Fundamentals of Fluid Mechanics for Chemical and Biomedical Engineers

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Lecture 16

Macroscopic Balances: Mass Conservation

In the last lecture we discussed about Reynolds Transport Theorem, which related the mass conservation and momentum conservation or what we derived it for any general variable which we called N, we could relate that control volume formulation with the system formulation whereas, the system is a entity which is a control mass, that is a fixed quantity of mass.

Whereas, in fluid mechanics or in this course, we are generally dealing with a control volume approach where we have a control volume, a fixed region space or a specified region in a space through which the mass can come in and go out. So, the conservation equations, the mass conservation and momentum conservation, energy conservation or angular momentum conservation all those conservation equations we know in terms of for the system, from rigid body dynamics.

So, what we are doing in this week or in this module is we learnt first that for a general variable N or an extensive property N and the corresponding intensive property η we derived the Reynolds Transport theorem, that how we can convert a derivative in the system form because all the equations we have dN/dt kind of term. So, how we can write that dN/dt in the system formulation, how can we convert into control volume formulas and now, we are going to apply this in this lecture for mass conjugation and then we will look at some examples.

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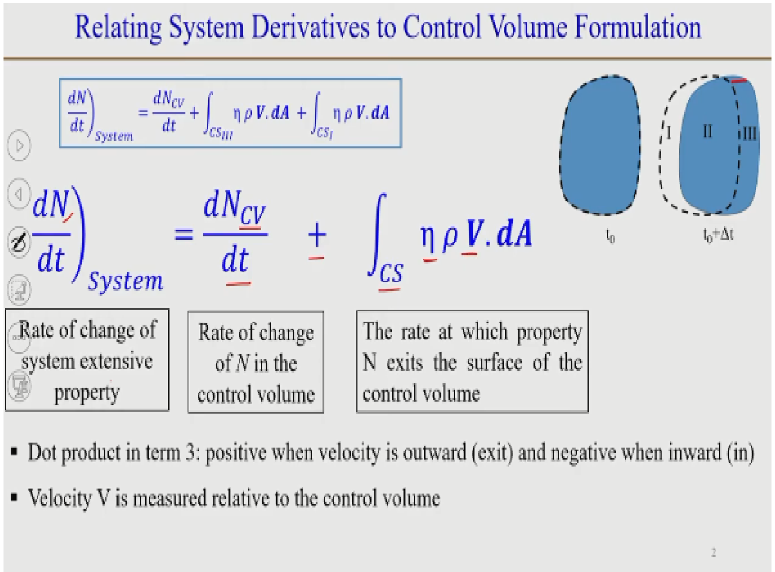

So, let us recall the equation that we derived, the equation states, the Reynolds Transport equation that the rate of change of a extensive property N is equal to rate of change of the property in the control volume. So, subscript CV refers to N in the control volume and then plus integral over CS which is control surface η ρ V. dA. So, η is the corresponding intensive property which is N divided by mass or N per unit mass. So, couple of things that we have this plus sign because we assume or we have derived it that the velocity is outward; if velocity is inward then this sign will become negative and this velocity V is measured relative to the control volume.

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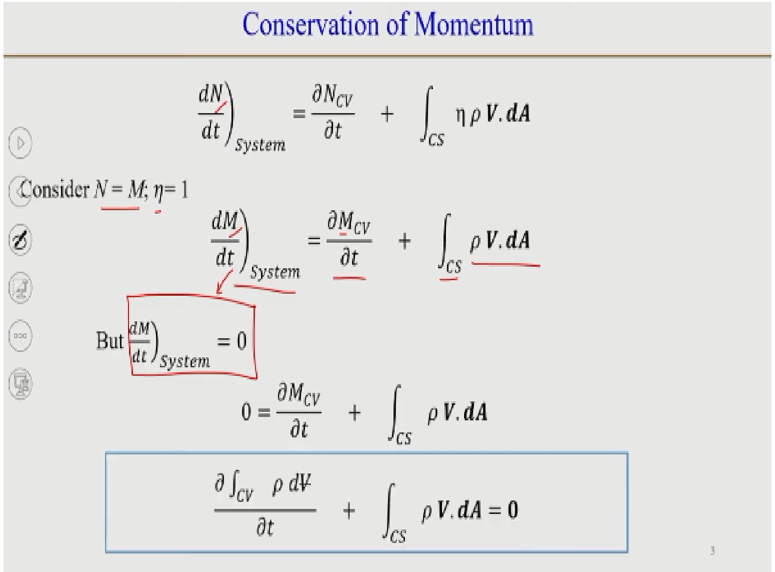

So, let us try to apply this to derive the mass conservation equation. So, when we want to derive mass conservation equation, we will consider that N =M or mass and correspondingly η will become M over M, so which gives us 1, so η is 1 and N equal to capital M and let us substitute it here. So, N is replaced by M, so dM/dt or rate of change of mass in the system that will be equal to ∂/∂ t MCV.

