Fundamental of Fluid Mechanics for Chemical and Biomedical Engineers
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Lecture 23
Mass Conservation: Differential Analysis

In the previous lecture we discussed about mass conservation, we derived differential form of mass conservation or continuity equation. Continuing with that, we are going to discuss the mass conservation equation in cylindrical coordinates, in the previous lecture, what we discussed is or derived the mass conservation equation, wherein we took a control volume of cuboid type having dimensions dx, dy and dz. And then we took the mass conservation equation, which we derived from Reynolds Transport Theorem and implemented or found the 2 terms, which was, one was the volume integral $\frac{\partial}{\partial t}$ or integral control volume, so that term. And then the integral of ρV.dA over the 6 faces of such a cuboid.

Now what we need to see is that we have derived the Cartesian coordinate equation and from that we could arrive the mass conservation equation in the vector form. But there are certain problems especially in Mechanical Engineering as well as Biomedical Engineering applications where we need to deal with flow in cylindrical pipes or cylindrical tubes, vessels etc. So, it is convenient to use a cylindrical coordinate systems for such cases.

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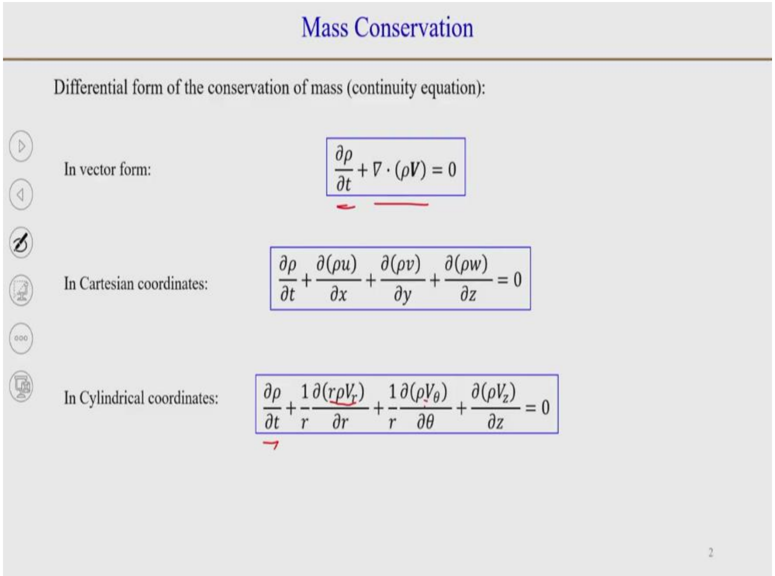

So we will look at the conservation equation in the cylindrical coordinate system here. So, if we write down the mass conservation equation in the vector form, $\frac{\partial \rho}{\partial t}$ + ∇.(ρV) = 0, which is the general form of continuity equation. In Cartesian coordinate, the first term of course will remain same, the second term, when you open up, you have $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}$.

In cylindrical coordinate system, when you expand the second term, the first term will remain same, unsteady term $\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho V_r)}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z}$.

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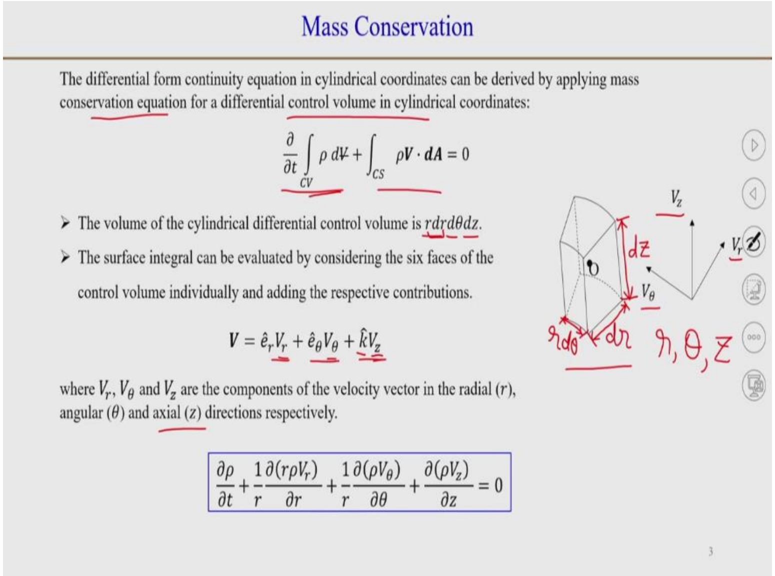

