2$.

So, this is μ $\partial^2/\partial y^2$ + μ $\partial^2/\partial z^2$ of u. So, we have written three terms. Then one term from here again, will come which will be μ $\partial^2/\partial x^2$ of u + the two remaining terms, μ $\partial^2/\partial x \partial y$ of v + μ $\partial^2/\partial x \partial z$ of w. So, we will combine the remaining term if we look at here, we can take or we can rewrite this in terms of μ ∂/∂x of ∂ u/∂ x + ∂ v/∂ y + ∂ w/∂ z, which is nothing but ∇.V.

And we know for an incompressible fluid ∇.V = 0, so this term will go away. So, we can write this = ρ ∂ u/∂ t + u del or ∂/∂x of u + v ∂/∂y of u + w ∂/∂z of u = the gravity term ρ g_x minus ∂/∂x of p + μ, $\partial^2/\partial x^2$ of u + $\partial^2/\partial y^2$ of u + $\partial^2/\partial z^2$ of u.

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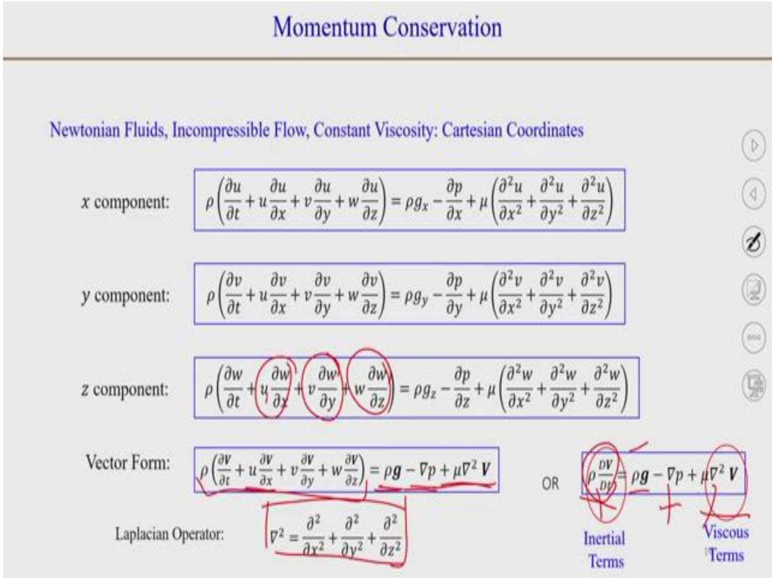

So, what we wrote there the exercise was done for x component of equation, but we can write similarly the y and z component of equation or in the vector form. So, we have already seen this term in the vector form that will be ρ DV/Dt. On the right hand side ρ g which is the body force minus ∇p + μ ∇^2V, ∇^2 is Laplacian operator when we were discussing the introduction to vectors we discussed the Laplacian operator.

And this is where it is important or useful for us to look into so this terms becomes μ ∇^2V, and this is the easier form to remember. If we understand the substantial derivative, if we understand the grad and we understand the Laplacian operator we will be able to expand this term at least in the Cartesian coordinate system. So, what we have been able to do in this is derive these system of equations which are called Navier strokes equations.

And in this Navier strokes equation the first term here is called the inertial term and this is the pressure term due to viscosity, so viscous term and the body force or here the gravitational term. Now, if we just summarize what we did, we took a differential fluid element of side dxdydz and we wrote down the second, Newton's second law of motion for this dF = dm into a and we took it for x component.

So, dm = ρ dxdydz and a we could write in terms of velocity field, which is DV/Dt or capital V, capital DV/Dt which is the substantial derivative. Now, the next task was to write down dF which is the combination of body forces and surface forces. The body force was simple because we could write ∂ x dydz into g + the surface forces. So, the surface forces we wrote down the surface forces in terms of the stresses on all the six elements of the fluid and we got Cauchy momentum equations there in terms of the stresses.

Then we substituted the stresses in terms of the relationship between stress and rate of a strain. So, what would have happened or what has happened here that the stress which was unknown we use the constitutive equation, which is the relationship between say, stress and the velocity gradients or the rate of deformation or the rate of a strain, and these are in terms of velocity field, so the stresses are replaced by μ multiplied by the expressions in terms of velocity or the derivatives of velocity.

So, those substituted and we finally what we got is Navier strokes equations. And it is important to understand this equation as well as if possible some form of this equation, so that you can visualize things. In the subsequent classes, we will try to solve some problems because these equations are non-linear equation. So, you might see here because the terms here these terms, they are multiplication of velocity.

So, you see here u into w. So, what happens that these becomes non-linear terms. And so this equation, the Navier strokes equation is a non-linear partial differential equation, and solving non-linear partial differential equation analytically is possible only for few simplifications. So, when we look at that how you can find the solution of it and the first thing is that how we can eliminate or is there any simplification by which we can eliminate the non-linear term.

So, eliminating non-linear term in the cases where you are able to eliminate the non-linear term you are able to solve these partial differential equations analytically. By analytically I mean that you can write down the equation and then integrate and find the solution which will be possible. When it is not possible to do this analytically, what one can do, one can transform these partial differential equation using numerical techniques they can be transformed into algebraic equation and solve the algebraic system of equations and that is what is called computational fluid dynamics.

So, because these equations are not possible to solve analytically for most of the cases, so a lot of effort in last say 50 years has gone in solving these equations using computers or using numerical methods and computational fluid dynamics now has become a mature field where you can solve any kind of fluid flow problem using computers by discretizing these equations. We will stop here. Thank you.