So, in place of N, we have now replaced with M, that is equal to integral over control surface, η =1, so it becomes ρ V. dA and we know that the mass conservation equation basically refers to that the mass in the system remains constant. So, that means the rate of change of mass in the system is 0, so we substitute that the value that we obtained or the expression that we obtained for dM/dt in this equation, then we will get the mass conservation equation, that dM/dt of our system =0.

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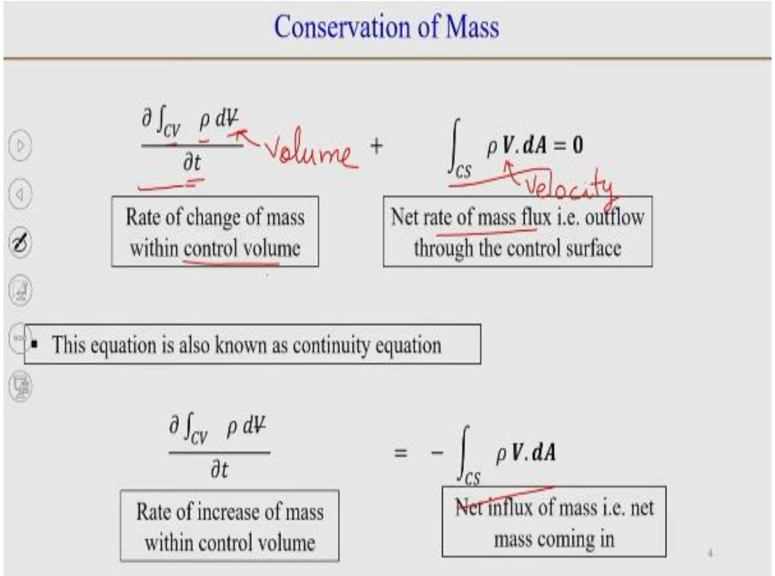

So, that is mass conservation equation which simply states this, this is ∂/∂ t of M in the control volume written in terms of ρ and V. So, if ρ is varying in the control volume, then we can put an integral, so ∂/∂ t integral over the control volume ρ and integrated over V. So, remember here this V a horizontal line cut this is the symbol that we are using for volume, because we have another V here which is V vector and this is velocity.

So, plus integral over the control surface which is the boundary of the control volume integral over ρ V. dA that will be equal to 0. So, this, the first term is rate of change of mass, so rate of change of mass within control volume ρ dV will be, when you integrate over the control volume, this will give us the mass in the control volume and its derivative with respect to time will be the rate of change of mass within control volume. And the second term gives us the net rate of mass flux or outflow through the control volume.

So, or the net rate of mass outflow that is what we call the flow or the mass flow that is going out of the control volume is given by this term. Now, we can call this also as continuity equation, so mass conservation equation is also known as continuity equation. We can write this, taking this term on the other side, so if we want to say that net rate of mass inflow, so then we can say the rate of change of mass within control volume is equal to net rate of mass inflow. So, the negative sign represents that the mass is coming in, so that is pretty obvious when you think that in a box, the mass that will be present at any time, that will be equal to mass coming in minus mass going out, if there is no accumulation.

If there is an accumulation, so the accumulation is equal to mass coming in minus mass going out. If we write that both the terms on one side, then we can say that the rate of increase in the control volume plus going out, so that will be equal to the net rate that is coming in. So, this equation is pretty simple to understand.

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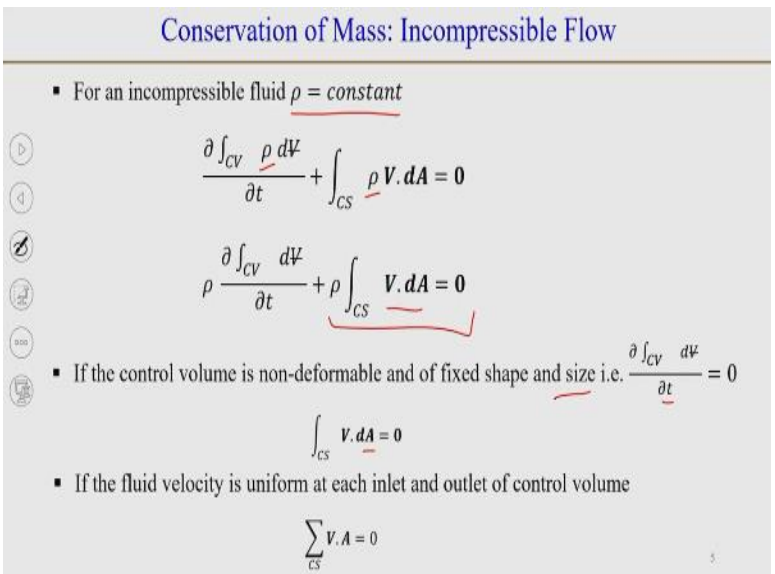

Now, if we consider a special case, that if the flow is incompressible, so we can treat ρ as constant. So, if the flow and the fluid is incompressible, then we can treat ρ as a constant, if ρ is constant then we can take ρ out of the integral sign in these places. So, ρ integral over the control volume or ∂/∂ t, so that is basically the rate of change of control volume plus ρ integral over the control surface area integral V. dA.

