Fundamentals of Fluid Mechanics for Chemical and Biomedical Engineers
Professor Dr. Raghvendra Gupta
Department of Chemical Engineering
Indian Institute of Technology, Guwahati
Lecture 21
Fluid Kinematics

Hello, so, in this second lecture on Fluid Kinematics, we are going to talk about the Rotation, Angular Deformation as well as Linear Deformation of a fluid element.

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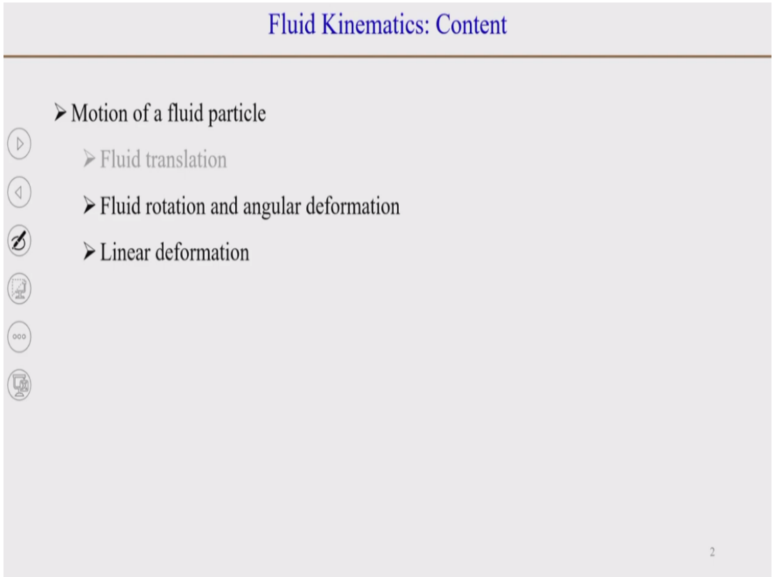

So, in the previous lecture we discussed about that, when a fluid element or when a fluid particle undergoes a motion its motion can be decomposed into the four components.

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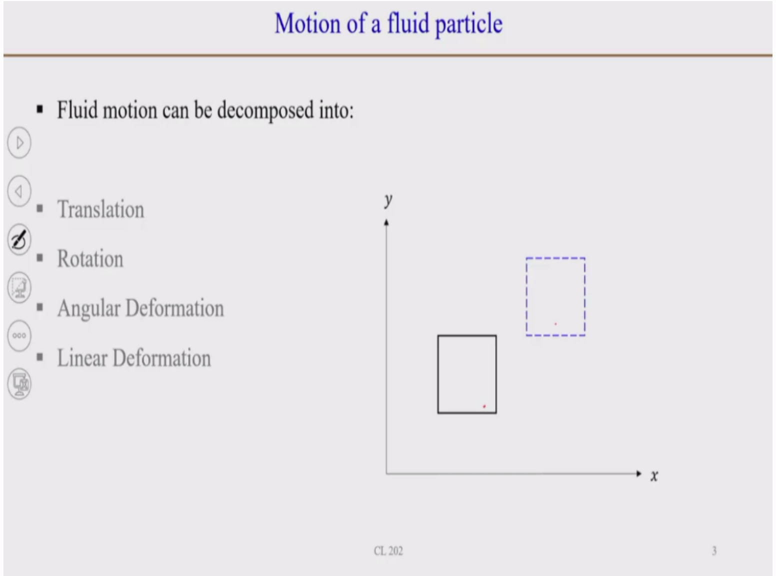

And those four components were translation. So, the translation refers to the motion of the particle from one location to another without actually any change in the shape of the fluid element. So, that is pure translation. In such a case, the distance between the corners of this rectangular fluid element does not change and it remains same.

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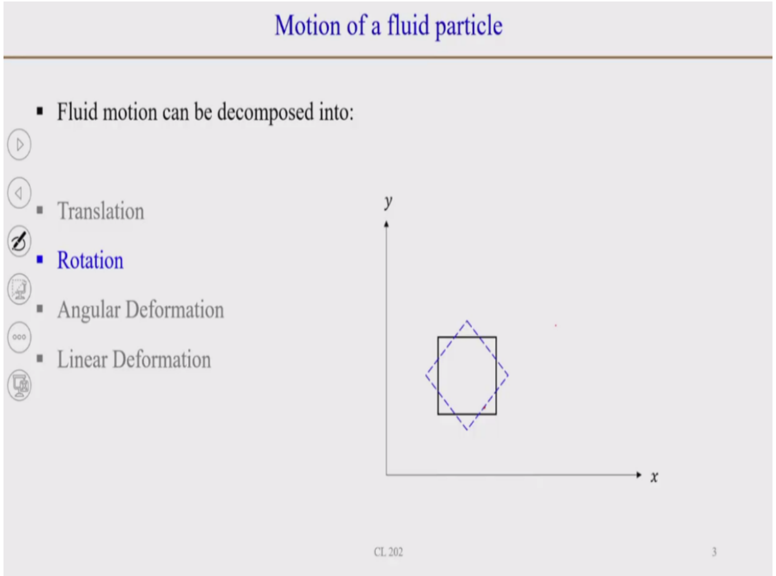

The second type of motion that we discussed was rotational motion. So, again, the shape of the fluid element does not change, but it just rotates about a axis normal to the fluid element here. So, that is pure rotation.

