Fundamental Fluid Mechanics for Chemical and Biomedical Engineer
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Force of Submerged Surfaces

Hello. So, this topic is fluid statics. In the previous lecture, we discussed about the hydrostatic force and derived the formula P = ρgH, which tells us how the pressure varies with depth in a fluid, that fluid might be a gas, or it might be a liquid.

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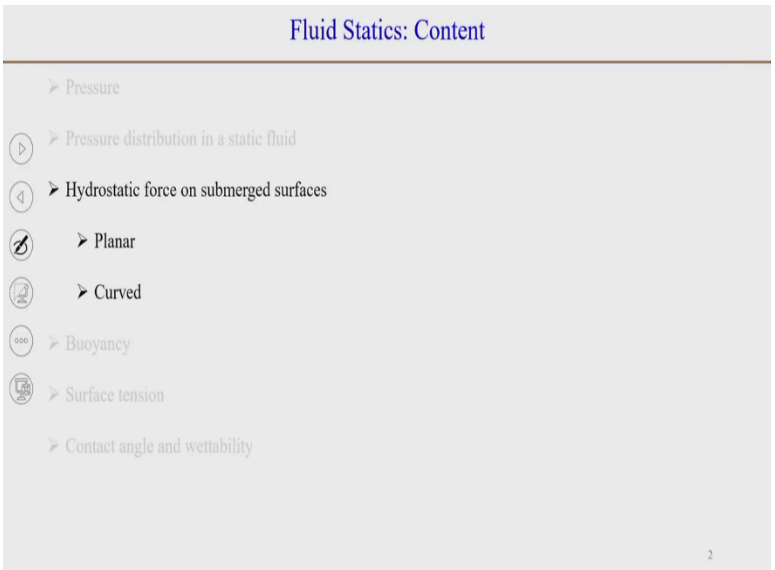

Now in today's lecture, what, we are going to look at what force does this hydrostatic pressure cause on the surfaces, which are submerged in a fluid. So, this surface can be a planar surface, or it can be a curved surface. So, for example, for a planar surface, you can think of a rectangular tank. And if you need to find out what is the force on the wall of this rectangular tank, then you will need to consider the hydrostatic force on the wall of tank.

So, you will need to find the magnitude of this force as well as the line of action of the resultant force. Then, you might have a cylindrical tank or a spherical tank. So, in such cases, your surface will not be a planar surface, rather it will be a curved surface. So, then you need to find the force on a curved surface.

So, in this lecture, we are going to look at how to find the hydrostatic force on a submerged surface, which might be a planar surface, or might be a curved surface. So, first, we will look at how to calculate the force on a planar surface, and then we will see how having learned that how we can calculate force on a planar surface. We will try to find how to calculate force on a surface. So, let us look at this.

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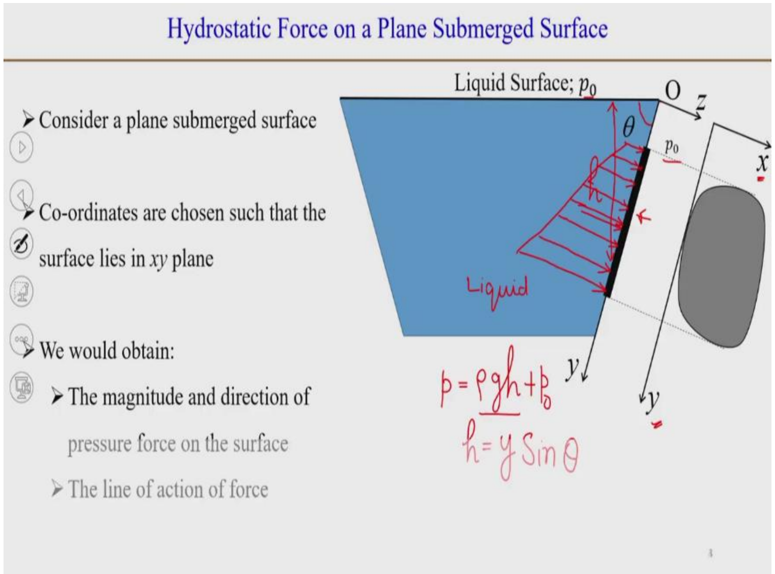

Now, if you can see there is a plane submerged surface. So, what we have is a liquid. So, this blue is liquid all liquid here. Now, we consider a surface, which is shown in the black thick, as a black thick surface. So, this is the surface. Now what we need to find is the force on this surface. The net force on the surface and what will be the point at which this resultant force will act. The pressure surrounding this surface at the top is P^o, and outside the surface on the other side of the surface is also pressure, surrounding pressure, which is atmospheric pressure let us say.