Now, if the control volume does not deform, so if the volume in the control volume, volume does not change if it is the rigid volume, then ∂/∂ t of control volume will be 0, if shape and size is fixed. So, then we can write ∂/∂ t over the control volume is equal to 0. So, then you up with only one term on the left hand side, so integral over the control surface V. dA =0.

And if the flow is uniform at the boundaries, so if over the entire control surface or wherever the flow is coming in or going out, if the velocity is same on an inlet or an outlet, then in place of integral you can simply write this in terms of summations, that summation over all the boundaries V. dA. So, we need to apply these equations depending on the case we have at hand.

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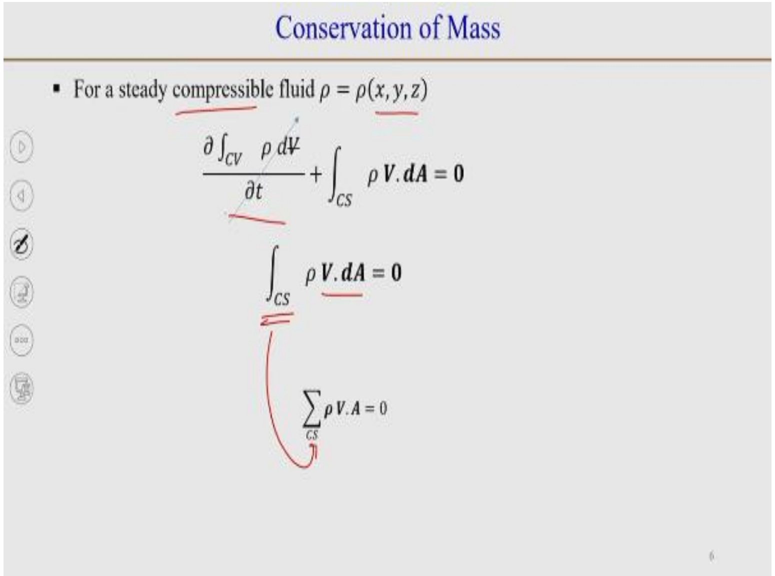

Now, if we look at the, if the flow is steady or it is compressible. So, if the flow is steady that means, the density does not change with time, but it may be a function of a space, so ρ is a function of ρ (x, y, z) then we have the mass conservation equation and because the flow is steady and if the control volume is fixed, then we can have this term is equal to 0. So, you will have integral over the control surface ρ V.dA =0.

If the velocities are uniform at the inlet and outlet boundaries, then integral you can turn it into summations. So, integral or summation over ρ V. dA =0, so summation means, you can have one boundary, two boundary or three boundaries you can write sigma over 1 ρ V. dA + ∑ + ρ_1 ρ V_2 . A_2 and so on and sum it over to get the answer.

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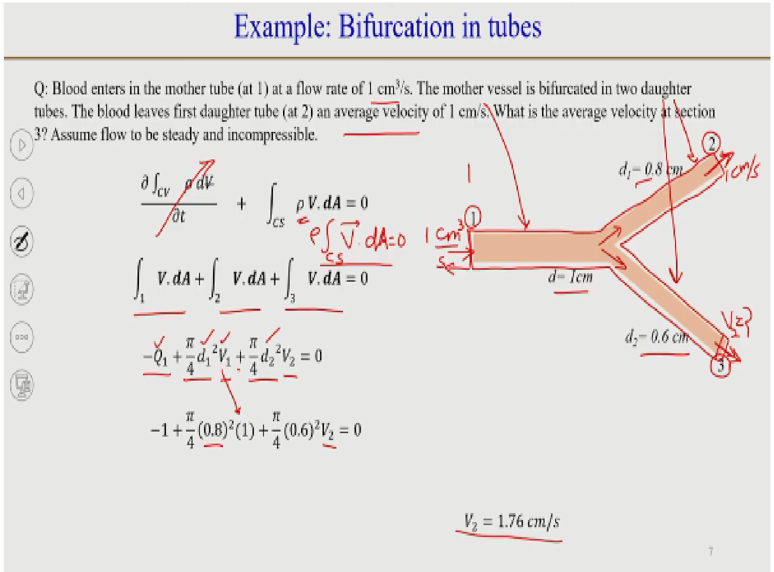

So, let us look at an example. So, this example states that blood enters in the mother tube at a flow rate, you have been given the flow rate and this mother vessel is bifurcated into two daughter tubes and the blood leaves the first daughter tube. So. let us look at the figure, the blood enters at this point and then this artery or you can say or a blood vessel it is bifurcated into two, so this is what is being referred to mother tube or mother vessel and then it has two daughter vessels, so daughter vessel 1, daughter vessel 2, the diameter of the daughter vessels are given, as well as what you have been given is the flow rate at which it enters.