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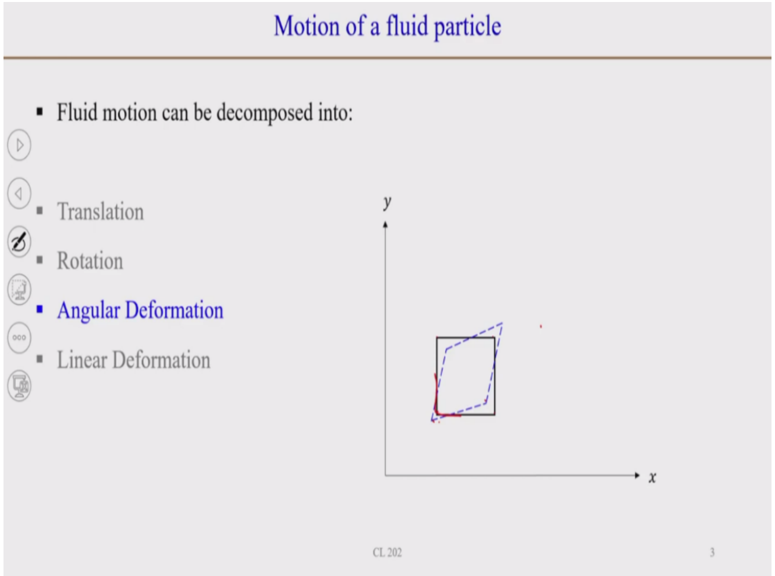

Now, the third, third most of the fluid element we discussed is angular deformation. So, in case of angular deformation, the size of the fluid particles if the fluid is incompressible does not change, the size still remains same, but the angle changes. So, the angular deformation refers to the change in the angles between these two sides from 90^o the change that happens from 90^o is what we will refer to angular deformation.

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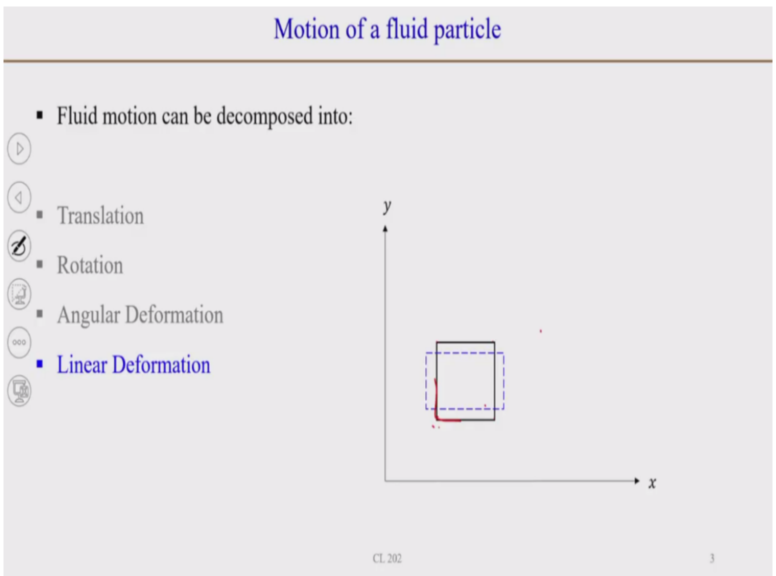

And the last one is linear deformations. So in the linear deformation we see that the size it may get stretched or it may get compressed the side of the element. So, all those motions can occur simultaneously or generally occur simultaneously. What we did in the last lecture was looking at the translational motion of the fluid particle and we derived an expression for the acceleration of the fluid particle for a known velocity field. We talked about the Eulerian and Lagrangian description and because the velocity field is given in the Eulerian framework, we derived the expression for the acceleration or what is called substantial derivative for a fluid particle.

In this lecture, what we are going to do is we are going to look at the rotation and angular deformation of the fluid particle first and then we will talk briefly about linear deformation of this fluid element.

