**Fundamental of Fluid Mechanics for Chemical and Biomedical Engineer**
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**Lecture 13**
**Buoyancy**

Hello. So in this lecture, we will continue our discussion of hydrostatics. So in the previous lectures, we looked at the derivation for the famous equation P = ρgh, where h is the depth below the fluid surface or the liquid surface in general, because we experienced the hydrostatic force more commonly in liquid because that is very predominant, but we also experienced in gases. So P = ρgh, first, we looked at, and in the next lecture, we looked at the forces on a surface, which is submerged in a fluid.

Now, this surface can be a plane surface or a curved surface. So we looked at that, what is the magnitude of this force and at what point will it act? So we looked at the force, its magnitude for a plane surface and we looked at what will be the magnitude of various components of the forces, we discussed it in terms of horizontal force, horizontal components of the force and the vertical component of force. And we saw that for a curved surface, if we take a projected area on the plane, we can treat the analysis as for a plane surface.

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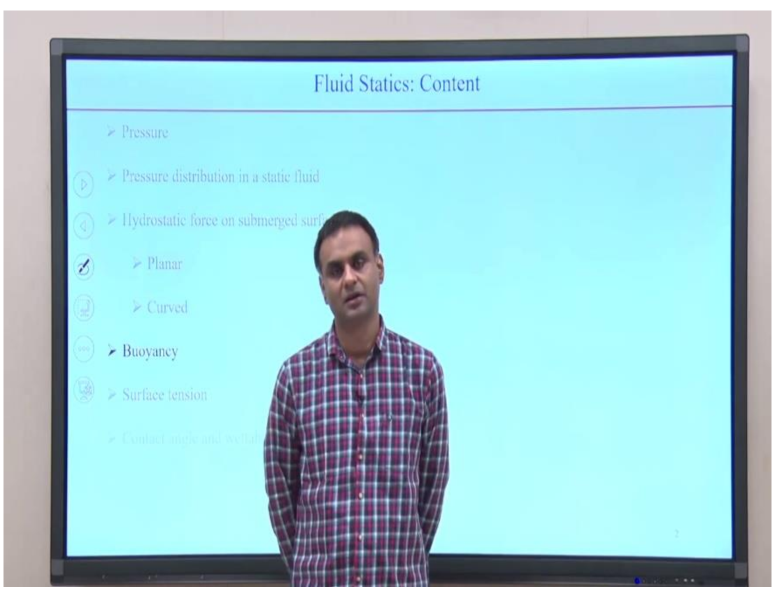


So now what we are going to look at buoyancy in today's lecture. And once we have discussed buoyancy, we will take-up a few examples for the submerged surfaces as well as one example for buoyancy. So all of us have been hearing about buoyancy or have been reading about buoyancy for a number of years since our school days, class eighth or ninth. The concept was introduced to us, what is buoyancy? We know as better as Archimedes' principle.

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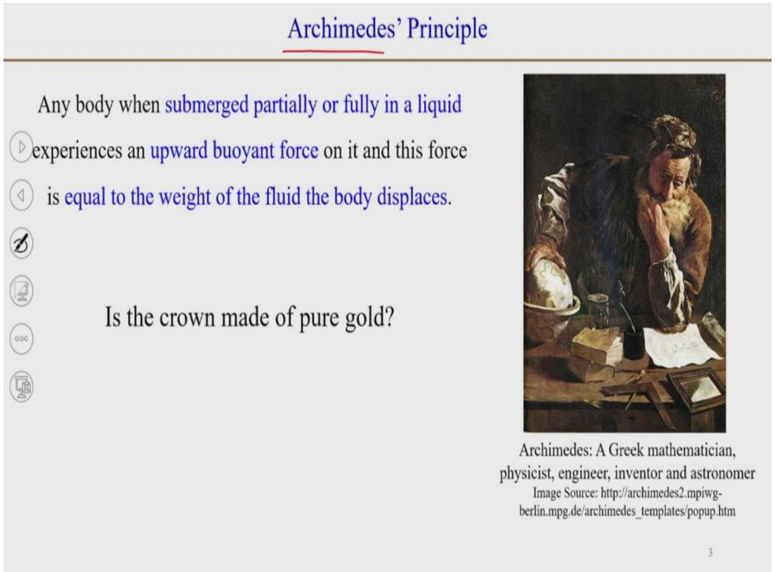


So it says that a body when submerged partially or fully in a liquid, the body experiences an upward buoyant force on it, and this force is equal to the weight of the fluid that the body displaces and the principle is known Archimedes' principle. So Archimedes was a Greek mathematician, physicist, engineer, inventor, and astronomer. So he did a number of things or he derived, developed a number of concepts in mathematics. And the most famous work of Archimedes is what we know as Archimedes' principle and that is about buoyancy.