So, 1 centimeter^3 per second, the diameters are also in centimeter and then it says the blood leaves first daughter tube at an average velocity of 1 centimeter per second, so the velocity here is 1 cm/s. Now, what we need to find out is the average velocity at this section. So, V at this, at section 3 is what is to be found, you can assume the flow to be steady and incompressible.

So, we start with the mass conservation equation and in this mass conservation equation because the flow is steady, so we can take this out, the velocity is given as average velocity, so we can write and then ρ is a constant, so we can take ρ out and the equation will be integral over the control surface ρ V. dA =0, we can divide by ρ, so it V have integral over the control surface of V. dA =0, we can take a control volume or we can have the, so we can draw a control volume which is following the just outside the vessels and this surface will be a control surface.

Now, the flow is coming in at three sections, section 1, section 2 and section 3. So, the flow is coming in or going out at these sections, everywhere else the flow is 0. So, when we apply this equation integral over control surface, we will have three surfaces, so integral over control surface 1 which is the section at one where the flow is coming in V. dA + V. dA at section 2, the exit of daughter vessel 1 and integral V. dA the exit of daughter vessel 2 the section 3.

So, V. dA the flow is coming in and the area vector is pointing out. So, V is pointing inward whereas A is pointing outward. So, you will get $\dot{V}$ dA with a - sign, so you will have - and V. dA is basically the volumetric flow rate at section 1, the value we know. So, Q_1 =1 cm^3/s, we write this in terms of letters, so Q_1.

Now, section 2, V. dA the flow is going out, so the velocity is outward, the area normal will also be pointing outward at the surface, so V dA will be positive, the area will be pi/4 d_1^2 and the velocity, mean velocity there is V_1 which is equal to 1 centimeter per second. So, we know V_1, we know Q_1, we know d_1. At section 3, again the flow of, we assume at least that the flow is going out, so if the flow is going out, then the area normal and the velocity vector they point in the same direction and the sign is positive. Area is pi/4 $d2^2$ into V_2, V_2 is the velocity that is the mean velocity at the exit of daughter tube 2.

So, let us substitute the values in because all the numbers are in terms of centimeters, so centimeter per second or $centimeter^3$ per second. So, we can use the same units, Q =- 1 + pi/4 0.8 centimeter is d1, so 0.8^2, V1 is 1 centimeter per second, we also know d2 $pi/4.6^2$ into V2. So, when we solve it, we can obtain the value of V_2 which will come out to be 1.76 centimeter per second. So, I suggest that you verify this value and check for yourself if this is correct or not.

So, this is a simple example and you could have just used A_1 V_1 = A_2 V_2 or A1 V1 = A_2 V_2 + A_3 V_3 to find this out. So, that will come out when you use this, basically this is a flow rate or volumetric flow rate, that is equal to the exit, the flow coming out from these two daughter tubes $A_1 V_1 + A_2 V_2$ that is equal to the total flow that is coming in. Now, suppose if this velocity the numbers would have been such that you get V_2 is equal to negative, so that simply means that the flow is coming in from the tube 3 rather than it is going out.

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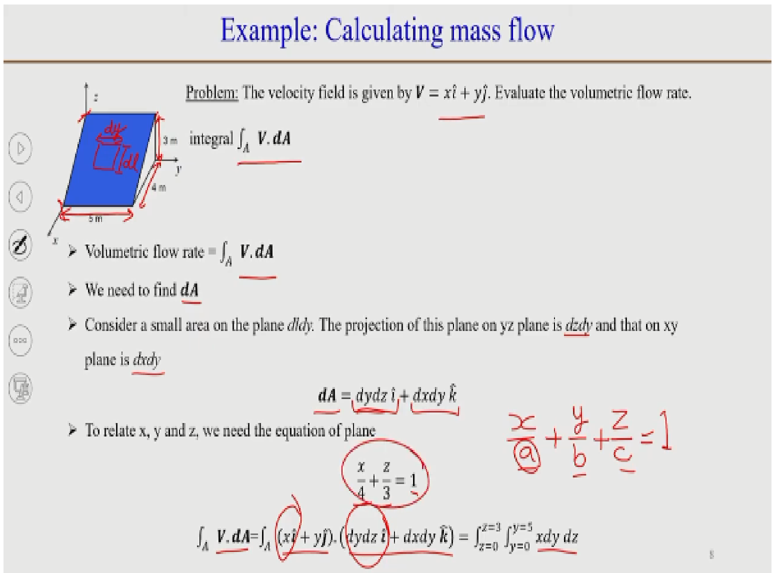