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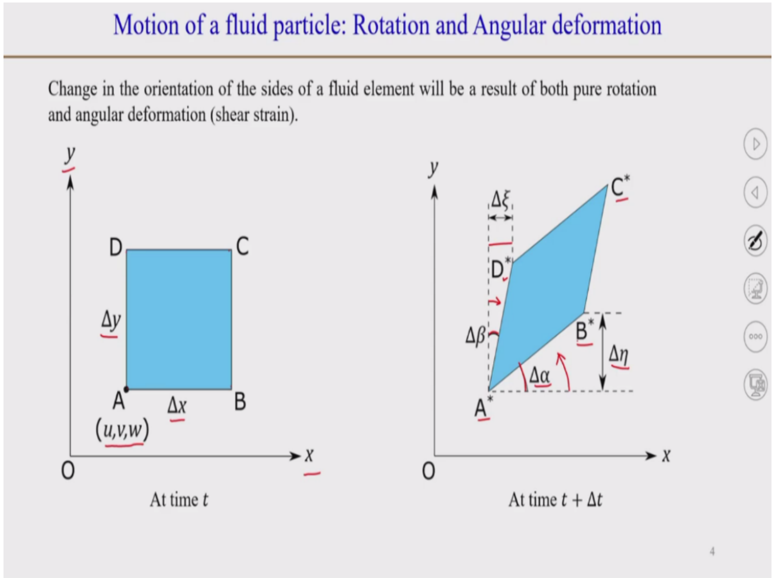

So, let us consider a fluid element which is rectangular, even though it appears to be square, we will assume it to be rectangular the side AB is of length Δx and side AD is of length Δy, the velocity components in the x, y and z direction at point A are u, v and w and this is a 2 dimensional fluid element in xy plane. It is named as, the corners of the fluid element are named as A, B, C and D, this is at time t.

Time t + Δt, this fluid element undergoes rotation as well as angular deformation and it deforms the corners A, B, C and D move to locations A*, B*, C* and D*, respectively. So, that happens at time t + Δt. And as we can see that the distance in the y direction or the distance that point B has traveled with respect to point A in the y direction is Δη.

Similarly, the distance that point D has traveled in the x direction with respect to point A is $\Delta\xi$. And the angles that these side make from the previous locations the side A star, B star it has rotated by an angle $\Delta\alpha$ and side AD star it has rotated by an angle $\Delta\beta$, but in the different direction. So, $\Delta\alpha$ rotation, the rotation of side A star, B star is in anti-clockwise direction. Whereas, rotation of side A star D star in the clockwise direction by an amount $\Delta\beta$.

And we can see that this fluid element has gone rotation as well as angular deformation. So, what we are going to look at try to decompose this motion of the fluid particle in rotation as well as angular deformation.

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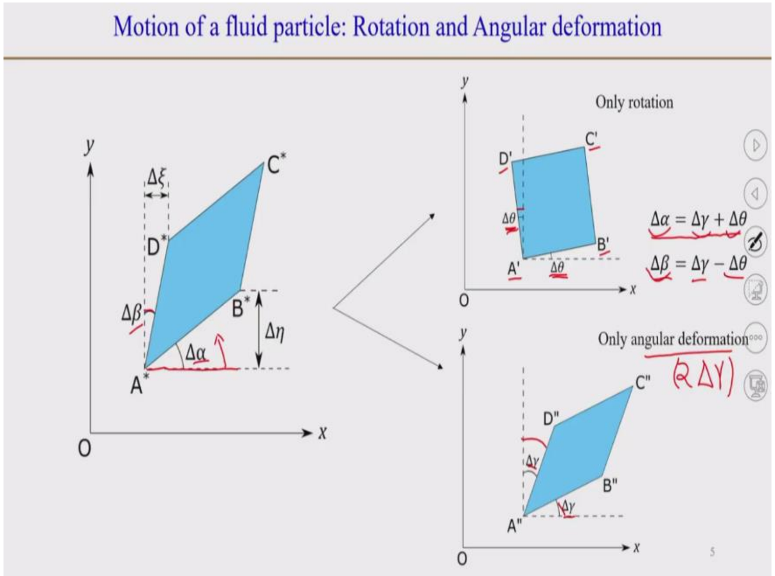

So, let us see if we assume that the pure rotation of fluid element or rotational component it has rotated let us say by an angle $\Delta\theta$ and if there would have been only rotation then these the points of the particle would have been on the location A´, B´, C´ and D´. Now, if it had been only angular deformation, then the angle would have changed by a magnitude of $2\ \gamma\ \Delta$.