Now this can be derived by doing the same thing what we did when we derived the continuity equation in the Cartesian coordinate system, so we can apply the mass conservation equation for a differential control volume in cylindrical coordinates. So we have this conservation equation and we can consider a control volume having dimensions in cylindrical coordinates. So, if we consider a cylindrical coordinate there are 3 directions, what we consider in cylindrical coordinates, r θ and z. So, the velocity component will be V_r, V_θ and V_z.

Now if we take a control volume, then in this control volume, the dimensions of this control volume will be the dimension along the redial direction, this will be dr and along the z direction which will remain same. z direction in the Cartesian as well as cylindrical coordinate, it will be same, so it will be dz. And the angular direction or azimuthal or θ direction, the angle that is has rotated is d θ, but this is just the angle, so the length of this will be rdθ.

Now on the other side, it will be r + dr into dθ and taking into account that we can find out the two terms and combine the terms, simplify it and we should be able to find the equation in the cylindrical coordinate system. The volume of this small control volume will be rdr into dθ into dz. So, basically dr into rdθ into dz, because such a small dimension we can approximate it to be a cuboid.

Now the surface integral can be evaluated by considering again the 6 faces of this, and find the control, the fluxes on the control surfaces and at the contributions. The velocity vector in this case will be e_r, V_r + e_θ, V_θ and kV_z. Now, we could have written in place of k, e_z, but considering that the z coordinate is same in Cartesian as well as cylindrical coordinate. So, I have written this as e_z = k. V_r, V_θ, V_z are of course the component along r, θ and z directions.

e_r, e_θ and k, they are the unit vectors along the radial direction, θ direction or angular direction and k is unit vector along the z direction. So, when we talk about cylindrical coordinates, so z direction is also generally referred as axial direction because it points or it is along the axial direction. So, when we put together we get this equation in the cylindrical coordinate. So, we are not going to do this, and I suggest that you do this exercise and try to derive the equation in the cylindrical coordinate. But we are going to look at a different perspective and try to find this equation from there.

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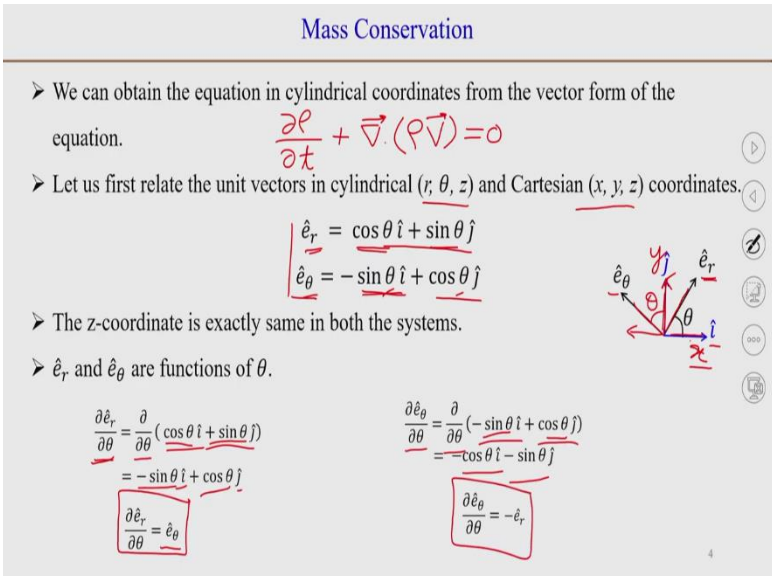

So, we can use the vector form of the equation, which is basically $\frac{\partial \rho}{\partial t}$ + ∇.(ρV) = 0. See, if we can write down the del operator in cylindrical coordinates and V in cylindrical coordinates and find the dot product, then we will be able to obtain the expanded form of continuity equation in cylindrical coordinates.

So, to do that we need to find and relate first, because we might need to understand the derivatives of unit vectors. So, first what we will do is, we will relate the unit vectors in cylindrical and Cartesian coordinates. So, if we draw the unit vectors, the unit vectors in the Cartesian coordinates, they are shown in the blue colours. Along the x direction you have unit vector i, along the y direction you have unit vector j.