So a story goes like this, that the King of Syracuse, he asks Archimedes' the place where Archimedes used to live. He asks Archimedes to find, if a crown that he has ordered for a temple to a jeweler and he suspected, he gave pure gold to the jeweler and the King suspected that there has been some amount of other metals mixed with the gold. So he asked Archimedes to find out if the crown is made up of pure gold.

So the way Archimedes wanted to find its purity was by looking at the density. So he could measure the weight of the crown. But because the crown is of irregular shape, so the question he asked himself was how I can measure the volume of this. And he was thinking about it. So one day, he entered in the bathtub to take bath and he realized that when he enters in the bathtub, the volume of water or the level of water goes up. So then he thought that, okay, this is an excellent method by which I can find the volume of a body of an irregular shape. So he then submerged the crown in the tub and then obtained the volume. So that is the story.

But then this part has been published in journals about the buoyancy force. So basically it says that the force that a body, when it is submerged in a liquid, is equal to the weight of the fluid that the volume of the body that is submerged displaces. So, now, having known the principle of hydrostatic, we will try to see or try to derive Archimedes' principle from what we have studied and we will do it, two different methods here.

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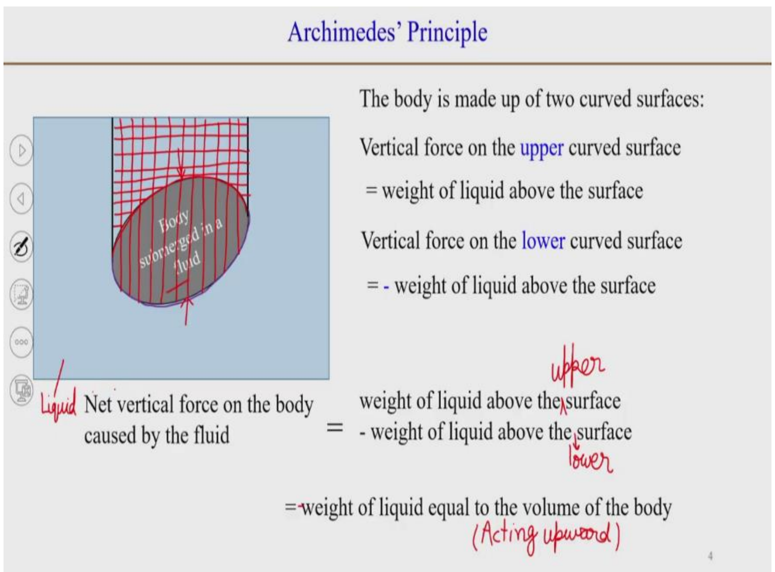


So let us look at a, let us say that in a fluid or in a liquid, the blue colored rectangle, you consider it as liquid. And in this, a body is submerged. Now, we can divide this body into two curved surfaces. One, we can say the upper curved surface and the other one, let us color it in a different one. So this can be divided into upper curved surface. Now, from the previous lectures, we know that the vertical force on this submerged surface, the upper curved surface will be equal to the weight of liquid that is above the surface.

So the vertical force on this surface will be acting downwards and will be equal to the weight of the liquid, let us just hatch this volume by the horizontal lines. So the volume of liquid that is above this curved surface pointing downwards will be the force on this curved surface. Now, we take the lower curved surface, then, what is the vertical force on this lower curved surface? So by seeing this, we know that the force on this curved surface will be acting upward. So it will be opposite to what is, opposite to the direction on the upper surface.

Now, the magnitude of this force will be equal to negative of this. So it will be, if we say that the force is positive in the vertically downward direction, then, this will be negative force and it will be equal to the weight of the liquid that is above this surface. So it will be equal and opposite in direction to the weight of the liquid that is above this surface. So this is the net force on the lower surface.

So if we combine the two, that the net vertical force on the body caused by the fluid will be the net vertical force on these two surfaces, which make the surface of the body. So that will be equal to the weight of the liquid above the surface minus weight of the liquid, weight of the liquid above the upper surface minus weight of the liquid above the lower surface. So that will be equal to the weight of the liquid equal to the volume of the body and that will be acting upward.

If we follow the sign from here, that will be a negative sign. So, weight of the liquid equal to the volume of the body. That will be the force and it will be acting upward. And that is what Archimedes' principle is. So, by simply looking at the force on the entire surface of the body, the hydrostatic force acting on the entire surface of the body, we could obtain Archimedes' principle. Now, we will do the same exercise, but in a slightly different manner.