So, $\gamma\Delta$ is the rotation of side AB and $\gamma\Delta$ in the other direction is the rotation of side AD. So, the angular deformation of this fluid element is $2\gamma\Delta$, sorry $2\Delta\gamma$. So, these are the two deformations of this element. Now, the total sum of or the combined motion which causes the change $\Delta\alpha$ and $\Delta\beta$, so we can write $\Delta\alpha$ and $\Delta\beta$ in terms of $\Delta\theta$ and $\Delta\gamma$.

So, we see that the rotation of line AB, which changes from previous location this dotted line it has undergone or it has rotated by a magnitude Δα in the anti-clockwise direction. So, this Δα will be equal to Δγ + Δθ and in both of these cases the line AB is being rotated by an angle Δθ and Δγ in the anti-clockwise direction. So, Δα = Δγ + Δθ.

Now, if we look at the rotation of line AD then the net rotation is Δβ, which is in the clockwise direction and this is being caused by a rotation of Δθ in anti-clockwise direction and angular deformation or rotation of side AD by an amount Δγ in the clockwise direction. So, Δβ will be equal to Δγ - Δθ. Note that this is in the anti-clockwise direction whereas other two rotations are in the clockwise direction. So, that is why we have Δβ = Δγ - Δθ.

Now, our goal is to relate these angular movement Δα and Δβ or thereby Δγ and Δθ because having known Δα and Δβ, we can solve these two equations and find Δγ and Δθ. So, we need to relate it with the velocity of the fluid or the velocity field of the fluid.

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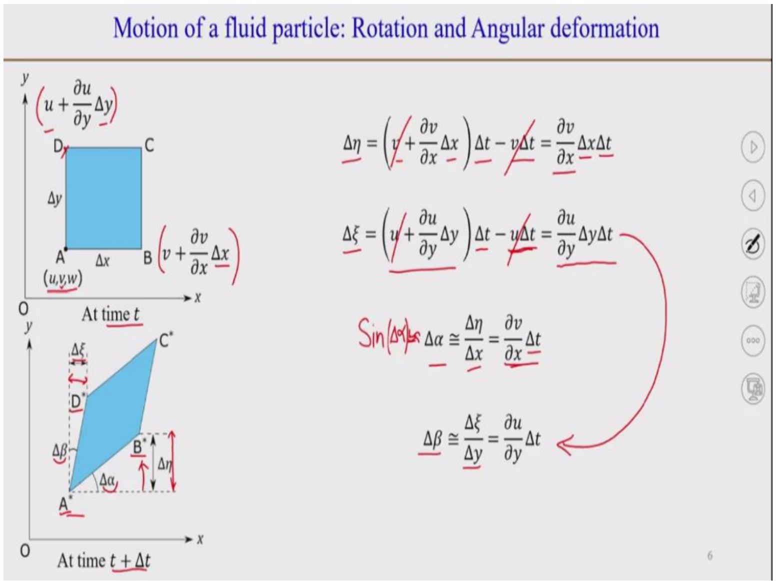

Now, as we said earlier that let us assume that the velocity of the fluid at point A is u, v, w and what we need to see is that what is the movement or what is the displacement of the fluid element or what is the displacement of point B from its previous position or with respect to point A. Similarly, we also need to see what is the movement of point D with respect to point A.

So, for point B the movement of point B with respect to point A is in y direction. So, we will need to consider the y component of velocity for point B and point A. So, we know that the y component of velocity at point A is v, by Taylor series we can write that the y component of velocity at point B will be $v + \partial v / \partial x$ that is the change in velocity so $\partial v / \partial x$ multiplied by the distance Δx. So, that is the velocity of point B.

So, in time Δt this position was at time t and the movement is after time $t + \Delta t$. So, we can find out that what is the net displacement of point B with respect to point A in time Δt. Similarly, for point D the velocity because the point D moves in the x direction, so, we will consider the velocity component in the x direction at point D and that will be from Taylor series expansion $u + \partial u / \partial y$ into Δy.

So, now, we can use these velocity components and try to relate the velocities with the net displacement $\Delta\eta$ and $\Delta\xi$. So, $\Delta\eta$ is the difference between the distance traveled in time Δt by point B star or by point B and point A. So, the distance traveled in the y direction by point B will be $v + \partial v / \partial x$ into Δx multiplied by Δt - the distance traveled by point A in time Δt which will be $v \Delta t$.

And we can see that $v \Delta t$, $v \Delta t$ will cancel out and what we will have is $\partial v / \partial x$ into Δx multiplied by Δt. So, that is $\Delta\eta$. Similarly, we can write an expression for $\Delta\xi$, which is the displacement or of point D with respect to point A in time Δt along the x direction. So, the distance traveled by point D in time Δt will be its velocity at point D multiplied by time Δt and the velocity is $u + \partial u / \partial y$ into Δy - the distance traveled by point A in time Δt along the x direction so that will be $u \Delta t$. So, we can again cancel out $u \Delta t$ terms. So, we get $\Delta\xi = \partial u / \partial y \, \Delta y$ into Δt.