If you take the unit vectors in the cylindrical coordinate system, the unit vector along the r direction is e_r and unit vector along the θ direction is e_θ. So, we can write down e_r in terms of i and j, so if we do that e_r is equal to the component along the x direction will be cosθ i, so along the component because the magnitude of this vector is 1 and the component along the x direction will be cosθ i. Along the y direction, this component will be sinθ j and we know that $\cos^2\theta + \sin^2\theta = 1$.

So the magnitude of e_r remains 1 and this is the vector form, we can write e_r in terms of i and j. Similarly we can write e_θ, so e_θ, if the angle between this line and this line is θ, they are normal to them, these two lines, the angle between them will also be θ. So, we can write e_θ is equal to along the i direction, which will be pointing out in the minus directions, so it will be minus sinθ i + cosθ j along this direction, positive direction.

So, we have been able to write e_r and e_θ in terms of i and j. And the important thing to notice here is, that e_r and e_θ, they are function of θ. So, if we take the derivative of e_r and e_θ, with respect to θ, it will not be 0. So, let us find out the derivatives of first e_r with respect to θ. So, $\frac{\partial e_r}{\partial \theta} = \frac{\partial e_r}{\partial \theta}$ which e_r is cosθ i + sinθ j.

So, when we differentiate cosθ, differentiation of course θ is minus sinθ i + differentiation of sinθ is cosθ j. So, if you look at e_θ, which is minus sinθ i + cosθ j, so $\frac{\partial e_r}{\partial \theta}$ is basically nothing but e_θ. So, this is a result that we will use when we find, when we expand the continuity equation in the cylindrical coordinate.

Now we will similarly find the derivative of unit vector e_θ with respect to θ. So, we write $\frac{\partial e_\theta}{\partial \theta} = \frac{\partial}{\partial \theta}$. e_θ is minus sinθ i + cosθ j. And when we differentiate it, minus sinθ, so when we differentiate minus sinθ, we will get minus cosθ and when we differentiate cosθ, we will get minus sinθ. So, minus cosθ i, minus sinθ j, which will be equal to minus er. So, these 2 results, we are going to use.

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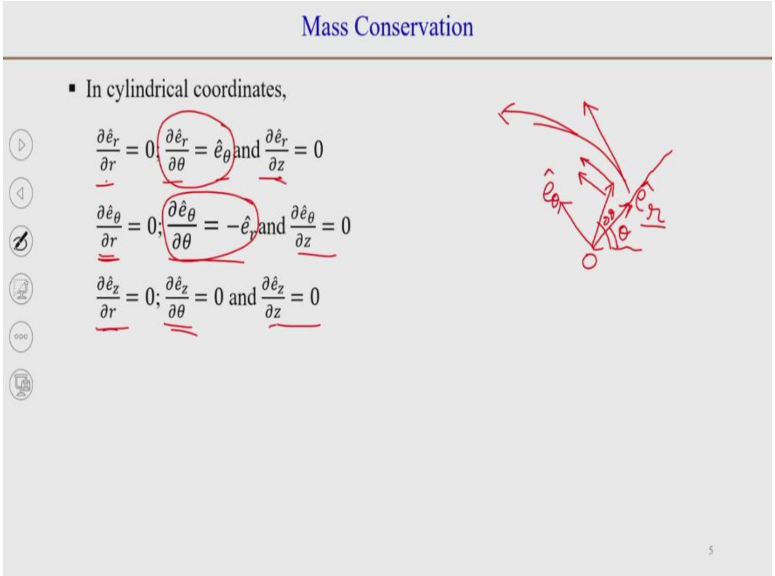

Now so if we write down the derivatives of all the variables, all the unit vectors in the cylindrical coordinates, so the unit vector, the differentiation of e_r with respect to r θ and z is 0. The derivatives or the partial derivative of e_θ with respect to r and z, they are 0. The partial derivative of e_z is 0 with respect to all the variables, r θ and z. There are only 2 partial derivatives which are nonzero here and we can understand that.

So if we look at a cylindrical coordinates system, let us say, this is my origin and I have er, e_θ and the normal vectors will be, the axial vector or the unit vector in the z direction will be normal to it. So, if we first look at the axial unit vector, e_z or k, then this unit vector is not changing, when you change the unit vector magnitude is always going to remain 1, if it is the unit vector along z direction, you change r or θ or z, the unit vector along the z direction is going to remain same.