So, let us look at next problem. So it talks that there is a plane which is drawn here and the velocity field is given by a vector V $x\hat{\imath} + y\hat{\jmath}$, so this is the velocity vector and it asks us to evaluate the volumetric flow rate. So, volumetric flow rate is nothing but the integral over the area of this plane V. dA. Now, we already have vector V, in such case or in this case what we need to find is the area vector dA first. So, to find volumetric flow rate, we need to calculate dA.

So, if we take a small area on this plane, you can see that this plane we can take a small area which has dimensions dl and dy. Now, this can be projected, if you project this area on the two planes, you can project it on yz plane and you can have a projection on xy plane. So, on the yz plane if you project it, then the dimension will become dz, so dl basically the projection of dl on the yz plane will be dz. So, the area, the projection will be dz, dy on the yz plane.

On the xz plane, sorry, xy plane the projection of dl will be dx, so you will get on the xy plane dx dy will be the length of this small area, where dx is the projection of dl on xy plane. So, we can write this area vector dA in terms of dy dz which is the projection on an x plane means, on x plane means the yz plane and its normal points in the x direction.

So, dydz$\hat{\imath}$ + dxdy $\hat{k}$, on the xy plane the normal to A, the normal vector to will be along the z direction, so unit vector k. So, we can write dA = dydz$\hat{\imath}$ + dxdy$\hat{k}$. Now, we know the area vector dA, so let us substitute V. dA. Other thing we will need is because we have taken an elemental area and its normal vector, but we will also need to relate when we integrate this, then we will need to relate x, y and z, so we need the equation of this plane.

So, if we write down equation of plane because the plane that has intersects x, y, z planes at let us say the intercepts are a, b and c we can write the equation of plane =x/a + y/b + z/c =1, where a, b and c are the intercepts on x, y and z axis respectively. So, if we look at this plane at x axis, it intersects at this point where the intercept is 4 meter, so x/4, y axis it is parallel to y axis, so it never intercepts it, so you can say intercept is at infinity, so that term becomes 0, y y b, b is infinity, so y y b will become 0.

And then you have the intercept at z axis at this point, so c is 3, so x/4 + z/3 =1 is the equation of this plane. Now, we can write down or substitute the values integral V. dA, so we can write in place of V x$\hat{\imath}$ + y$\hat{\jmath}$ dot the area vector dydz$\hat{\imath}$ + dxdy $\hat{k}$. Now, as we look at when we take dot product $\hat{\imath}$. $\hat{\imath}$ is 1, $\hat{\imath}$. $\hat{k}$ because they are normal to each other, so that will be 0, $\hat{\jmath}$. $\hat{\imath}$ normal to each other, so that product will be 0 $\hat{\jmath}$. $\hat{k}$ the product will be 0, so we will have only one term where we have $\hat{\imath}$. $\hat{\imath}$.

So, that means, this term and this term, the product of these terms you will have inside the area, so xdy dz $\hat{\imath}$. $\hat{\imath}$ is 1. And integral over the areas, so if we look at these over the plane, then the limits go from y =0 to y =5 and the other limit for z from z =0 to z =3, because we have xdy dz. Now, we know that x in terms of z can be written from the equation of this plane.

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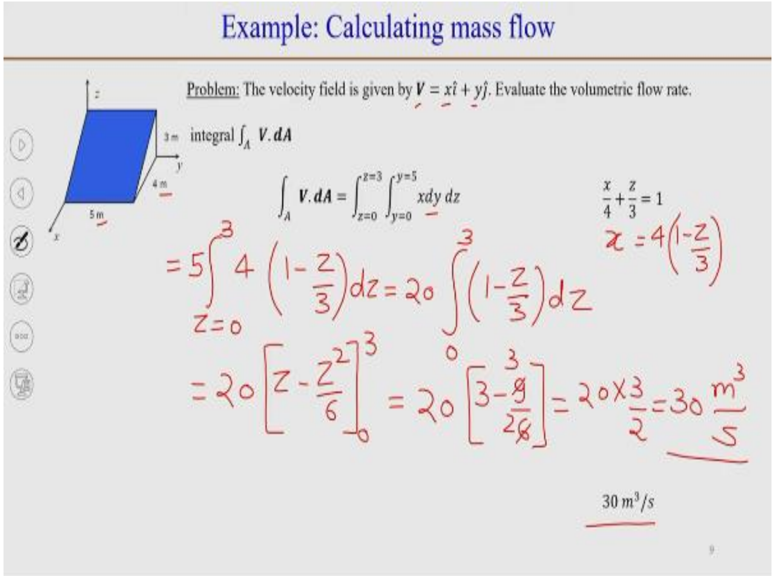