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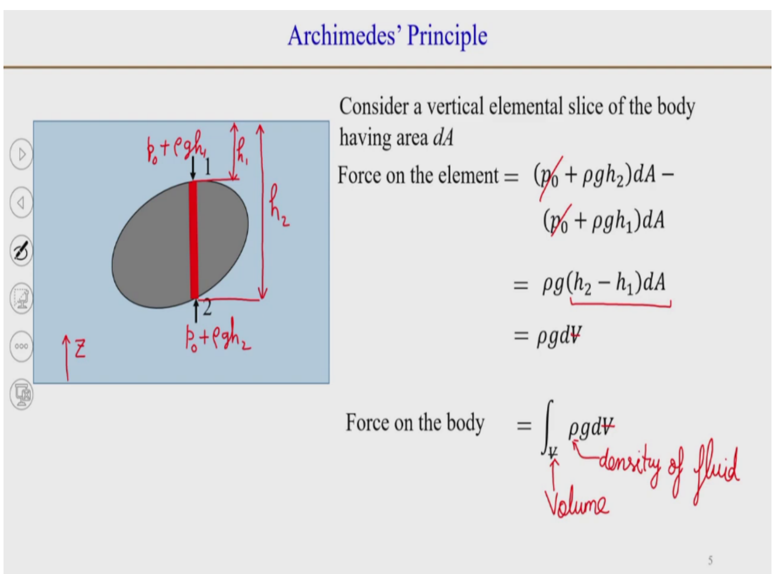

So if you, again, take the body and take an element slice having an area dA, and you take two points on this body, or a small slice point 1 and point 2 on the upper and lower surfaces of this body. Then, so the force on the lower surface will be $p^o + \rho g h_2$, where $h_2$ is this depth. And $h_1$ is the depth of point 1, the force at point 1 will be $p^o + \rho g h_1$. So if we say that the Z direction or the positive direction is pointing upward, then the net force on the element is $p^o + \rho g h_2$ dA, which is acting upward minus $p^o + \rho g h_1$ dA, which is pointing downwards.

So the net force on this will be $p^o$ and $p^o$ will cancel out, because area is same. So what you have is $\rho g h_2$ minus $h_1$ dA and that will be acting upward. Now $h_2$ minus $h_1$ into dA is nothing but the volume of this body of this small rectangular body. So $h_2$ minus $h_1$ is the height and dA is the cross-sectional area. So that will be equal to ρg, the volume of this body and so the force in this body is the integral of ρg d~~V~~. So we will use ~~V~~ strikethrough for volume because we also use V for velocity.

So in order not to confuse ourselves, we will try to use ~~V~~ strikethrough as a symbol for volume. So here, ~~V~~ strikethrough is the volume of the body. So again, ρis, remember it is density of the fluid, not that of the body. So, again, the force on the body is equal to the weight of the fluid that is equal to the volume of the body. So again, we have been able to derive Archimedes' principle. So now in hindsight, we see that Archimedes' principle is nothing but an extension of the principle of hydrostatics that $P = \rho g h$ at a particular depth h in a fluid. So now, we will look at a problem.

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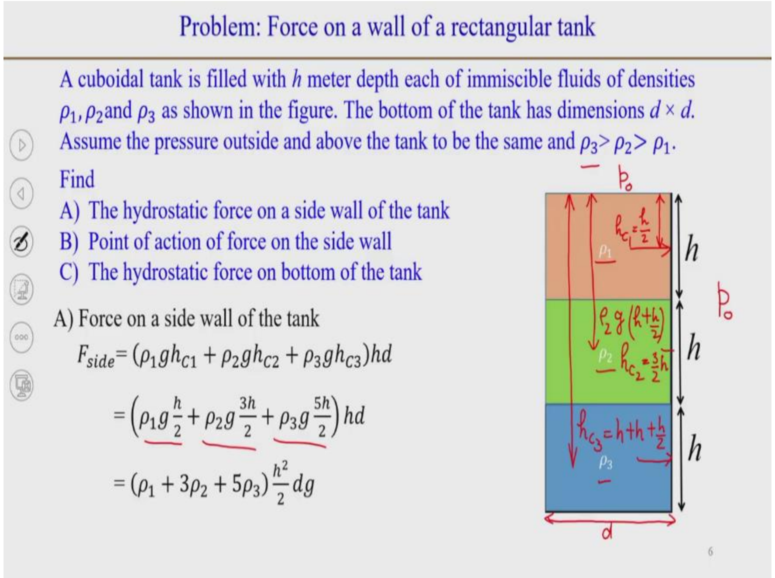


So, let us read the statement of the problem that there is a cuboidal tank. And this tank is filled with three different fluids having densities $\rho_1$, $\rho_2$ and $\rho_3$ and $\rho_3$, which is filled at the bottom has the highest value and then it is greater than $\rho_2$. And $\rho_2$ is greater than $\rho_1$. The height of these fluids is same, h for each and the bottom of tank is square shape. So the dimension, this dimension is d and the dimension normal to the screen or perpendicular to the screen is also d. We can assume that the pressure outside above the liquid surface is $p^o$ and outside is $p^o$.