If you look at er, so er, the unit vector, the magnitude is always 1, but if you are looking along the r direction, the direction will also remain same. So, that is why the derivative of unit vector e_r with respect to r is 0. But if you look at unit vector r, if this is θ and you turn it by a small angle, delta θ, let us say then the unit vector direction of e_r direction will change. So, that means with the change in θ, e_r is changing that is why we have this nonzero. And we can see with a bit of vector geometry, we can see that it is e_θ. We have already seen in terms of i and j that this comes out to be a θ.

Similarly, when you change z, e_r direction does not get changed, with the same argument e_θ does not change with r. If you have e_θ at one r, e_θ is pointing here at another r also e_θ will be pointing here. But if you change along a curve, if you have e_θ pointing out here, next point it will be pointing out here. Again, there is a change in the direction. That is why you have dou θ of e_θ is minus r. And again the last term, the vector does not change direction with z so this is 0.

So, that might just motivate us or we would like to look at that what happens when we talk about the unit vectors in x, y and z coordinates. So, in Cartesian coordinates, the unit vectors i, j and k their direction remain same if you change, x or y or z. So, $\frac{\partial}{\partial x}$ of i, j and k, 0 $\frac{\partial}{\partial y}$ of i, j, k is 0 and $\frac{\partial}{\partial z}$ of i, j, k and i, j and k is 0. That is what we have here.

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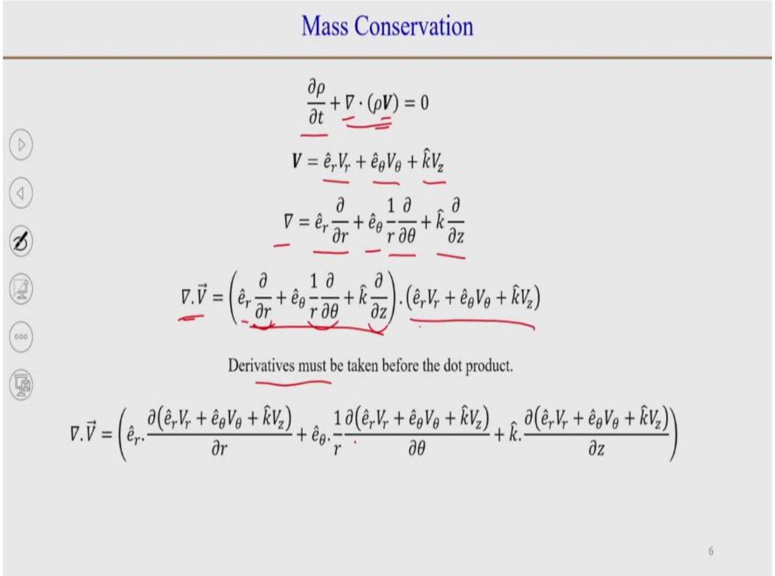

So, with this, now we can try to substitute or we can try to write this equation in terms of unit vectors in cylindrical coordinates. So, we have 2 vectors here, the del operator, and vector V. So, vector V, we can write e_r V_r + e_θ V_θ + kV_z. And the del operator is e_r $\frac{\partial}{\partial r}$ + e_θ 1/r , $\frac{\partial}{\partial \theta}$ + k $\frac{\partial}{\partial z}$.

Now I am not going to look at how this expression for del operator comes by when we are expanding it in cylindrical coordinates. You can look into it and it can be derived from the definition of del operator in the cylindrical coordinates and then you can write r in terms of say $\sqrt{x^2 + y^2}$ and θ = $\tan^{-1}$ y/yx. You can take the derivatives and substitute and you will be able to find. So you can write $\frac{\partial}{\partial x}$ in terms of r θ z, $\frac{\partial}{\partial y}$ in terms of r θ z, or the partial derivatives in the respective coordinates and you will be able to obtain it. But we are not going to do that here.

So, when we expand or when we substitute here, the first term will remain same because it does not have any vector. We need to find $\nabla.(\rho V)$, but for simplification, I will just write down this in terms of $\nabla.V$ and then later on when we find, we can replace v with ρ V.

