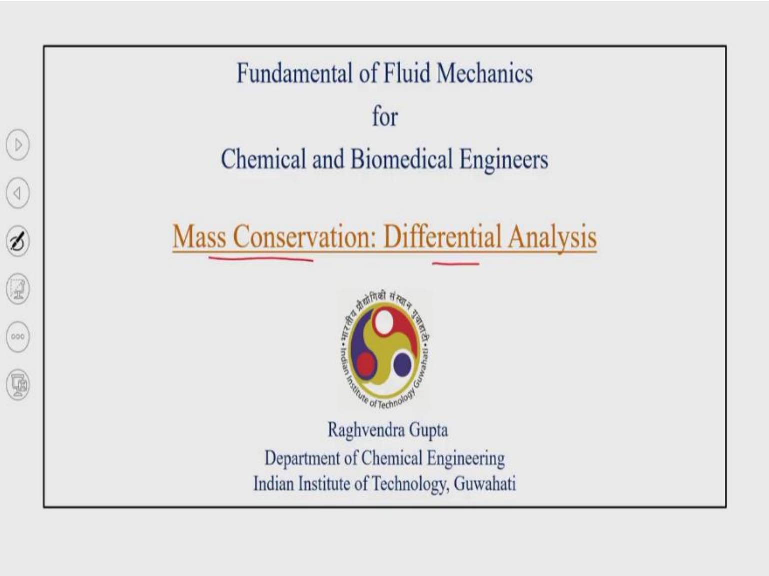
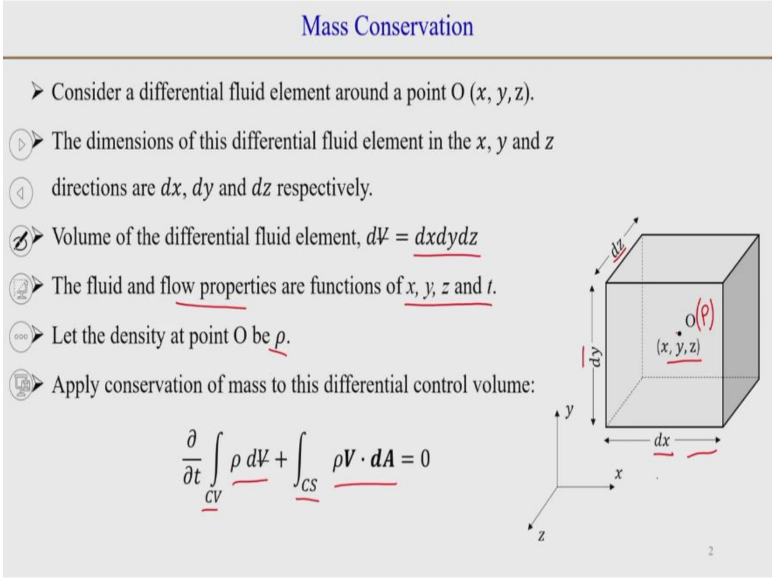
Fundamental of Fluid Mechanics for Chemical and Biomedical Engineers
Professor Dr. Raghvendra Gupta
Department of Chemical Engineering
Indian Institute of Technology, Guwahati
Lecture 22
Mass Conservation: Differential Analysis

(Refer Slide Time: 00:40)



In this class we will derive the mass conservation or continuity equation in the differential form and then use it to solve a problem. We will do this Cartesian coordinate system. We have already seen the mass conservation in integral form where we are doing macroscopic balances. So there we used the Reynolds Transport Theorem to find the mass conservation equation where we replace the general extensive property N with m and the intensive property η, which is N /m that is equal to 1.

(Refer Slide Time: 01:32)



So, again what we will do is, we will take a small control volume which is differential control volume, we can say. And this differential control volume, we will assume it to be of a cuboid form which have dimensions dx, dy, and dz along the xyz coordinate. So, we will consider Cartesian coordinate system and then we will derive the mass conservation equation in the differential form for cartesian coordinate system. We will be able to write it in the vector form and this vector form can also be used or can be expanded in the cylindrical coordinate system.

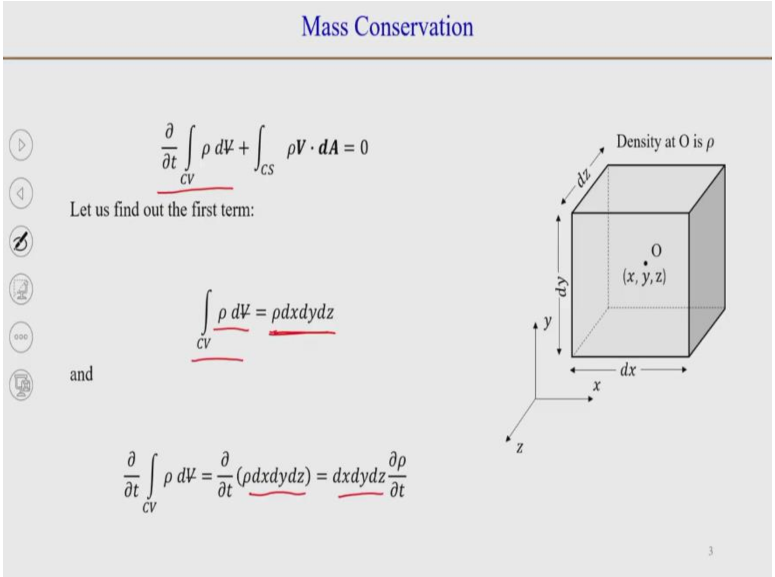
So, let us look at this, we have the differential volume, differential control volume, the point at its centre is O which has coordinates x, y and z. So, the point o is at a distance dx/2 from the x face on the left side and dx/2 from the left face on the right side and so on. So, it is at the centre of this cube.

Now, this cube, the dimensions are dx, dy, dz. So, the volume of this cube will be not cube, it is cuboid because the dimensions are different, dx, dy and dz and they are not necessarily equal. So we have dv = dxdydz. We will not assume anything about the independence of the fluid or flow property. So when I talk about fluid properties, I am talking about fluid properties such as density and viscosity.

So, density can be a function of space that means the density can be different at x, y, z, 1, 1, 1 and x, y, z 1, 2, 3. Density can also be different with time so density can be a function of time similarly the flow properties which can be the pressure, which can be the stress or which can be the velocity field. So, here when we derive mass conversation equation, we will be concerned with density and velocity. So the density and velocity we will assume that they are function of space and time.