So when we consider hydrostatic pressure, if we write $p^o + \rho gh$, so from the other side of the surface, we will again experience pressure of $p^o$. So we will consider only the hydrostatic pressure caused by the liquid column while solving this problem. So the questions are that we need to find the hydrostatic force on a sidewall of the tank. So the tank is cuboid shaped and it has square cross-sections. So the area of all the sidewalls will be the same. It will have four sidewalls. And the question asks us to find the hydrostatic force on any of the sidewall of the tank.

Then, the next question is we need to also find the point of action of this force on the sidewall. And the third question is what is the hydrostatic force on the bottom wall of the tank or on the bottom of the tank? So let us look at the first part, that the hydrostatic force on the sidewall of a tank. Now, we see here that the density is different in the three fluids, $\rho_1$, $\rho_2$ and $\rho_3$.

So we cannot directly use our relationship that we derived that $P = \rho gh$ at the centroid for a column of liquid. It is not varying continuously, so we can divide it into three different liquid columns, or we can treat these as three different liquid columns, and obtain the net force on the sidewall. So that is what we will do, that the side force in the first liquid column at the top will be acting on the centroid of this liquid column, so that will be $h_{c1}$, which is equal to h / 2.

So remember that this is not the point of action of this force. It is just to find the magnitude of force we can find that P because of the first column is $\rho g h_{c1}$, $h_{c1}$ is the centroid. So $\rho_1$ $ghc_1$ for the first column. Similarly, for the second liquid column $hc_2$, if this is $h_{c2}$. So the pressure will be $\rho_2 g h_{c2}$ so $\rho_2 g h_{c2}$ will be equal to h, which is the height above + h / 2. So $h_{c2}$ will be equal to 3 / 2 of h. And the force on the bottom column will be the pressure $\rho_3$ $gh_{c3}$ into the area of that particular, the wall that is in touch with this column.

So $hc_3$ will be equal to h + h + h / 2. So that will be equal to 5h / 2. And the area for each case will be because the height or depth of the column is h multiplied by the dimension normal to the screen. That will be the area so hd. So we can substitute those values here. And we see that for the first column, the force will be $\rho_1$ gh / 2 into hd. For the second column, $\rho_2$g 3h / 2 into hd, for the third column, row 3g 5h / 2 into hd.

And we can take gh / 2 out of the bracket. So we will have $\rho_1$ + 3 $\rho_2$ + 5 $\rho_3$ into $h^2$ / 2 dg.  Now, if we take simplify this, that $\rho_1 = \rho_2 = \rho_3$, then, what will we have equal to 9 $\rho h^2$ / 2 gd. So 9 / 2 is 4.5, which will be the centroid of this. So that will be the net force on the sidewall of a tank.

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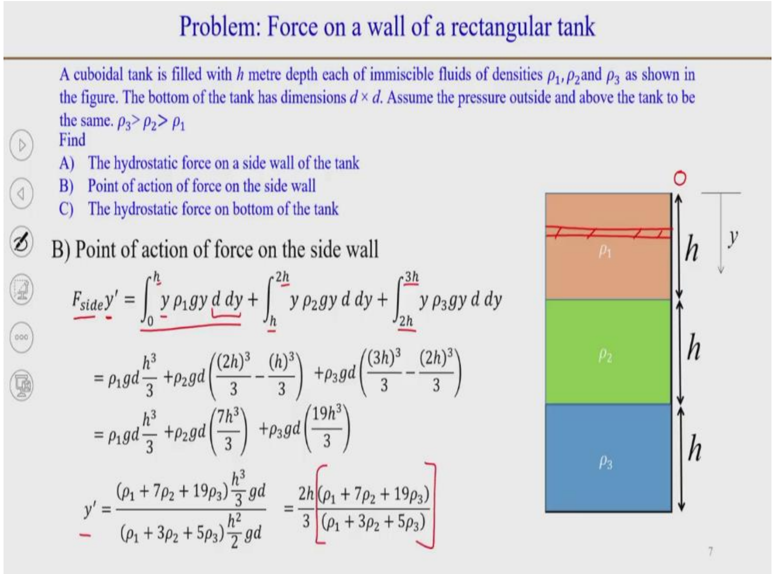


Now, the next question is, the point of action of force on the sidewall. So, the point of action of this force on the sidewall, we can obtain by taking the moment let us say on this point about this point O. And we can assume that the general depth is y from the top of this column. So, we can have, we can write a moment equation that for the resultant force $F_{side}$ into y´, so y´ is the depth below the liquid level at which the force will act. Of course, in the direction normal to the screen, this will be symmetric.