So, let us write down del, the same definition and the definition of V here. But, the one thing we need to note here is, before we take the dot product, we need to derive or we need to take the derivative or we need to differentiate this. Now, when we were working with cylindrical, sorry, Cartesian coordinates, or when we work with Cartesian coordinates, we can write it $\frac{\partial}{\partial x}$ i or i $\frac{\partial}{\partial x}$, it does not make a difference. But here why I have written in this manner or I have tried to follow this convention because of this region, before you take a dot product, first you need to take the derivatives.

So, let us take the derivative for each case and see what do we get. So, what we substitute here is, $e_r \frac{\partial}{\partial r}$ of the velocity vector + e_θ 1/r $\frac{\partial}{\partial \theta}$ of the velocity vector + k . $\frac{\partial}{\partial z}$ of velocity vector.

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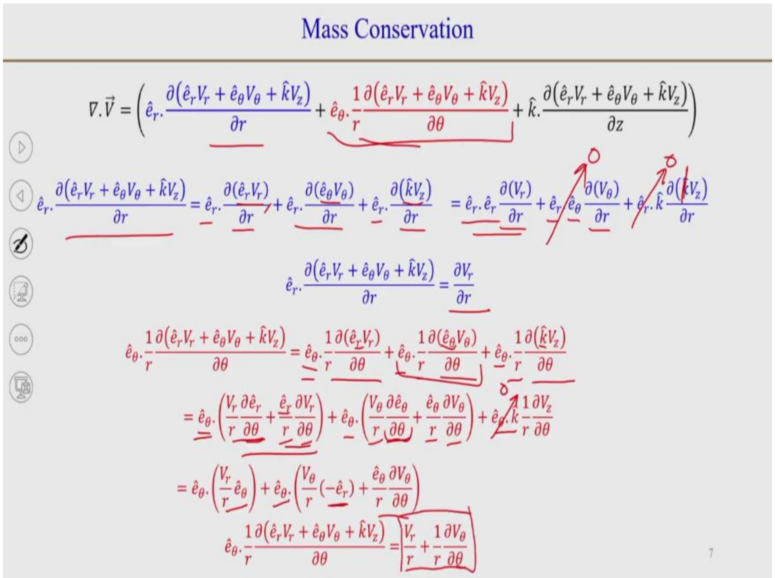

Now we will look at each term in succession and simplify. So, I have coloured the 3 terms, so we will look each term one by one. So, let us look at the first term, which is $e_r .(\frac{\partial}{\partial r}(e_r V_r + e_\theta V_\theta + kV_z))$. So, the first term when we look at $e_r .\frac{\partial}{\partial r}(e_r V_r)$.

The second term, $e_r .\frac{\partial}{\partial r}(e_\theta V_\theta)$. and the last term $e_r .\frac{\partial}{\partial r}(kV_z)$. Now in all the brackets we have a product, so we will expand it, when we will expand this term, because e_r does not depend on r so it can come out, so we can write $e_r.e_r$ and $\frac{\partial V_r}{\partial r}$.

Similarly, e_θ also does not depend on r, so we can write $e_r.e_\theta$, $\frac{\partial V_\theta}{\partial r}$, + e_r.k $\frac{\partial V_z}{\partial r}$. We do not have a k here now. So, $e_r.e_\theta$ will be 0 because e_r and e_θ are the 2 unit vectors in normal direction, r and θ. Similarly, e_r.k will also be 0. So, we will have only this term, $e_r.e_r = 1$. So, this will give us the first term is simplify to $\frac{\partial V_r}{\partial r}$, which is radial component of velocity.

Let us look at the second term, so when we expand it, we will get e_θ. $\frac{1}{r}\frac{\partial e_r V_r}{\partial \theta}$, e_θ $\frac{1}{r}\frac{\partial e_\theta V_\theta}{\partial \theta}$ + e_θ $\frac{1}{r}$, $\frac{\partial k V_z}{\partial \theta}$. Now, remember e_r and e_θ, they are function of θ. So, we need to expand these products so we will look into it, e_θ dot this term we expand in this bracket here. So we can write V_r this product of, differentiation of a product so V_r and $\frac{1}{r}$, so $\frac{V_r}{r}\frac{\partial e_r}{\partial \theta}$+ $\frac{e_r}{r}\frac{\partial V_r}{\partial \theta}$.