Now let us say the density at this point O is ρ. And then we can apply the Reynolds Transport Theorem or the mass conservation equation in this differential control volume. We will assume that the flow can come in or go from this control volume. So, the differential equation or the mass conservation equation is ∂/∂ t of integral over the control volume ρ d$\forall$ + integral over the control surface or area integral ρ V.dA. So now our task remains to evaluate these 2 terms for the differential control volume that we have considered.

(Refer Slide Time: 05:17)

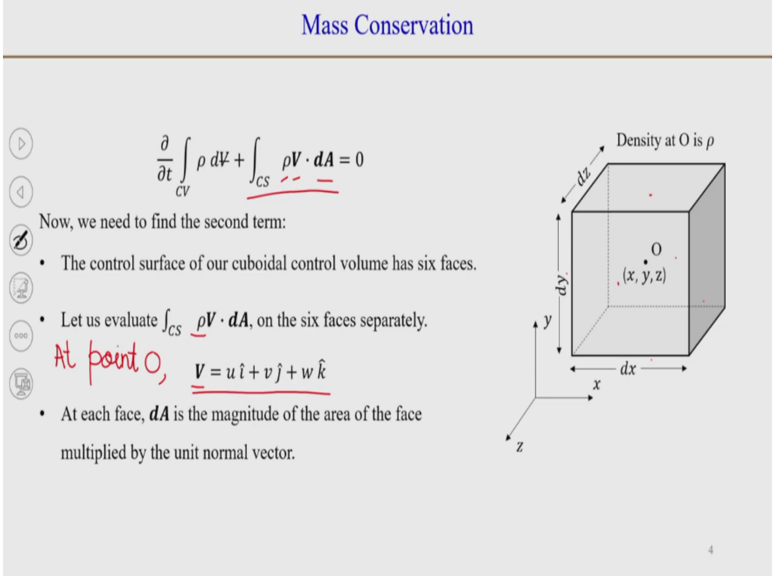


So, let us look at the first term of this equation. Now, the first term is basically integrating density over this control volume. So because the density will be a continuous function and we can write Taylor expansion for density and then it will be varying in all the directions, but when we do the summation or the integral over the control volume, then we will have that is equal to ρ dxdydz.

So, in this small control volume the density is ρ, which is the density at point O into dx, dy, dz. When we will be finding out the terms for the second term or when we will be evaluating the fluxes at different places then we will consider the variation of density because the density will be varying say, for example, x to x +, dx/2 or x to x - dx/2. In this case because we are taking the integral over the entire control volume so when you take the density at the 2 symmetric faces that will cancel out and finally you will end up with this expression.

So, you can do that exercise for yourself or to convince yourself. Now because the first term is time derivative or partial derivative with respect to time of this integral over the control volume ρ d$\forall$ so when we replace integral ρ d$\forall$/ρ dx dy dz and take its time derivative. Now the control volume is not going to change, its boundaries are not going to change the volume, it is going to remain fixed so dx, dy, dz can be out of this differential or this will not be dependent on time. So dx, dy, dz come out of the differential term and $\partial \rho/\partial t$ remain because the density can be a function of time or density is a function of time, we have considered that. So, that is our first term.

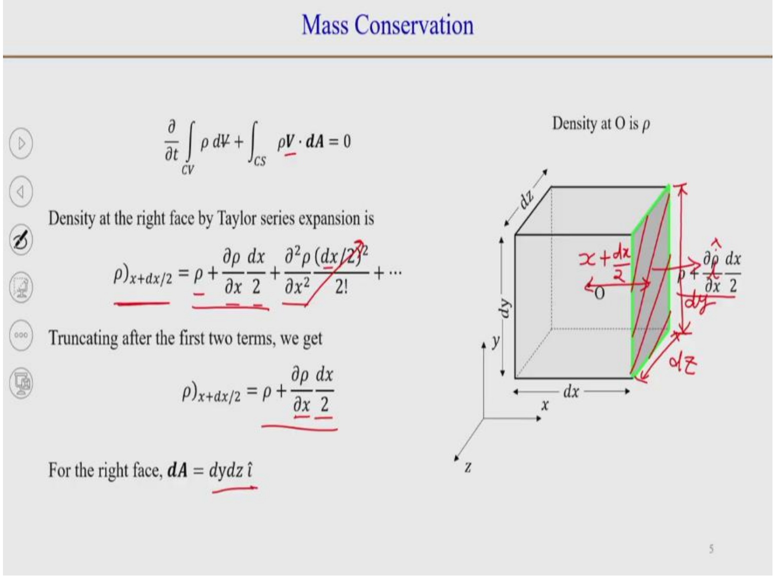
(Refer Slide Time: 08:04)



Now we will try to find out the second term, which is the area integral over the control surface. So, for this control volume, all its faces, there are 6 faces in this cube 1, 2, 3, 4, 5 and 6. So, left and right faces along the x direction as we can see, top and bottom faces along the y direction and front and back faces along the z direction. So, we will need to evaluate the fluxes, which is basically ρ V.dA, so we need to evaluate each of these term ρ, V and dA for each of these areas and then substitute in this so that we can find out the second term.

So, we need to evaluate on the 6 faces, the velocity, this velocity vector. Let us assume that at point O, the velocity vector V =ui + vj + wk, which are the u, v, w, are the x, y and z component of velocity at point O. So, density we know that at point O is ρ, velocity at point O is ui + vj + wk. And now we also need to find the magnitude of area vector dA and the direction will be normal to that particular face depending on it is x plane, or y plane or z plane and it is pointing out in the positive or negative direction.