So it will be at the middle of the surface. What we need to find is only y´. So F side into y´ is equal to, for the first column, we will have y into d F or pdA. So y is the distance for a small, if we take an elemental volume. So y into $\rho_1$gy, which is the pressure into dd is the side normal to the screen into dy is the small or the height of this or depth of this elemental volume. So this becomes d dy is basically dA. Similarly, we will have for the second column, where $\rho_1$ will be replaced by $\rho_2$ and for the third column, $\rho_2$ will be replaced by $\rho_3$. And the integration limits for the first column from 0 to h, for the second column h to 2h, and for the third column 2h to 3h.

So, when we do this integration, for the first part, we will have $\rho_1$gd, which are all constant, $y^2$ dy. So when you integrate $y^2$ dy, you will get $y^3$ / 3. And if we substitute the limits from 0 to h, so $h^3$ / 3 minus 0. So from the first term we get $\rho_1$gd $h^3$ / 3, from the second term, similarly, we will get $\rho_2$ gd $2h^3$ / 3 minus $h^3$ 3 and for the third term $\rho_3$ gd $3h^3$ / 3 minus $2h^3$ / 3. So, let us simplify first term will be same, the next term will be $\rho_2$gd $7h^3$ / 3 and the third term will be $\rho_3$gd 19 $h^3$ / 3.

We can further simplify and substitute the value of the net force that we obtain just a few minutes back. So, y´ = $\rho_1$ + 7 $\rho_2$ + 19 $\rho_3$ into $h^3$ / 3gd / $\rho_1$ + 3 $\rho_2$ + 5 $\rho_3$ into $h^2$ / 2gd. So that is our y´. We can cancel out gd / gd and $h^2$. So we will get 2 / 3 h into $\rho_1$ + 7 $\rho_2$ + 19 $\rho_3$ / $\rho_1$ + 3 $\rho_2$ + 5 $\rho_3$. So that is the location of the line of action of this force, which will be 2h / 3 into this whole expression in terms of the densities of the fluids.

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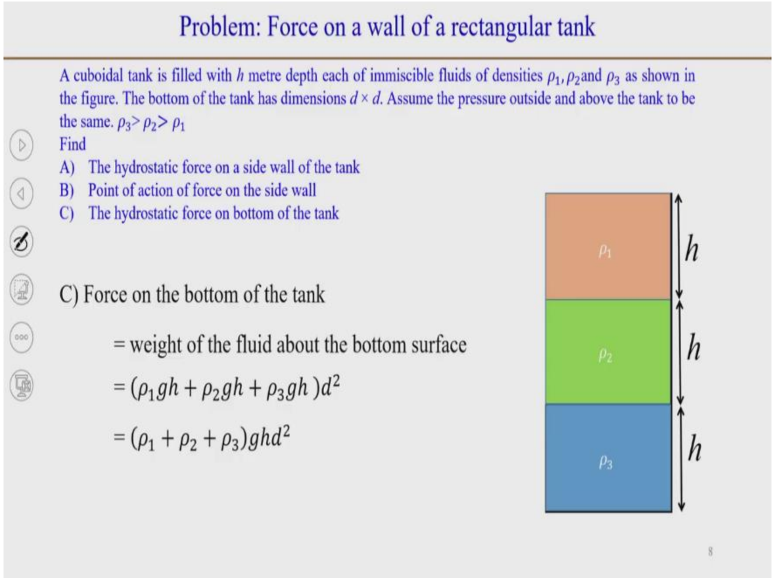


So, now the third part is simple that what we need to find is the hydrostatic force on the bottom of the tank, which will be equal to the weight of the fluid or fluids above this tank. So that will be the weight of the entire fluid column. So $\rho_1$ gh + $\rho_2$ gh + $\rho_3$ gh multiplied by the area of the bottom, which will be d into d so $d^2$. So, if we take gh, out from the bracket, we will get $\rho_1 + \rho_2 + \rho_3$ within the bracket multiplied by gh $d^2$. So, in this, we have an assumption that the heaviest fluid, which is $\rho_3$ is at the bottom, and then the next heavier fluid, and then the lightest fluid $\rho_1$ at the top.