Now, we have obtained $\Delta\eta$ and $\Delta\xi$ in terms of partial derivatives of velocity components. Now, we can relate what we need to find is $\Delta\alpha$ and $\Delta\beta$ because once we find that we can find $\Delta\theta$ and $\Delta\gamma$, which is the rotation of the fluid element and the angular deformation of the fluid element.

So, now we can write that $\Delta\alpha = \Delta\eta / \Delta x$, because, if the $\Delta\alpha$ is small then in that limit, we can write that sine of $\Delta\alpha$ will be equal to $\Delta\alpha$ and that will be equal to $\Delta\eta / \Delta x$, the length of AB. So, $\Delta\eta / \Delta x$ and from this expression we can substitute that $\Delta\eta / \Delta x$ will be $\partial v / \partial x$ into Δt. Similarly, we can write that $\Delta\beta$ will be equal to $\Delta\xi / \Delta y$ and from this we can obtain that $\Delta\xi / \Delta y = \partial u / \partial y$ into Δt.

So, we have obtained $\Delta\alpha$ and $\Delta\beta$ in terms of the velocity gradient and time Δt or we can obtain $\Delta\alpha / \Delta t$ and $\Delta\beta / \Delta t$ they are equal to $\partial v / \partial x$ and $\partial u / \partial y$ respectively. Now, our goal is to find $\Delta\theta$ and $\Delta\gamma$.

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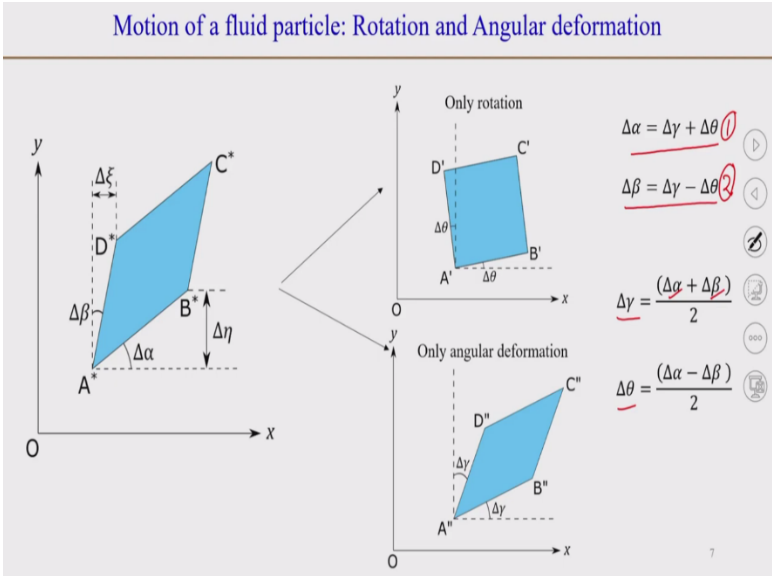

So, let us substitute those we had these relationships $\Delta\alpha = \Delta\gamma + \Delta\theta$ and $\Delta\beta = \Delta\gamma - \Delta\theta$. If we add these two equations, let us say these are equations 1 and 2, then we will get $2\,\Delta\gamma = \Delta\alpha + \Delta\beta$ or $\Delta\gamma = (\Delta\alpha + \Delta\beta) / 2$. And similarly, $\Delta\theta = (\Delta\alpha - \Delta\beta) / 2$.

So, we can substitute the values of $\Delta\alpha$ and $\Delta\beta$ as we have found just now in terms of $\partial u / \partial x$ and $\partial u / \partial y$.

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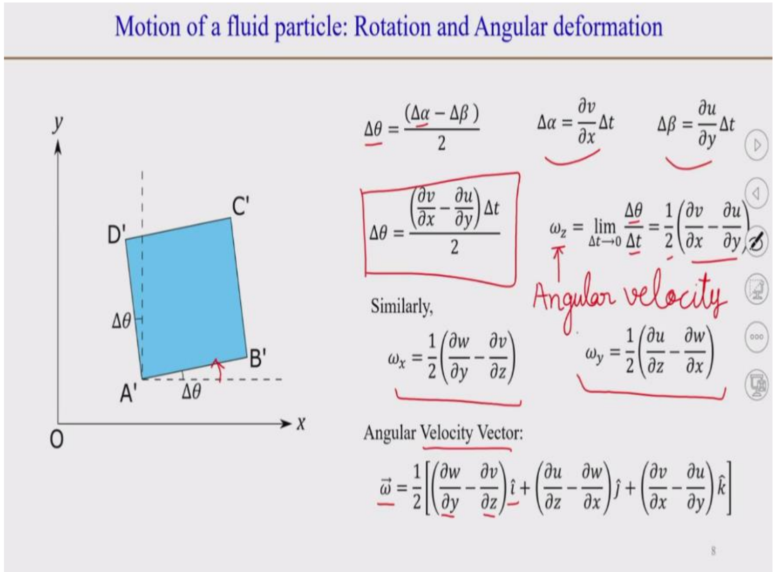