(Refer Slide Time: 10:04)



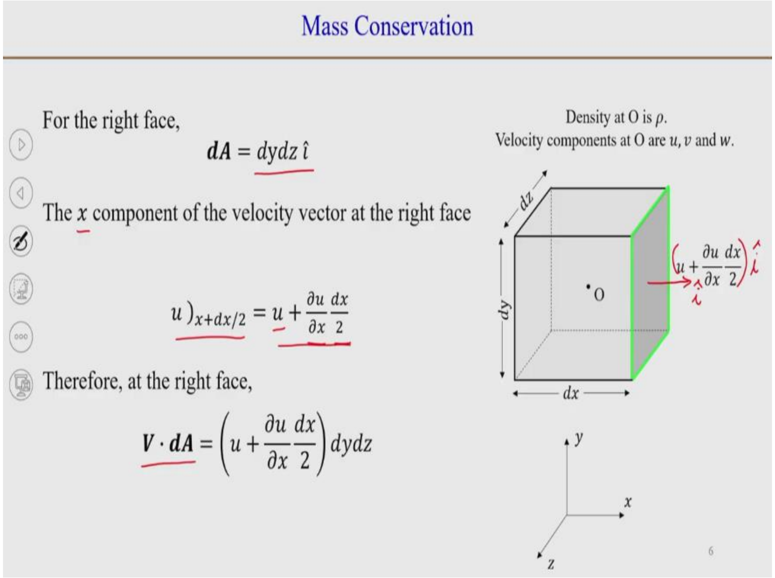
So, let us do that exercise and we will start with the first face, which is shown by a green boundary here. So this is the right face and the normal to it will be in the positive x direction so it will be i direction and the dimensions of this face, this is, a x-plane so yz it will be in yz plane. So, this dimension is dy and the other dimension is dz, so dA will be dy, dz into i.

Now we also need to find the density and velocity at this face. So the density we can write because this face is at a distance, so if we take centre to centre distance between the face and the centre of this cuboid that will be x + dx/2. So, using Taylor series, we can write an expression for the density at this face, which will be ρ at x + dx/2 =ρ + ∂ ρ/∂ x into dx/2 + second derivative of ρ with respect to x multiplied by 1/factorial 2 into dx/2 whole square and so on. So, you will have our terms which are higher order.

Now as we know that we have assumed dx, dy, dz to be small so the higher order terms dx squared and higher dx cube etc., they will be smaller when we compare them with respect to dx. So, those terms we will neglect so we will consider only the first 2 terms, so this will give us the density on the right face, which is ρ + ∂/∂x of ρ into dx/2. So, if you look at that is the density at point O + the variation of density or the density gradient multiplied by dx/2 so basically considering the linear variation of density that is the density on the right face. Now the area as we have already discussed is dydz i.

Next we need to find what is the velocity vector at this face V.dA because the velocity V.d a in that only the normal component of velocity will be relevant because the other 2 terms of velocities, when you take the dot product with the area vector, which has i that will become 0.

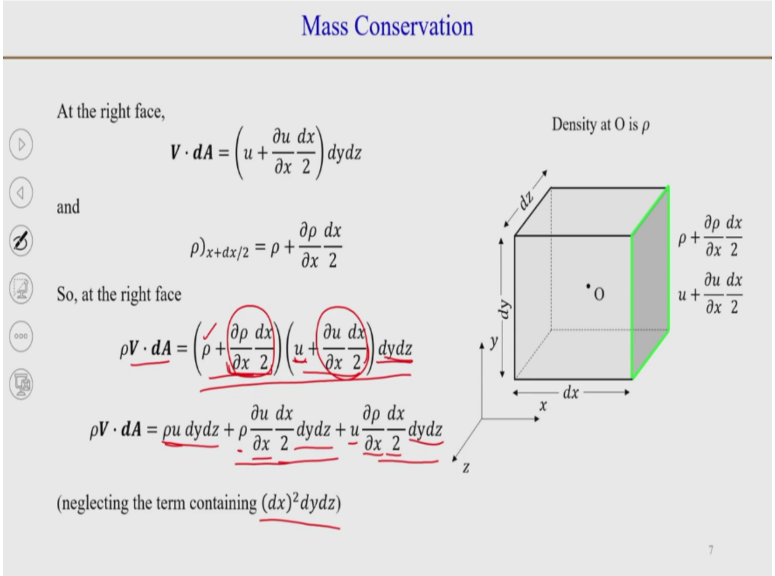
(Refer Slide Time: 13:43)



So we will take the u component of velocity at this phase, or x component of velocity rather. So, x component if velocity u, small u that will be at x + dx/2 that will be equal to u + ∂ u/∂ x into dx/2 using the same argument that we used for finding out the density we will, we have done Taylor series expansion for u and then neglected terms dx square or higher powers of dx.

So, this is again a linear variation you can say of u along x direction, so now we know the variation or the density at the face we know the area vector and we know u. So at the right face we can write V.dA in the integral that will be equal to, because the area vector point out normal to the face in the positive y direction and u also pointing out in the positive i direction. So, when we take the integral, i.i, will be equal to 1, so you will have u + ∂ u/∂ x dx/2 into dy/dz.

(Refer Slide Time: 15:19)

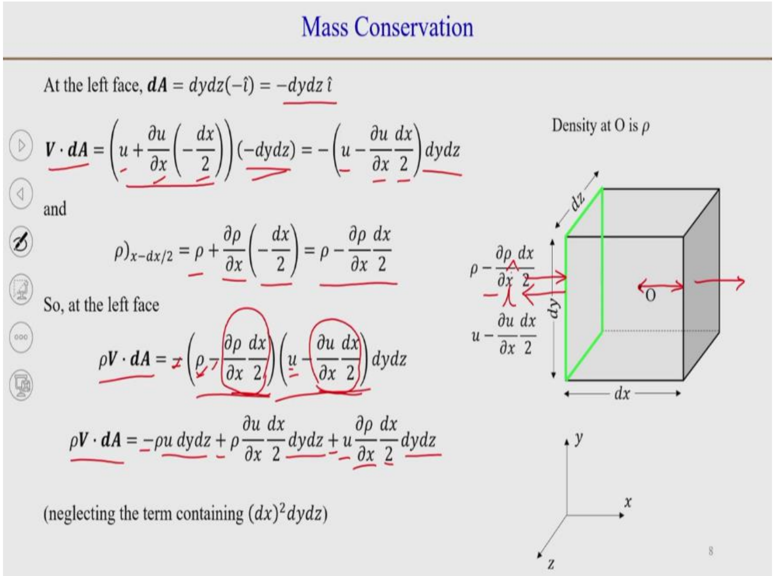


So, now we can multiply density to it, so when you combine the density and V.dA, this is what you have. Now, there are 2 terms in the first bracket and 2 terms in the second bracket. So, let us multiply and simplify it. So, if you multiply ρ with the first term, so you will get ρ u and this is multiplied by dy, dz so this is your first term. Then you multiply ρ with the second term here, so you will get ρ ∂u/∂x into dx/2 multiplied by this area dydz.

Then let us multiply the second term here with the first term, so you will have u, which is coming from this and ∂/∂x of ρ into dx/2 multiplied by dydz. Then you will have another term which will be multiplication of the second term of both the brackets. So you will have ∂/∂x into dx/2 squared into dy/dz.