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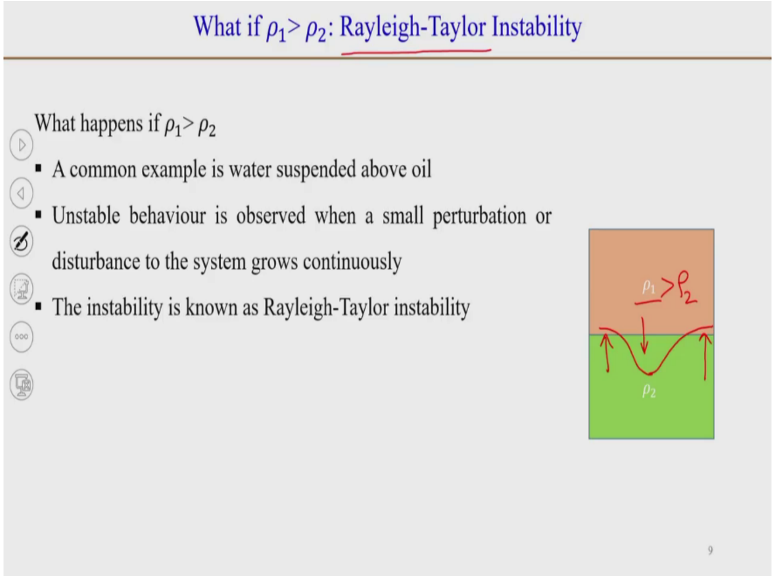


Now that is of course, obvious that if we do not have such an arrangement, what will happen if the fluid at the top is heavier than the fluid at the bottom. So if $\rho_1$ is greater than $\rho_2$, what will happen that the liquid will start or the heavier liquid will start coming down and the lower liquid, or the lighter liquid at the bottom will start going up, and this will rise, give rise to an instability. So, one of the common examples, which we can experience, or we can do experiments also. You can take two glasses of liquid, one filled with water and another having a liquid of lower density.

So, for example, you can take an oil, which has a density lighter than water, you can fill a lighter than that of water. You can fill both the glasses. Put some surface, solid surface, or plane surface, or a paper on the top of oil, turn up the water on it and then slowly remove the separation, which is a paper or metal sheet or whatever you have. And you will realize or you will see that the fluid, if they are immiscible, then, the oil will start coming up and the water will start coming down because water is heavier than the oil. And this is, this phenomenon is known as Rayleigh-Taylor instability.

So, the instability is a common name. When you see that, the instability refers, that if you disturb a system in equilibrium, or if you give a small perturbation to the system, and if this perturbation grows continuously with time, then the system becomes unstable. So such phenomena are known as instability. And there are a number of fluid instabilities or a number of instabilities that are occur in fluid mechanics. So the name this phenomenon is given the name as Rayleigh-Taylor instability.

So this is just because we are talking about the fluids of different densities on one above the other. So I thought I will introduce that term to you. There is whole lot of mathematical description based around it, because this has application in a number of areas, especially in atmospheric fluid mechanics or fluid dynamics. The system of course will not remain static anymore, because the fluids will start moving.

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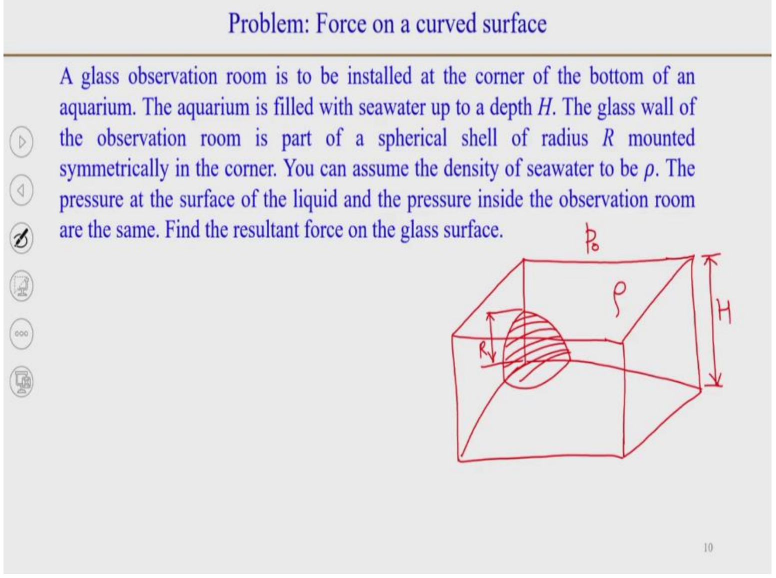


Now, we will look at the force on a curved surface problem. So, what this talks about, that there is a glass observation room that is to be installed at the corner of the bottom of an aquarium. So if you go and visit an aquarium, some of the aquariums have which are close to see, there you can see the faces, especially the bigger faces such as dolphins from the bottom. And you can see from them, them from a glass enclosure.