So, we can substitute those values. So, first we substitute or first we find $\Delta\theta$, where $\Delta\theta$ is the rotation of the fluid element. So, $\Delta\theta = (\Delta\alpha - \Delta\beta) / 2$ and we can substitute these two values of $\Delta\alpha$ and $\Delta\beta$ in here. So, we get when we substitute for $\Delta\alpha$ $\partial v / \partial x$ into Δt - $\Delta\beta = \partial u / \partial y$ into Δt.

So, we get this expression for $\Delta\theta$ or in the limit we can bring Δt here, so we get $\Delta\theta / \Delta t = (\partial v / \partial x - \partial u / \partial y) / 2$. So, $\Delta\theta / \Delta t$ that is angular displacement per unit time, which is nothing but angular velocity. So, the rotation is in the z direction or the vector will be pointing out in the z direction. So, z component of angular velocity will be equal to ω_z = half of $\partial v / \partial x - \partial u / \partial y$.

So, we have obtained the angular velocity or the rotational velocity of the fluid element similarly, we can do the same exercise for the rotation in zx and yz plane and obtained the other two components of velocity. So, we get ω_x by similar exercise we can find ω x = half of ∂ / ∂ y of w which is z component of velocity - ∂ / ∂ z of v and ω_y = half of $\partial u / \partial z - \partial / \partial x$ of w.

And we can write this in the vector form the angular velocity vector will be ω = half of ∂ / ∂ y of w - $\partial v / \partial z$ $\hat{\imath}$ unit vector along the x direction + $\partial u / \partial z - \partial w / \partial x$ $\hat{\jmath}$ + $\partial v / \partial x - \partial u / \partial y$ $\hat{k}$. So, we can obtain the angular velocity vectors from the partial derivatives of the velocity components.

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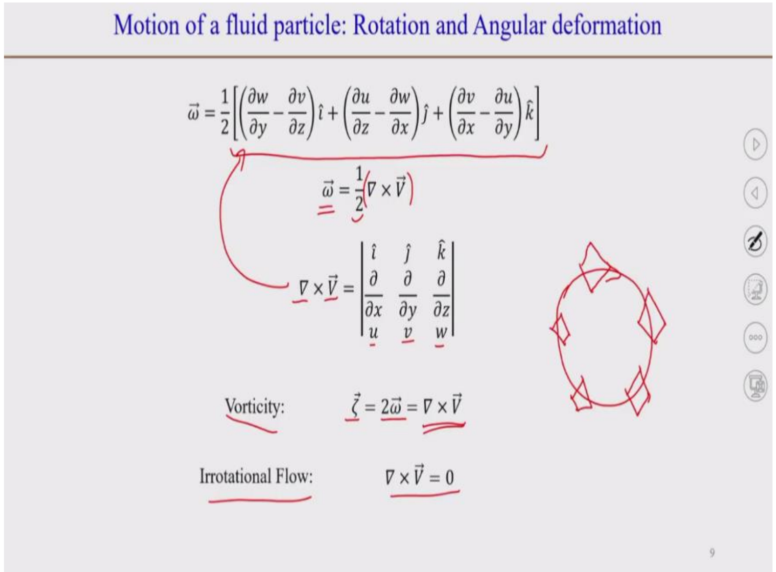

Now, if we look at this expression closely we see that this is nothing but the expression in bracket is nothing but $\nabla \times v$. So, angular velocity ω of a fluid element is half of $\nabla \times v$, where v is the velocity vector. So, we can write this in the vector notation in terms of a matrix for a cross product that $\nabla \times v$ = matrix the first row $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ $\partial / \partial x$, $\partial / \partial y$, $\partial / \partial z$, u, v, w and when we simplify this matrix, we will get the terms in this bracket.

So, the angular velocity is simply in vector form it is easy to remember half of $\nabla \times v$. Now, because this half term is there, so to eliminate this half term a new quantity or new term is defined what is called Vorticity which is twice of angular velocity and that is called $\nabla \times v$. So, $\nabla \times v$ is called Vorticity and the term Vorticity comes from Vortex. So, Vortex from that Vorticity comes.