So, we can substitute that. So, let us write this we can write x =1 - z/3 multiplied by 4, so x/4 =1 - z/3 and x =4 into 1 - z/3. So, we can substitute this here and we will get integral z =0 to z =3, x is not a function of y, so we can integrate with respect to y straight away and we will get y which will be 5 - 0, if you put the limit, so we will get this 5 x, so in place of x we can write 4 into 1 - z/3 dz. So, when we simplify it 5 into 4 is 20, so we will have a 20 integral 0 to 3 1 - z/3 dz.

Let us integrate it, so that will be equal to 20 within bracket when you integrate 1 you will get z -, z when you integrate, so $z^2/2$ and 1/3. So, $z^2/6$ integral, sorry, the limits from 0 to 3. So, 20 into and if you substitute the limits, so when you put z equal to 3 that becomes 3 - $z^2/6$, so $3^2/6$ that means 9/6 and - 0.

So, 3 - 9/6 which is basically 3/2, so 3 - 3/2 20 into 3/2 or 1/half times of 20 which is 30 and unit will be, we have not been given unit here. So, we have been given the units in terms of meters and let us say that x and y in these are in meters and the velocities in m/s. So, if that is the case, then the answer comes out to be 30 m^3/s. So, you can write the units here also, 30 m/s.

So, what we need to look at here is when you have been given such a plane, the effort goes into finding out the unit normal vector and if we need to relate the x and y in such cases, then we need to do that a bit of vector algebra or a bit of geometry where we need to find the plane equations etcetera.

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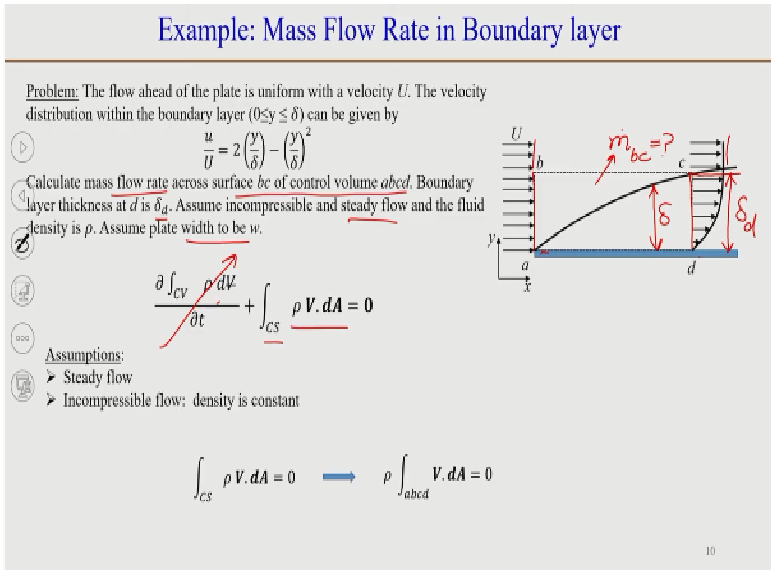

So, let us look at another example. We have already discussed what is a boundary layer, boundary layer is the layer near the wall in which the viscous effects are important. So, when you have flow over a flat plate as has been shown here, when the flow enters the uniform flow enters over this plate and because of the no slip boundary condition at this wall, the velocity at the wall will become 0 and there will be gradient in velocity inside the boundary layer, as you go over boundary layer the velocity becomes uniform, so it has uniform velocity.

Of course, when this change is happening in the velocity profile, because the velocity here and here it is same, there will be some mass that has to go out of this control volume. So, a control volume a b c d has been drawn here and what we need to find is the mass that is going through, calculate mass flow rate across surface bc of control volume a b c d. So, let us say what we need to find is $\dot{m}$ bc.

Now, the velocity profile because this is at the section ab velocity profile is uniform or the velocity is same across all from 0 to y, whatever y we take whereas at section cd, we see a change in velocity, so this velocity has been given as a function of $u/U = 2\ y/\delta - y/\delta^2$, where δ is at any location, δ is the thickness of boundary layer.

At point d this thickness has been told that this thickness of boundary layer at point d is δ d, what we see here is two dimensional picture and the plate width which is normal to this, normal to the board is w. So, the area of the plate if we take, then the length multiplied by the width. So, let us apply mass conservation equation here, we write down the mass conservation equation in the form that we have just derived. So, ∂/∂ t over integral over the control volume ρ dV + integral over the control surface or the area integral ρ V. dA =0.