So, let us say that the problem talks about having a, so it says that there is a glass enclosure, and let us say that this glass enclosure is somewhere here, and we can just hatch this surface so that it is easily identified. So this will be a 1 8th part of a spherical shell. So, you assume a hollow sphere and cut from three planes. So what you will have is 1 8th of a sphere. So, the aquarium is it has at the corner of the bottom an observation room, which is a spherical shell of radius R, so the radius of this is spherical shell is R, height of this aquarium is given to be H.

This is symmetrically placed in the corner. So that is why it will have 1 8th of a sphere. You can assume the density of the water or seawater to be ρ and the pressure at the surface of the liquid and pressure inside the observation room to be the same. So, the pressure here is $p^o$ and pressure on the other side of the observation room, where there is air is also $p^o$. So we do not need to consider the pressure in our analysis or $p^o$ in our analysis.

Now, what we need to find is the resultant force on the glass surface. So, we can have, this glass surface, of course will have three projections in all three directions, on the two sidewalls and the bottom wall. So, we can have a force on the one horizontal surface and another horizontal surface so let us say that these are x and y surfaces, which are normal to each other. So the direction will be once in the x direction, the force will be acting and the other force will be acting in the y direction.

But their magnitude will be same, because the projections on it is placed, the glass surface is placed symmetrically. So the area projected on x surface and area projected on the y surface, they will be equal. So if we are able to obtain the force in the x direction, we will be also, the magnitude of force in the y direction will be same. The third surface or third area projected will be vertically downward or the bottom surface. So we can obtain the forces.

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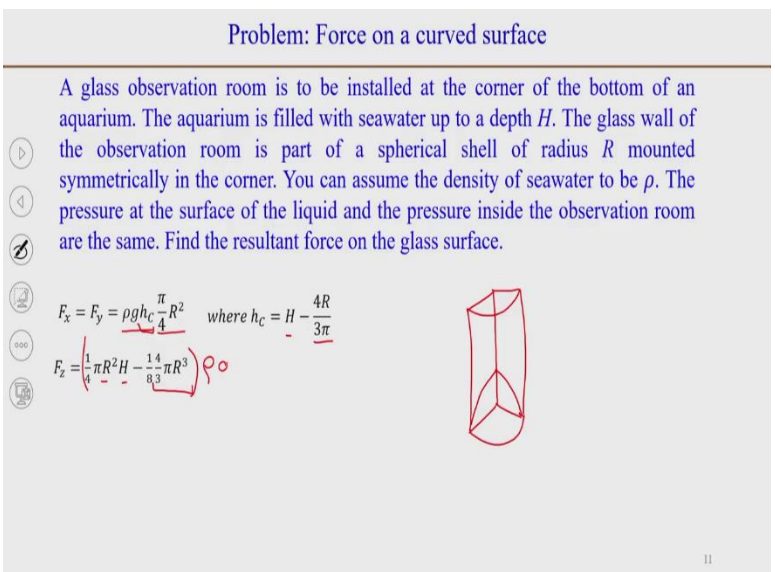


So, let us see the $F_x$ and $F_y$ they will be equal to pressure into the area. So the area of this glass surface will be one half of the cross-section or so $\pi R^2$ will be the if we cut this sphere in the middle, the area will be $\pi R^2$, but the projection will be only one quarter of it, so the area will be $\pi / 4 R^2$. Now, the pressure will be $\rho g h_c$, where $h_c$ is the centroid. And from our knowledge of mechanics, we can find, or we can look into the books that for a semi-circular surface, the centroid is located at 4R / 3π from the center.

So, the distance of the centroid from the top surface will be H, which is the depth of the liquid, or the distance from the liquid level to the center minus the location of the centroid. So 4R / 3 π, so H minus 4R / 3 π is $h_c$. So that will be the force in x and y direction. And the force in z direction will be equal to the weight of the liquid, above this liquid column. Now, the liquid column will have, so at the bottom, it is circular.