The next term, this term we look into now, so e_θ dot the differentiation of e_θ, so $\frac{\partial e_\theta}{\partial \theta}$ into V_θ / r + e_θ, unit vector e_θ / r $\frac{\partial V_\theta}{\partial \theta}$. The last term because k unit vector does not depend on θ, so we can write e_θ . k 1/r $\frac{\partial V_z}{\partial \theta}$ and this will be 0.

Now if we will look at here, we have $\frac{\partial e_r}{\partial \theta}$, which = e_θ. So, this term will give us V_r / r, e_θ unit vector. In this term we have a e_r so when we take a dot product of e_θ and er, that will be 0. So, this term will be eliminated. So, now when we come here the differentiation of e_θ is equal minus e_r so that is what we have written here, V_θ / r of minus e_r + e_θ / r $\frac{\partial V_\theta}{\partial \theta}$.

Now e_θ . er, that will be 0, so this term will be eliminated and you will have this term because $e_\theta.e_\theta$ will be equal to 1. So, finally, this term you will have, when you simplify the second term here, then you will have 2 terms here, V_r /r + 1/r $\frac{\partial V_\theta}{\partial \theta}$.

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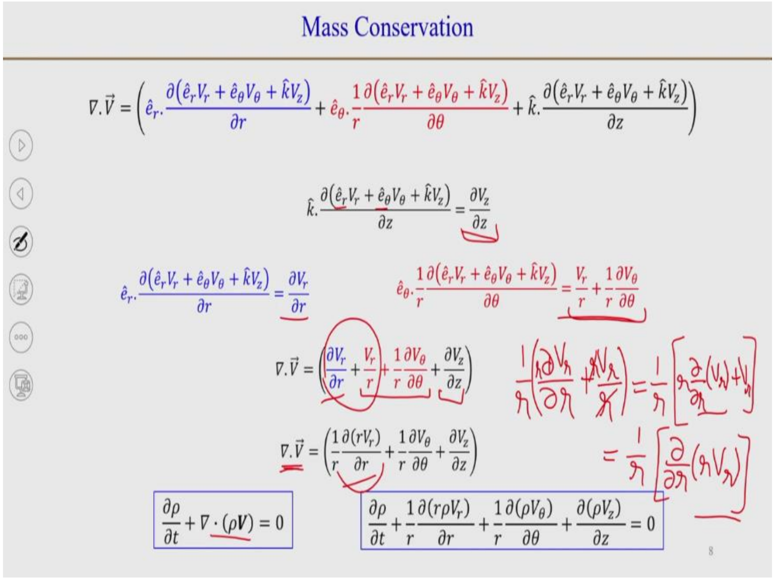

The last term is simple because you have $\frac{\partial}{\partial z}$ and neither e_r nor e_θ depend on z so the dot product of k . e_r will be 0, k.e_θ will be 0, so you will have only the last term k . k is = 1, so $\frac{\partial V_z}{\partial z}$ is this term. So, now we can replace all the 3 terms here, the simplified form of all the 3 terms in this ∇.V. So, when we substitute ∇.V is equal to, then we can write the first term will be $\frac{\partial V_r}{\partial r} + \frac{V_r}{r}, \frac{1}{r}, \frac{\partial V_\theta}{\partial \theta}$ and the last term $\frac{\partial V_z}{\partial z}$.

So, we have written ∇.V and we can combine these two terms here to get this. So, let us just see how do we get it, if we write $\frac{\partial V_r}{\partial r} + \frac{V_r}{r}$and we can multiply it with $\frac{1}{r}$ so we will get, when we multiply with 1/r outside, then we will have to multiply with r inside. So, we will get $\frac{1}{r}$ and this r and r will cancel out so you will have $r\frac{\partial V_r}{\partial r} + 1$.

Or you can write this, 1/r and the term in bracket, if you look into it carefully it is $\frac{\partial(rV_r)}{\partial r}$. When you expand r V_r of r $\frac{\partial V_r}{\partial r}$ +, sorry this should have V_r here + V_r into 1, so this is what we have written replacing these two terms. So, this is 1/r $\frac{\partial rV_r}{\partial r}$. So, this gives us del operated on vector V and we can substitute this in the vector form of continuity equation. So as we said earlier that in place of ∇.V, we will write ∇.(ρV) and that will be our expanded form in the cylindrical coordinates.