So, again we will neglect the higher powers of dx, so dx squared because this term will be 1 or will be lower than when you compare with the other 3 terms. So, we can neglect the multiplication of these small terms. So, in this we will have 3 terms, ρ V.dA. So, now we have written down ρ V.dA for the right face.

(Refer Slide Time: 17:24)



So, now we will do the same exercise for the left face. So the area vector on the left face will be again the magnitude of area will be same, dydz as that of the right face, but the area vector will be pointing outward normal to this control surface. So, now it will be pointing out in the -i direction so you will have at the left face area equal to -dydz into i.

Now we will need to write V.dA, so dA we already found and v will be pointing out in the positive x direction or the x component of velocity because when we are taking V.dA, what we need to find out at this phase is x component of velocity. So it will be pointing out in the positive x direction. So we will have these 2 combined and now the velocity is u + ∂u/∂x -dx/2.

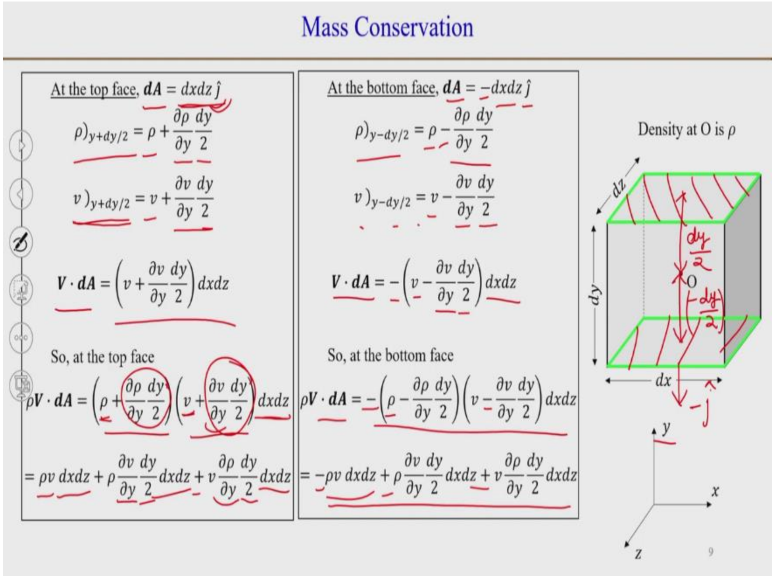
So, earlier, because it was at a + dx/2 distance so the property was of varying or the flow property u was u + ∂u/∂x, dx/2 on the right face. On the left face it is on the - side or in the -x direction so you will have u + ∂u/∂x into -dx/2 and multiplied by the area. So, this will be equal to -u -∂u/∂x, dx/2 into dydz, that will be V.dA, multiply by ρ, so ρ on this face will be ρ + ∂ ρ over, ∂ρ/∂x into -dx/2, same as we did for u on the left face. So the density on the left face will be ρ -∂ρ/∂z into dx/2.

We can combine again all this, so we will get ρ V.dA and we can take the multiplication and neglect the term which has dx square in it. So, when you multiply the first 2 terms, you will have ρ u dydz and a -sign because of this. Then you can multiply ρ with the second term here, so you will have -into -that will become +, so ρ ∂u/∂x, dx/2 into dydz.

The next term will be product of first term in the second bracket which is u into second term in the first bracket, this term. So, when you multiply, there is a -sign outside and this -sign, so you will have this sign to be + u ∂/∂x of ρ into dx/2 into dydz. We will neglect the multiplication of the second term of both the brackets because it will give you dx square whose magnitude will be smaller than all 3 terms. So, this is our ρ V.dA on the left face, so we have looked at on both the faces.

Now one can have a doubt in the mind that we have taken the variation only along the x direction of ρ and u and why not along the y direction. So, as I said, for the density when we were talking about volume integral, if you take the variation along the y direction, those variations, when you take the summation or the integral over it, they will eventually cancel out and you will end up with the same thing.

(Refer Slide Time: 21:53)



So, next we can do the same exercise for the top and bottom faces so that is what we are going to look at. So, let us look at first the top face, which is pointing out along the positive y direction and the distance of this face from the point O is dy/2. So, we will have at the top face, the area will now be because it is xz plane or pointing out in the positive y direction so dx, dz that will be the magnitude of area and the direction will be positive y direction. So, the unit vector is j, the density there, ρ y + dy/2 because the phase is at a distance dy/2 from centre O, so that will be ρ + ∂/∂ y into dy/2.

So, now we are considering the variation along the y direction. So, similarly the v component of velocity because now we are looking at the area, which is pointing out in the positive y direction so the u component of velocity or the x component of velocity and z component of velocity, when we will have dot product with dA, those terms will become 0. So we will find only the y component of velocity. So v at the location y + dy/2 which is the top surface that will be equal to v + ∂/∂y of v multiplied by dy/2.

So, we can write V.dA, these terms and find out ρ V.dA at this top plane, so you will have a ρ being ρ + ∂/∂y of ρ into dy/2 into v, V.dA will be positive because both the velocity component and the area vector, they will be pointing out in the positive direction. So, you will have v ∂/∂y into dy/2 into the area dxdz and we can expand the terms and neglect the multiplication of the 2 second term. Because now it will have dy squared into dxdz so again the magnitude of this fourth term will be smaller than the other 3 term.

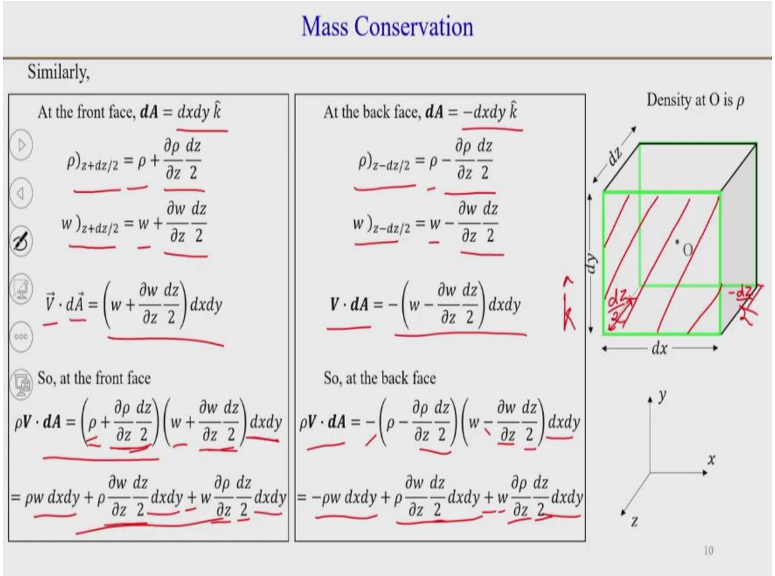
So we will neglect that term, the first term which will be multiplication, the first term of both the brackets, ρ v dxdz, the next term ρ multiplied by ∂v/dy into dy/2 multiplied by area dxdz and the third term will be v multiplied by ∂/∂y of ρ into dy/2 multiplied by area dxdz.