But, the Vorticity is not necessarily related to that there is circulating fluid or there is a vortex and that will cause the Vorticity. The Vorticity represents the rotation of the fluid element itself. So, if there is a fluid element and if it does not rotate during its motion, let us say if it keeps it position constant during this circular motion then it may not be rotating. So, if $\nabla \times v = 0$, then the flow will be irrotational.

And irrotational flow as we will see later on that irrotational flow as a important tool to understand or to study inviscid flow because the mathematics involved simplifies and there has been a lot of development in the expressions for irrotational flows. Now, irrotational flow as we saw that can happen when the fluid element rotates, and what can cause the fluid element to rotate?

The fluid element can rotate because of two reasons one, if it is rotating initially, then the fluid element may keep rotating and it will be rotating fluid or it will have rotation. Or if there is viscosity, because the other forces such as pressure force or the normal stress will not be able to cause the rotational motion or the body force will not be able to cause rotation. So, the rotational force comes only from the shear stress and as we saw earlier in the definition of viscosity or in the definition of fluid that the shear stress in a fluid is caused because of viscous property of the fluid.

So, only for the viscous fluids we can have rotational motion. So, for an inviscid fluid the motion can be or the flow can be irrotational. So, until now we have discussed what is $\Delta\theta$. Now, we need to find $\Delta\gamma$, which is the angular deformation of the fluid element.

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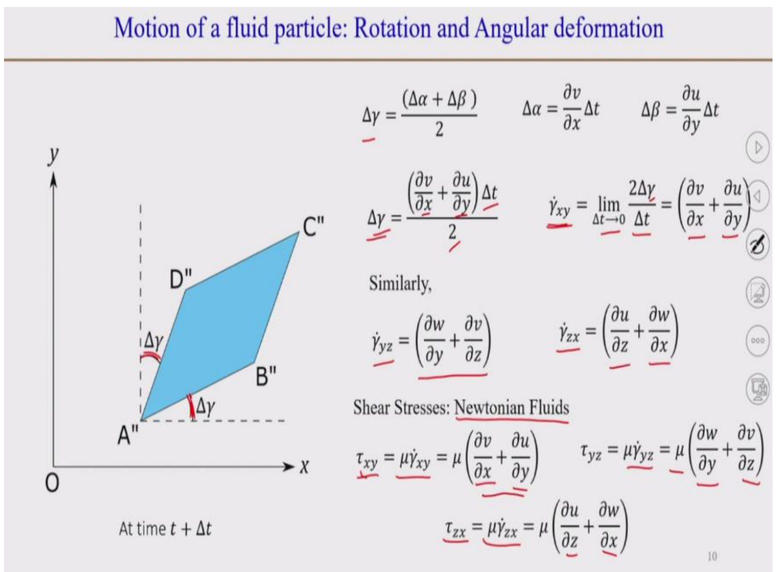

So, if we go back to our previous picture and look at the only angular deformation of the fluid element. So, this fluid element has gone angular deformation of 2 $\Delta\gamma$ that is a change in angle between line AB and AD from 90^o. So, that is 2 $\Delta\gamma$. Now, $\Delta\gamma$ as we obtain is $\Delta\alpha$ + $\Delta\beta$ / 2. So, we can substitute the expressions that we derived for $\Delta\alpha$ and $\Delta\beta$ in terms of partial derivatives of velocity components u and v.

So, when we substitute we get $\Delta\gamma$ = ∂ v / ∂ x + ∂ u / ∂ y into Δt/ 2 or $\Delta\gamma$ / Δt is ∂ v / ∂ x + ∂ u / ∂ y / 2, but we know that the angular deformation is 2 $\Delta\gamma$. So, that is why we have written here that the angular deformation or the rate of angular deformation of the fluid element in the xy plane $\dot{\gamma}_{xy}$ = 2 $\Delta\gamma$, which is the angular deformation divided by Δt in the limit Δt tending to 0.

So, that will be equal to $\partial v / \partial x + \partial u / \partial y$. Similarly, we can obtain the angular deformation in the yz and zx plane. So, $\dot{\gamma}$ or the rate of angular deformation in the yz plane that will be equal to $\partial w / \partial y + \partial v / \partial z$ and $\dot{\gamma}_{zx} = (\partial u / \partial z + \partial w / \partial x)$. Now, this is what the components of strain rate or shear rate or rate of angular deformation are in xy, yz and zx plane and these we can use when we derive the momentum equations and obtain the expressions for shear stress.

So, for a Newtonian fluid, the generalized form of stresses can be written $\tau_{xy} = \mu\dot{\gamma}_{xy}$ which $= \mu (\partial v / \partial x + \partial u / \partial y)$. Similarly, $\tau_{yz} = \mu\dot{\gamma}_{yz} = \mu(\partial w / \partial y + \partial v / \partial z)$ and the third component or in zx plane $\tau_{zx} = \mu\dot{\gamma}_{zx} = \mu(\partial u / \partial z + \partial w / \partial x)$.