Now, the flow is, we can assume the flow to be steady and we can assume the flow to be incompressible. So, density is constant, flow is steady, so we can take ρ is constant and the first term becomes 0. So, we end up with the simplified equation ρ V. dA =0 integrated over the control surface. We can take ρ out of the integral because the flow is incompressible. Now, what we can do, for this control volume a b c d we can expand and write down the flows or the V. dA term over each part of this control surface.

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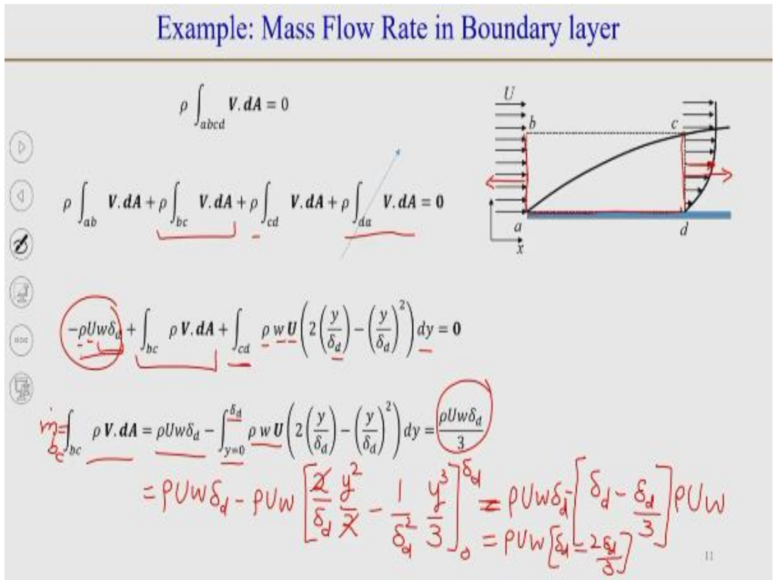

So, let us look at ab first, so if we look at ab the flow is coming in and the area vector will be pointing outward. So, the first term will be a negative, because the flow is coming in V and dA they are in opposite direction and so this term will give you ρ multiplied by U, U is the velocity and the area will be w into δ d. So, the area through which the flow enters is δ d and w which is the width normal to the board.

Now, let us look at bc. So, bc is the surface on which we need to find mass flow rate, so V. dA is the volumetric flow rate, so we can say that this is the mass flow rate through bc and this is what we need to find. Let us look at the third surface cd, so the flow is going out, that is the velocity direction and that area vector will also be pointing outward, so we will have ρ, you can take ρ outside or inside, so ρ and velocity, so velocity is U/U we had been given.

So, we have written this small u in terms of capital U into 2 y/δ at this particular location δ =δ d. So, we have written 2 y/δ d - y/δ t^2 into w into dy, because the velocity profile is varying with the distance from 0 to y, so we will integrate it from 0 to y.

Now, the last surface da, because this is wall and this is non porous, so the flow from this wall will be 0, so we have neglected this term. Now, we need to find out $\dot{m}$. So, basically what we need to do is integrate from 0 to y and then rearrange the terms to find $\dot{m}$. So, we take this term on the left hand side, which basically $\dot{m}$ bc that is equal to, we take this term on the right hand side, the first term, so this will be positive ρ U w δ d and the second term or the term under the integration which is mass flow rate at section cd this will become negative, minus and we can substitute the limits for y.

So, y =0 to δ d ρ w U all three of these are constant within bracket 2 y/δ d - y/δ d^2 into dy. So, we can simplify this and when we simplify, we will get ρ U w δ d - ρ U w. And let us integrate, so when we integrate we will get 2/δ d into $y^2/2$.

So, we will get 2/δ d, both of them are constant and 2 $y^2/2$, and 2 and 2 will cancel out, - 1/δ d^2 which is constant with respect to y, because δ d is varying with respect to x and when we integrate y^2 we will get y^3 over 3 and we can substitute the limits from 0 to δ d. So, what that will be equal to ρ U w δ d into so y^2 that will become δ d^2/δ d, so you will have δ d minus, substitute δ d in place of y, so δ d^3 /δ d^2 and divided by 3, so you will have δ d/3.

And when you substitute the limit at 0, because of that the terms will become 0. So, you have this, ρ U w δ d minus this multiplied by ρ U w. So, if you look at what you get this ρ U w δ d - 2 δ d/3 ρ U w, so ρ U w δ d - 2 δ d/3 and if you solve what you get the final answer δ d - 2 δ/3 is δ d/3, so you will get ρ U w δ d/3. So, that is your final answer.