Similarly, we will write the terms for the bottom face, which is this one, and the normal vector will be pointing out in the -j direction or the unit vector normal to it will be -j. And the distance of this bottom face will be at a distance -dy/2 from point O. So we can write the ρ v dA individually dA will be dxdz, but pointing out in the -j direction. ρ will be here, ρ -∂/∂y of ρ into dy/2, this - comes because it is in the negative direction. Similarly for v, v will be v - ∂/∂y of v into dy/2.

We will find out or we can find out V.dA, one is pointing out in the - j direction another in positive z so dA - j and v is positive j. So when you take multiplication, you will have - v - ∂/∂y of v into dy/2 and multiply by the magnitude of area dxdz, that will be V.dA and then we can collect all the terms ρ V.dA and take the multiplication, neglect the term having dy squared so then we will have all 3 terms. You can see the sign here because of this negative, this sign will be minus, then you multiply the first term from here, so this minus, - will become +, so remaining 2 terms become positive, but the first term becomes negative here.

So, now we have done the fluxes or we have obtained the mass flux at x faces and y faces and we are left with the 2 faces along the z direction of front face.

(Refer Slide Time: 27:40)

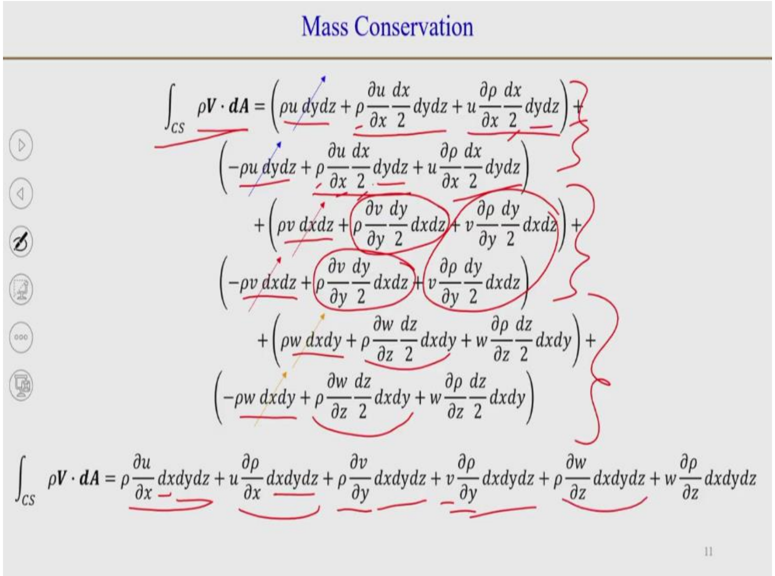


So, we can do the similar exercise at the front face here, so at the front face which is at a distance of dz/2 and the unit normal is pointing out in the positive k direction here. So, the area vector will be dxdy because it is a xy plane or dxdy into the area vector or the area normal, unit normal k there. It is in the positive z direction so ρ at this phase will be ρ + ∂/∂z of ρ into dz/2. Same for w, w + ∂/∂z, dz/2 and we can combine all this together. So we will have a ρ which is ρ + ∂ ρ/∂ z, dz/2, w + ∂/∂ z of w, dz/2 dxdy.

And when we do that because both the vectors are pointing out in the positive direction so their dot product will be positive, so we can write down, the first term will be ρ w multiplied by area dxdy, the next term will be ρ ∂/∂z of w dz/t into dxdy and the third term will be w ∂/∂z of ρ into dz/2 into dxdy. The fourth term which will be multiplication of these 2 small terms that we will neglect because that will have dz square and the term will be smaller in magnitude when you compare with these 3 terms.

The same exercise for the sixth face, which is at the back at a distance of - dz/2 from point O and the normal to it will be pointing towards - k direction or the negative z direction. So that dA will be dxdy into unit area vector or unit vector k, ρ will be at z - z/2, will be the z coordinate on the back plane or the back face so that will be ρ - ∂/∂z, g/2. Similarly w will be w - ∂/∂z w into dz/2 and we can write all this, V.dA will be negative and we combine all the terms so the multiplication will be - ρ w dxdy + ρ ∂/∂z into dz/2 dxdy. And the third term will be - - will become +. W into ∂ρ/∂z into dz/2 into dx, dy. So now we have this integral ρ V.dA on all the faces.

(Refer Slide Time: 31:09)

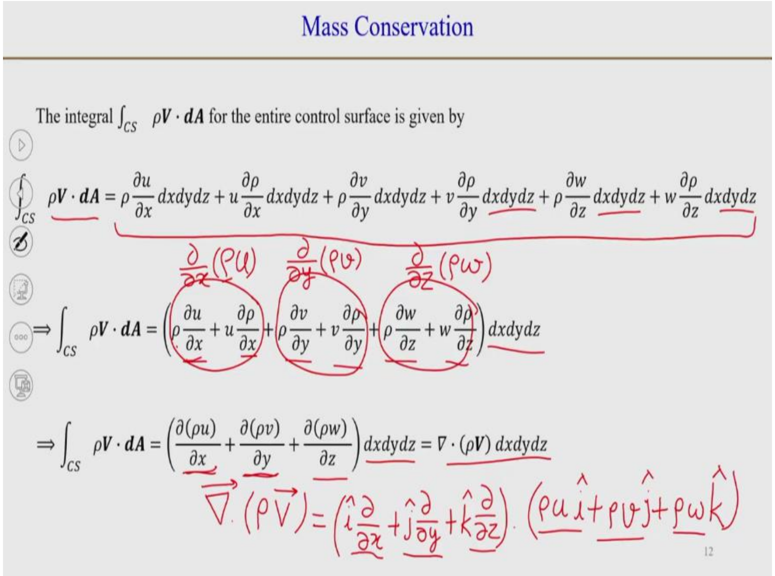


We can sum this term to find the second term in the mass conservation, so which is integral over the control surface which has 6 faces ρ V.dA. So we can write all the 6 terms that we have obtained, the 2 terms for x faces, the next 2 terms for y faces and the last 2 terms for z faces. So, we can see that the first terms here in each case they are positive and negative, on the respective say, left and right faces positive and negative, top and bottom and front and back.