So, we can write the shear stress in terms of strain rates in a 3 dimensional flow for a Newtonian fluid. So, now, we have obtained the rate of rotation or rotational velocity or what we called in terms of Vorticity and the rate of angular deformation or a strain rate or shear rate for a fluid element. Now, we need to look at the linear deformation of the fluid element.

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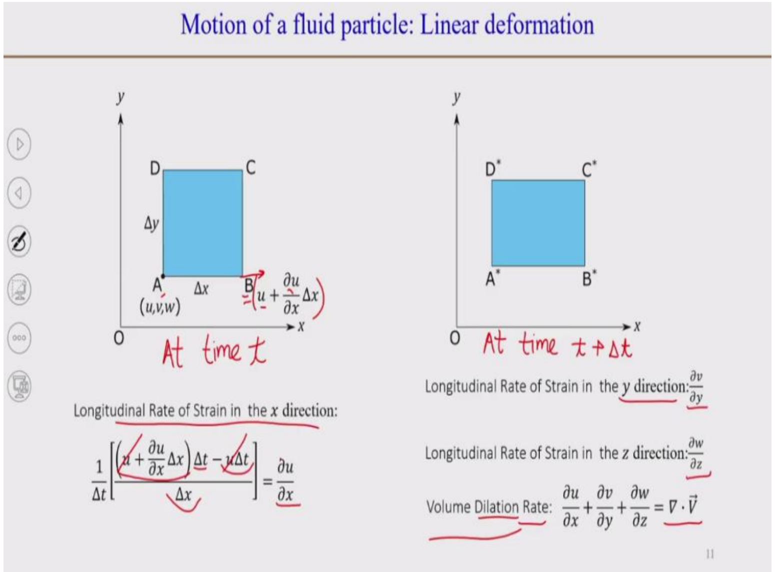

So, if we look at, let us say if we see the fluid element at time t and let us say if it has gone some linear deformation at time t + Δt then we can see the shape of the fluid element remains rectangle, so the angle between AB and AD sides is still 90^o. However, the length of the side AB has increased. So, the linear deformation, in the linear deformation what we will be looking at the strain or the change in the length of side AB, so that we can find that how much point B has moved with respect to point A.

So, the u component of velocity of point B because we are now concerned in the motion of point B along the x direction. So, u component of velocity of point B will be u + ∂ u / ∂ x into Δx that is the velocity u component of velocity at point B. So, the longitudinal rate of strain or rate of the strain in x direction is the distance moved.

So, the strain is the change in length of side AB the change in length of side AB is the distance moved by point B, the distance moved by point B in time Δt will be time Δt multiplied by the velocity of point B which is u + ∂ u / ∂ x into Δx - the distance moved by point A in time Δt that will be u Δt.

So, we can see that u Δt, u Δt will cancel out and this divided by Δx which is the initial length of line AB, so, we will get a strain and as we will see that that ∂ u / ∂ x into ∂ t into Δt will be the strain when we divide it by Δt we will get the rate of strain. So, the rate of strain in the x direction will be ∂ u / ∂ x. Similarly, we can find longitudinal rate of strain in the y direction will be ∂ v / ∂ y and the longitudinal rate of strain in the z direction ∂ w / ∂ z.

So, that is the rate of change of the sides of this fluid element in the x direction ∂ u / ∂ x ∂ v / ∂ y and ∂ w / ∂ z. Now, that will cause the change in the length of the sides will cause a total change in the volume and that change in volume is what is called dilation. So, we can show, from simple calculus that the change in volume will be some of the change in the sides.

So, that will be ∂ u / ∂ x + ∂ v / ∂ y + ∂ w / ∂ z into Δt that will be the dilation in time Δt or if we write in terms of rate then volume dilation rate will be ∂ u / ∂ x + ∂ v / ∂ y + ∂ w / ∂ z, or if we look at this equation closely we can identify that this is nothing but dot product of operator ∇ and V. So, ∇.V is the volume dilation rate.

And as we will see, when we see the continuity equation that ∇.V = 0 is the condition for the flow to be incompressible. So, that means that ∇.V = 0 that means, that the flow the fluid particles do not stretch or do not get compressed that means they do not get their volume they do not dilate and that simply means that the flow is incompressible.

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So, we will stop here, but before we do that, I would like to extend my thanks to Doctor Amit Kumar, a colleague in the Department of Chemical Engineering who has helped in the preparation of these slides, we teach the course at IIT, Guwahati together. So, his help in preparation of these slides is gratefully acknowledged. Thank you.