So if we look at the liquid column, that will be part of a 1 4th of a cylinder of radius R and height H. So the volume of the cylinder or volume of liquid in this cylinder will be π / 4 $R^2$ H, which is cylinder volume divided by 4 and we will have to subtract the volume of this sphere. So sphere volume is 4 / 3 π $R^3$ and it is only 1 8th of this sphere. So 1 / 8th of 4 π $R^3$. So, when we multiply it with ρg, that will give us $F_z$, So we have obtained the three components and that is what has been asked.

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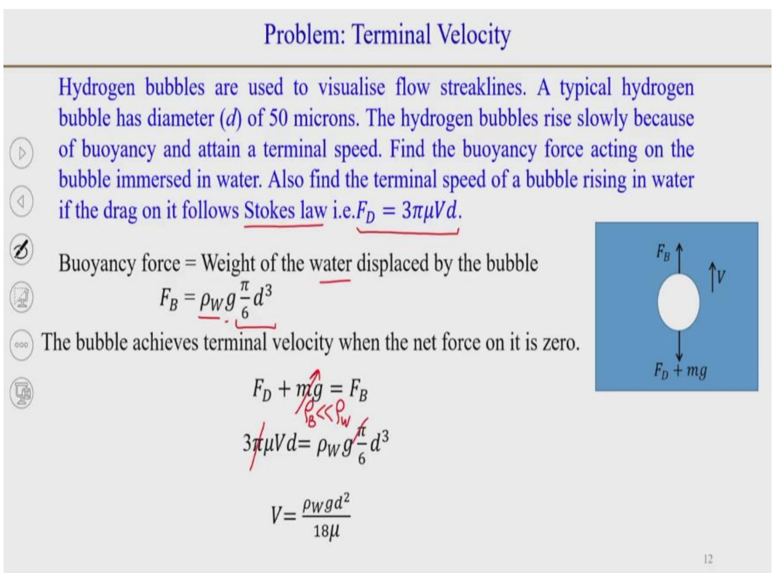


So, let us look at the next problem, which is about buoyancy. So it says that hydrogen bubbles are used to visualize flow streaklines. A typical hydrogen bubble has a diameter of 50 microns and the hydrogen bubbles rise slowly because of buoyancy and attain a terminal speed. What we need to find is the buoyancy force acting on the bubbles immersed in water. And we also need to find the terminal speed of a bubble rising in water, if the drag on the force is this.

So, when we need to measure the velocity and one of the common techniques to measure velocity to introduce tracers in the fluid. And these tracers should be such that they can follow the fluid faithfully. So, the hydrogen bubble talked about in this example is one such tracer, that it is very small in diameter, and it will quickly relax or its velocity will be equal to the fluid velocity within no time.

The Stokes flow generally refers to the flow when the Reynolds number is very small. So when the fluid velocity is very small, the length scale in this case, the bubble diameter is very, very small. So, the Reynolds number is sufficiently lower than 1. Then, we can call the flow as Stokes flow. We will discuss it later on. And you would have studied in your school that the drag force on a bubble rising, or a liquid dropping in a stationary fluid or a quiescent fluid is $3 \pi \mu V$ into d, where V is the velocity or the terminal velocity.

Because when a bubble rises, it will start rising with the velocity and then because of the drag experienced by the bubble, the velocity will slow down and finally the two forces, the buoyancy force and the drag force. Of course, there will be some mass, but that can be negligible when compared with buoyancy force. So, when all the forces on the bubble, they are in equilibrium, it will have 0 acceleration and it will achieve a constant velocity and that velocity is what we call terminal velocity.

So, the first question, it asks that, what is the buoyancy force acting on this bubble? So, that is simple. Buoyancy force will be weight of the water displaced by this bubble. So the volume of this bubble will be $\pi / 6\ d^3$, where d is the diameter of the bubble and multiplied by $\rho W$, density of water into gravity. So, that will be weight of the water displaced by the bubble and that will be equal to the buoyancy force.

The forces are acting on this will be the buoyancy force, the drag force, which will be opposite in the direction of, opposite to the direction of motion of bubble and some mass of the bubble. So, the bubble will achieve a terminal velocity, if these forces are equal, because we can neglect this, because $\rho$ of bubbles or $\rho$ of hydrogen is very small when you compare that of water about three orders of magnitude small, so we can neglect that.

And if we substitute the value of $F_D$, which has been given $3 \pi \mu Vd$, and substitute the value of $F_B$ buoyancy force, which we have just calculated $\rho wg\ \pi / 6\ d^3$. And by canceling out the term, so

$\Pi$ will cancel out and d will cancel out. So you will have $d^2$ here. So you will get $V = \rho w g\, d^2 / 18\, \mu$. So that is the terminal velocity of the bubble. So, with that, we come to the end of this lecture. In the next lecture, we will look at surface tension and related concepts. Thank you.