So, the first term in all the brackets will cancel out and we can put together all these terms and try to see what do we get. So, if we look at, you have the second term of the brackets on the left and right faces, you have same terms ρ ∂u/∂x dx/2, dy/dz so, when you add them together, you will have ρ ∂u/dx, dx/2 + dx/2 will become dx into dydz.

Similarly, the next term which is u, ∂ρ/∂x into dx/2 dydz in both the places, so dx/2 other components being same, so you will have dx/2 + dx/2 will become dx. So, the next term will be u ∂/∂x, dxdydz. Similarly when you combine the second term for the y faces then you will see, you get ρ ∂/∂y into dxdydz. The last terms when you combine them, you will get v ∂/∂y, dxdydz. And doing the same thing for the front and back faces, you will get ρ ∂w/∂z, dxdydz + w ∂ρ/∂z, dxdydz.

(Refer Slide Time: 33:34)



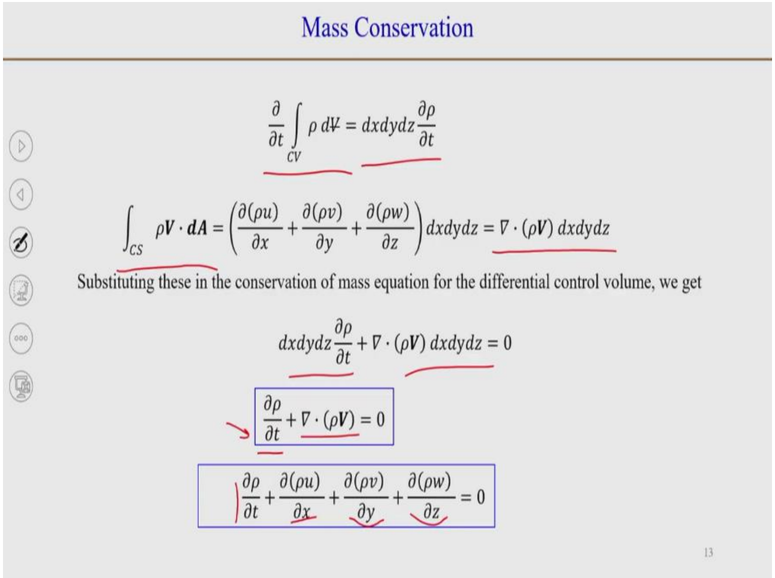
So, let us see if we can further simplify this, so this is the summation or the integral over ρ V.dA. So, if we combine this, then we see first that all of these, all 6 terms will have dxdydz so that can come out and now we have 6 terms ρ ∂u/∂x + u ρ/∂x + ρ ∂/∂y of v + v ∂/∂y of ρ + ρ ∂/∂z of w + w ∂/∂z of ρ.

So, let us look at the first 2 terms here, and that is nothing but when you expand ∂/∂x of ρ. So, if you write ∂/∂x of ρ u, then the first term, when you expand it the product of this differentiation, so you will take ρ as it is and the differentiation of u, so ρ ∂u/∂x + u ∂ρ/∂x. So the same thing you can see the next 2 terms will be ∂/∂y of ρ v and the last 2 terms will be ∂/∂z of ρ w.

So, we can write this, we will have now ∂/∂x of ρ u + ∂/∂y of ρ v + ∂/∂z of ρ w multiplied by dxdydz. Or if you look at this term carefully then, this is basically ∇ operated on ρ velocity. So, if you remember the ∇ operator is $\hat{\imath}$ ∂/∂x + $\hat{\jmath}$ ∂/∂y + unit vector $\hat{k}$ ∂/∂z, that is ∇ operator and take its dot product with ρ v.

So dot product with ρ u$\hat{\imath}$ + ρ v$\hat{\jmath}$ + ρ w$\hat{k}$ and when you do this exercise only $\hat{\imath}.\hat{\imath}$, $\hat{\jmath}.\hat{\jmath}$ and $\hat{k}.\hat{k}$ will be nonzero, other multiplication terms will become 0, so you will have $\hat{\imath}.\hat{\imath}$ so ∂/∂x of ρ u, that is your first term, ∂/∂y so ρ v second term, ∂/∂z of ρ w. So, you can write this in terms of vectors so ∇.ρ V, dxdydz.

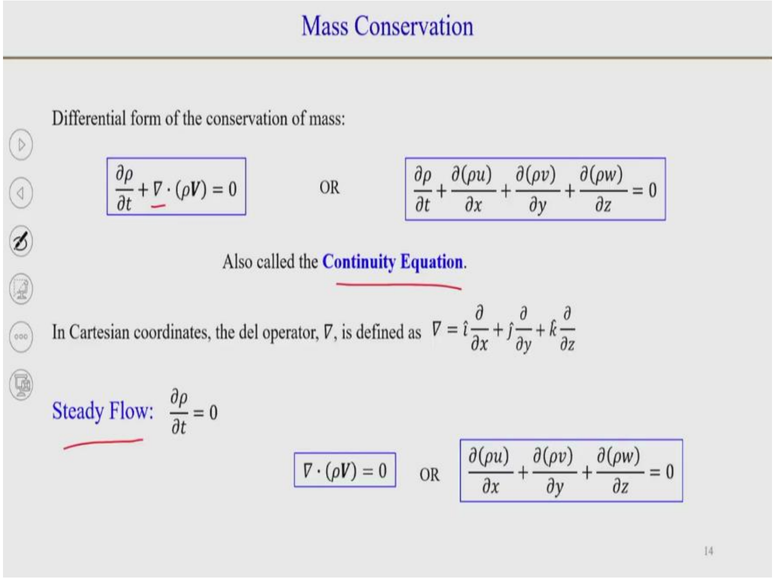
(Refer Slide Time: 37:07)



So, we have found the first term and second term, put them together. So the first term in the mass conservation equation was the partial derivative with respect to time of volume integral of ρ dV and the second term, the surface integral over the control surface of ρ V.dA and we have simplified. So first term is dxdydz ∂ρ/∂t and the next term is ∇ operated or ∇.ρ v into dxdydz.