Now, just, sorry, this has overlapped, so what you can see is how will the problem change if the plate is porous, so if the plate at ad is porous, you can think of because the flow rate at the surface ad will not be 0, so you will need to take into account the flow rate of surface ad also.

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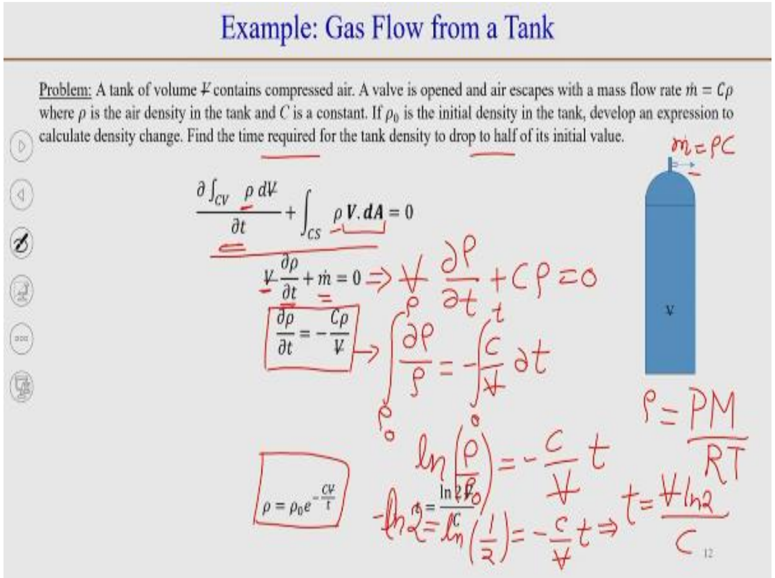

Another example say we are looking at gas flow from a tank, in chemical engineering applications as well as in medical applications you often have a gas tank and from which the flow is coming out or going in. So, if you look at this, a tank volume ¥ and this has compressed air in it. A valve is there at the tank and this valve is opened and the air starts escaping, the mass flow rate of the air escaping =ρ into C, where ρ is air density in the tank and C is a constant.

They suggest that the initial density, when the valve is opened, the density in the tank is ρ^o. So, what we need to do is develop an expression to find out the density change with respect to time, because when the gas escapes from it, the density which is if you take ideal gas law, if you assume the gas to be ideal ρ =PM/RT, so the density, pressure is going to change and because of that the density will change.

So, then the next part is that you need to find the time required in which the tank density will drop to half its initial value. So, let us write down the conservation equation. Now, the density is not constant with respect to time, so this term will be nonzero now, and there is only one control surface through which the gas goes out. So, the density is not constant, with respect to time, it is changing with respect to time, but we can assume the density inside the cylinder to be uniform.

So, over the volume, when you integrate over the control volume ρ is constant, so we can get ρ out of the integrals. So, ∂ ρ over ∂ t and when you do integral d$\forall$ you will get the volume, so $\forall$ ∂ ρ over ∂ t, this is the first term and the flow is going out, so this will be positive and if we write down in terms of mass flow rate, because we have been given mass flow rate. So, V. dA is flow rate or volumetric flow rate multiplied by density, so what we have is mass flow rate, so this term becomes $\dot{m}$, so we can write $\forall$ ∂ ρ over ∂ t + $\dot{m}$, this is the equation.

Now, we can substitute $\dot{m}$ =ρ into C and take it other side, so what we could do is write $\forall$ ∂ ρ over ∂ t + C into ρ =0, or ∂ ρ over ∂ t =- ρ C over $\forall$. So, that is the expression for rate of change of density with respect to time, now we can integrate this equation and we will get this final expression.

So, let us integrate it, we reshuffle it and we get ∂ ρ over ρ =- C/$\forall$ ∂ t and we integrate, say initial density at time t =0, ρ =ρ 0 and then at time t it becomes ρ, so when you integrate you will get ln and you substitute limit ln ρ. So, ln ρ - ρ 0 or ln ρ - ln ρ 0, which you can write ln ρ over ρ 0, this =- C over volume of the tank into t or t - 0.

So, when you write this in the exponential form you are going to get this. Now, you can find out the time that will be required for it to become 50 percent by substituting that ρ =$\rho^o/2$, so you write that ρ/ρ^o =half, so ln 1/2 =- (C over V)t or we can find ln half =- ln 2, so we can write t =(V ln 2) over C, that will be the time required for the tank density to drop to half of its initial value, sorry, so this has gone inside, I will just leave at that.

So, what we have discussed today is we applied the Reynolds Transport Theorem to derive a mass conservation equation and then we have looked at four different problems that how we can apply the mass conservation principle or the macroscopic mass balance to solve problems and find the mass flow rates etcetera. So we will stop here. Thank you.