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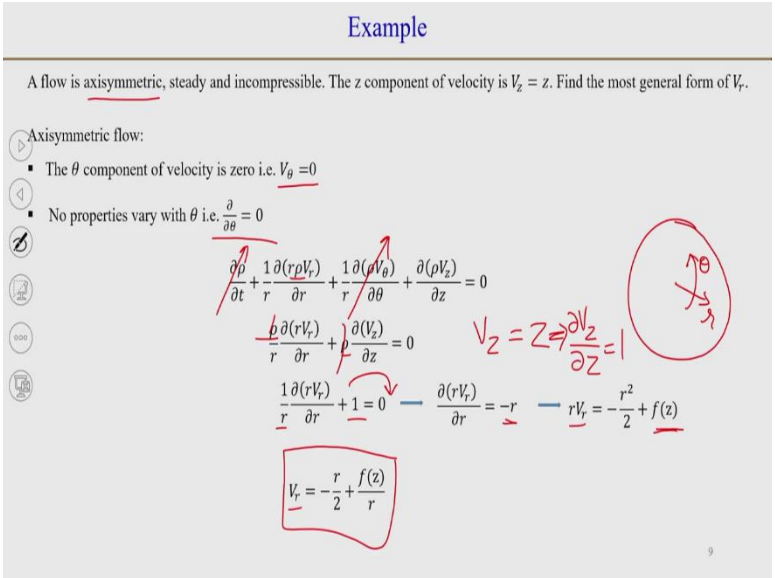

So, now we will look at a simple example, which asks us that if flow is axisymmetric, it is steady and incompressible. The z component of velocity has been given which is V_z = z and we have to find the form of V_r or the radial component of velocity. So, we already know what is a steady flow? That means $\frac{\partial}{\partial t}$ is 0, so the unsteady term is 0, when the flow is incompressible, density is constant.

Now the axisymmetric flow, so axisymmetric as the name suggest simply means that the flow is symmetric about the axis. So, if you look at a cylindrical pipe where the cross section is circular, this is the r direction and this is the angular or θ direction. So, when things do not change along the θ direction, then we can say that the flow is symmetric about the axis or flow is axisymmetric.

So, when the flow is axisymmetric, then V_θ = 0 or θ component of velocity is 0 and there is no variation in flow properties along the θ directions. So, that means if we take the partial derivatives with respect to θ, that is $\frac{\partial}{\partial \theta}$ = 0. So that is our axisymmetric flow, that means the θ component of velocity is 0, now we can write the continuity equation because the flow is steady, so this term is 0. The flow is axisymmetric so this term the $\frac{\partial}{\partial \theta}$ term and the term containing V_θ is 0.

Now we end up with these two terms because the flow is incompressible. So, ρ is constant and we can take it out of the derivative or we can also divide by ρ so it will cancel out, so we will have the equation in the form of $\frac{1}{r}\frac{\partial}{\partial \mathrm{r}}$ + $\frac{\partial\, V_z}{\partial \mathrm{z}}$. Now V_z = z so $\frac{\partial\, V_z}{\partial \mathrm{z}}$ = 1 so that is what we have substituted here.

Now we need to integrate this equation, so we have to integrate this equation, we can do $\frac{\partial}{\partial \mathrm{r}}$ of r V_r, we take this term on the other side so that it will be minus 1 and multiplied by r so that will be equal to minus r. Now when we integrate this to r, then we will get rV_r = - r^2 + a constant. This will be constant with respect to r, but it can still be a function of z because $\frac{\partial}{\partial \theta}$ = 0, the flow is steady so it will not be a function of time, it will not be a function of θ, but the constant can still be a function of z.

So, we will just take r on the other side and that will give us the general form of V_r where V_r = minus r / 2 + f(z) / r, of course if we are asked, what is the simplest form of this equation when f(z) becomes 0 then we get the simplest form which will be V_r = -r / 2.

So, that is all for this lecture, in this lecture we saw some of the properties of cylindrical coordinate unit vectors. So, we saw the unit vectors, the derivatives with respect to r and θ, derivative of e_r and e_θ with respect to θ, they are nonzero whereas the other derivatives of unit vectors are 0 and we used that to find ∇.V so we will use the cylindrical coordinate to solve problems quite often.

Let us stop here, thank you.