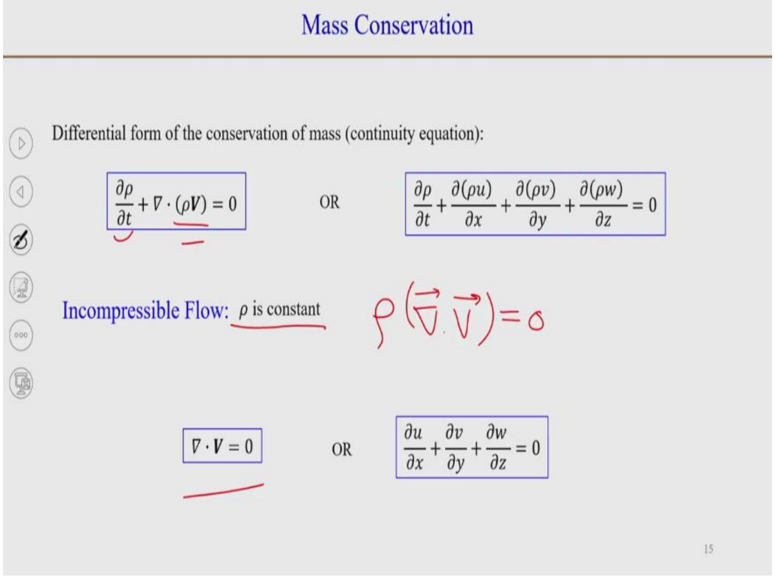
So, let us sum those together so you will get the mass conservation equation and you have dxdydz, so we can divide by dxdydz and the equation in vector form we will get is ∂/∂t of ρ + ∇.ρ v =0. Or if you expand it in the Cartesian coordinate, yon can write ∂/∂ t or ∂/∂t of ρ + ∂/∂x of ρ u + ∂/∂y of ρ v + ∂/∂z of ρ w, it is easier to remember in this form and it is useful to remember the continuity equation when you are studying a fluid mechanism course.

(Refer Slide Time: 38:36)



So, we can look at some of the variation into it, this is also called continuity equation and we already saw that this ∇ in Cartesian coordinate, we can expand it in this manner. If the flow is steady so we know that this steady flow means that density is not going to change with time, the velocity is not going to change with time, so the first term ∂/∂t of ρ =0. So we will have ∂dot ρ v = 0 or you can write in the expanded form.

(Refer Slide Time: 39:25)

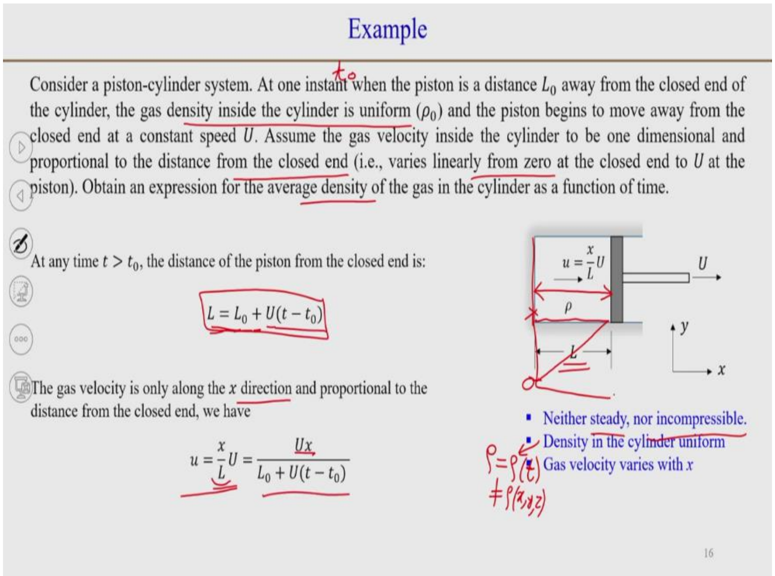


Now another variation, we will look at, say if the flow is incompressible, so when the flow is incompressible then the density is constant. So, again the first term will become 0, ∂/∂t of ρ is 0 because ρ is constant and now ρ is constant, so it does not depend on x, y and z coordinates. So this can come out of the brackets, so you will have basically ρ into ∇.V =0.

So, you can divide by ρ which is constant and you have ∇.V =0 is your continuity equation for incompressible flow or if you expand it then you will have ∂/∂x, u + ∂y ∂y, v + ∂/∂z, w =0. So, for an incompressible flow, the first term will anyway be zero, if the flow is steady or unsteady and because we will be dealing with incompressible flow most of the time in this course so ∇.V will be the thing that we will be using.

If the flow is unsteady again, the first term becomes 0 but then ρ is not necessarily, because we have said that if the fluid is steady, it is not necessarily incompressible, so if the flow is compressible then ρ can still be a function of x, y and z. So, the equation will be ∇.V as we saw in the previous slide, so that is the difference in the 0 variations that we saw for steady and incompressible flows.

(Refer Slide Time: 41:15)



Now let us look at a simple example, so we consider a piston cylinder system as shown in the figure here, at one instant when the piston is at a distance L0 away from the closed end of the cylinder, the gas density inside the cylinder is uniform which is ρ 0. So, let us say that this particular instant that one is talking about is t^{o} or t0 and the piston begins to move away from the closed end at a speed, constant speed u. So at t0, at t is greater than t0, the piston starts moving with a speed u.

Assume the gas velocity inside the cylinder to be 1 dimensional and proportional to the distance from the closed end. So, the gas velocity inside the cylinder is proportional to the distance from the cylinder and what we need to find is an expression for the average density of the gas in the cylinder as a function of time.

So, if we look at this flow, the flow is not steady because the density when it is expanding the volume is expanding and the flow or the gas is inside the system so it is going to change with time. So the flow is not steady and density is also not constant so flow is not also incompressible. But the flow is 1 dimensional it says and the density in the cylinder can be assumed to be uniform so that means that ρ is a function of ρ (t) only and is not a function of x, y or z.

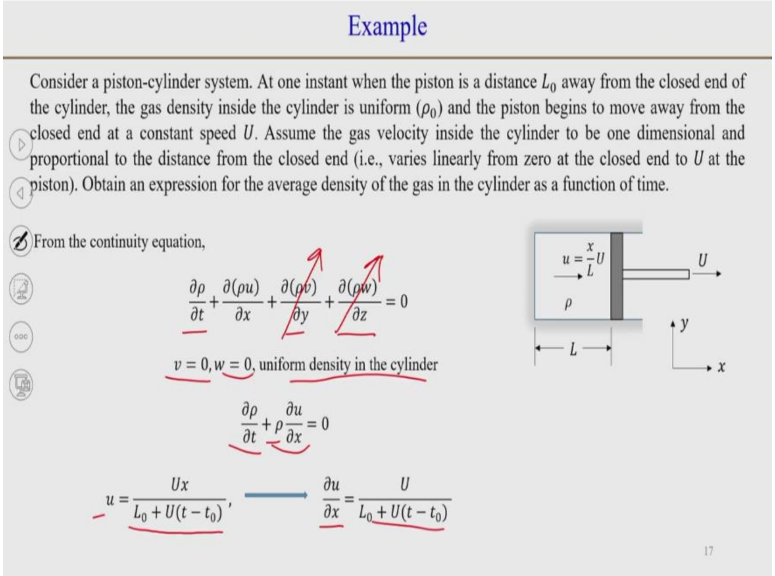
And gas velocity varies with x so it says the gas velocity, the flow inside the cylinder is 1 dimensional, but the gas velocity varies from its, gas velocity inside the cylinder to be 1 dimensional and proportional to the distance from the closed end, it varies linearly from 0 at the closed end so the velocity is 0 at this end.

So, it will be varying, so this will be 0 here and it varies up to the pistons, at the piston that will be equal to the piston velocity. So, we can write down that, at any point of time, what is the distance between the piston and cylinder because that is varying. So let us say that this distance in general at any time is L and at time t =t0, the distance is L^o and then it moves with a constant speed u, away from the piston so that will be L =L^o + u into time and that time will be t - t^o.

So, that is the expression for the distance between the closed end of this cylinder and the piston, at any time t is greater than t^o. We can see that at time t =t^o, this will give us L =L^o. The gas velocity, we can find now the gas velocity because the gas velocity is proportional to x direction. So we can write this, u =x/L, so it is proportional to x and multiplied by u/L, so it will become 0 at the closed end and at the piston that will become equal to x =L at the piston. So it will become equal to piston velocity and gas velocity at the piston.

So, now this L is not constant, it is a function of time, so we can substitute L from here so that will be ux /L^o + ut - t^o which is the expression L.

(Refer Slide Time: 46:02)

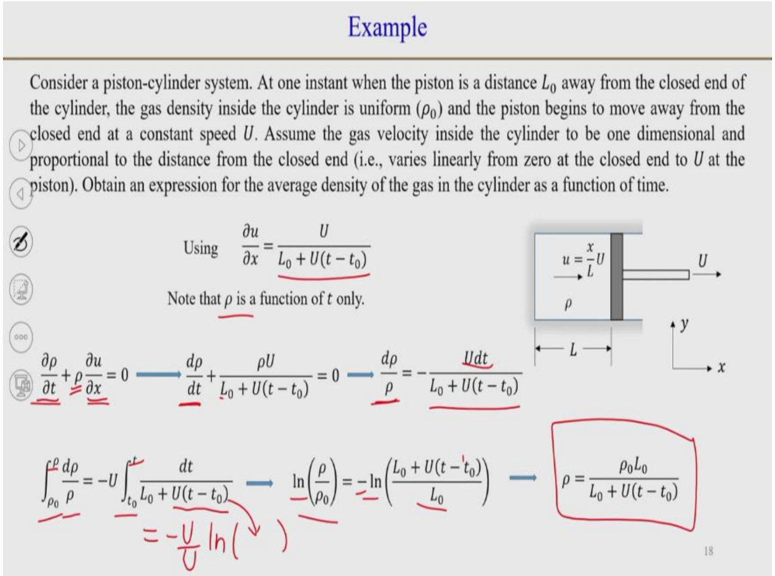


Now we can substitute this in the mass conservation equation. So let us write down the differential form of mass conservation equation ∂ρ/∂t because we need to find the variation of density with time so we can use this equation and the flow is 1 dimensional so that means the y component of velocity, which is v and z component of velocity which is w, both of them are 0.

The density is uniform in the cylinder so ρ is not a function of x, so we can write this equation or simplify this equation, this term, the third term and fourth term, they become 0 and you can write ∂/∂t of ρ + ρ ∂u/∂x. ρ can come out because ρ is not a function of x, so ρ can come out of the differential or differentiation. So now we already had a function of u, u as a function of x so we need to find ∂/∂x of u.

So, we can differentiate u with respect to x so ∂/∂x of u differentiate with respect to x, the denominator is not a function of x, so that will be equal to u /L^o + u, t - t^o.

(Refer Slide Time: 47:36)



Now we can substitute it there so we can substitute ∂u/∂x in the continuity equation and because ρ is a function of time only, so this can become, this is a partial derivative but ρ is a function of time only so we can write as a total derivative d ρ/dt now, ∂ρ/∂t becomes d ρ/dt + ρ times ∂u/∂x. So we can substitute the value of ∂/∂x of u which is u /L^o + u, t/t - t^o and we can integrate this equation to simplify.

So first we can take the variables on each side so ρ becomes, so the left hand side can become d ρ/ρ, that will be equal to - u times dt /L^o + u t - t^o and we can integrate it in the limit from time t0 to t and at time t0 we know that the density is ρ^o and at time t we can say the density is ρ, so we will get ρ as a function of time. When you integrate the first term d ρ/ρ will be ln ρ and substituting the limit we will get L n ρ/ρ^o.

The next term when you integrate it, will be - u and when you integrate this term so you will get 1/u, ln of this term. So you will have - ln, of l^o + u, t - t^o and the limits from t^o to t, so when you substitute limit as t you will get L^o + u t - t^o and at time t^o, the second term will become 0 because t^o - t^o =0. So, you will have divided/L^o only.

So, you have Ln on both sides and there is a negative sign there so you will have ρ/ρ^o =1/this term or this term will be reversed. So you will have ρ/ρ^o = L^o /L^o + u, t - t^o or ρ =ρ^o L^o /L^o + u into t - t^o. So, this example, we had flow to be unsteady and compressible and we used a 1 dimensional differential form of continuity equation. I think all of us who are taking this course, we should remember the differential form of continuity equation, which is ∂/∂t of ρ + ∇.ρ V =0.

Let us stop here, thank you.