So, now we choose our coordinates such that this surface lies in xy plane. So, what we see here is that this, the origin has been chosen at point O in this figure and the coordinates are xy, in y and z plane. Now, if we see the surface normal to this, from this point, then that is what we get x and y planes. So, this is the surface on xy plane. So, we have both the views xy plane, and yz plane. Now, we need to find the force on this surface. We know that the pressure will vary P = ρgh, where h is a typical depth at any point. So, h is the distance from the top point.

Now, this will be the hydrostatic pressure caused by the liquid + atmospheric pressure P^o, so that will either at any point, pressure on the surface and the pressure will act normal to this surface. So, the direction of the pressure force will be normal to this. But this force will not be uniform throughout the surface, because as we can see that in our chosen coordinate, h = y sinθ, where θ is the angle that as shown here, the angle between OY or y axis and the liquid level.

So, we can see that the pressure on this or the pressure force on this surface will be minimum here and then as we go deeper and deeper, the pressure will keep increasing, or the pressure force will keep increasing. It will be acting. So, we can see that it would be very linearly p = ρgh, or h = ysinθ. So, p = ρgysinθ. So, we need to find out what is the resultant force. Now and we also need to find that what will be the line of action of this force or what will be the point at which this force will act? So, let us look at this now.

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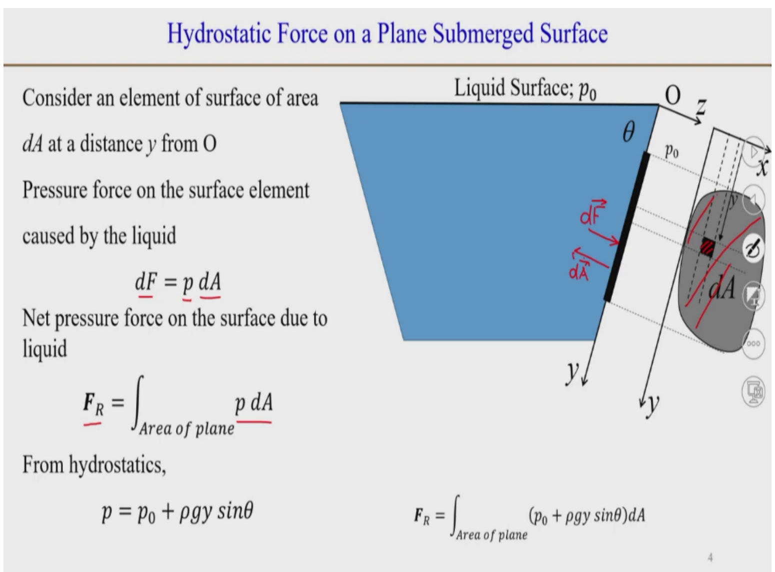

If we consider the elemental surface, because in this case, the force is not uniform with y or uniform with h, it is varying. So, we will need to consider a differential element. So, let us say, we consider a small area element, which is called dA and this surface dA is at a distance y from the origin and we consider what is the force on the surface. So, the force on this surface element caused by the liquid only, remember that right now, we are not considering what is the force that is being exerted on this surface or the surface experiences because of the fluid on the other side or because of the atmosphere on the other side.

So, the fluid that the force that we are considering is $dF = p\,dA$. Now, because we are considering here only the magnitude of the force, so, we do not write this $dF = -\,p\,dA$. But if you are considering in the vector form, as we will do later in this lecture, then we will need to consider the sign also. So, we have that the force on this element = pressure multiplied by the area. Of course, the force will act normal to this surface and the area vector will be outward normal to this. So, this is area vector, whereas this is the direction of force. So, the force will be dF p dA.

Now, we know what is p in terms of the hydrostatic pressure. So, we can substitute that and integrate to find the area. So, the total force on this surface F_R or F resultant will be, we integrate p dA over the area of the plane, so over the entire area of this plane. If we consider, then, we will be able to obtain the net resultant force.

Now from hydrostatic, we know that $p = p^o + \rho gh$, where $h = y\sin\theta$, so we will write $p = p^o + \rho gy \sin\theta$, and area will be dx, dy. So, if we substitute that, then, we get the resultant force integrated over the area of plane $p^o + \rho gy\sin\theta\ dA$. I would like to emphasize again, that this force is only from the liquid side. We do not consider right now the force from the other side, which will be, p^o A in this case. So, the resultant force is integrated over the area $p^o + \rho gy \sin\theta\ dA$.

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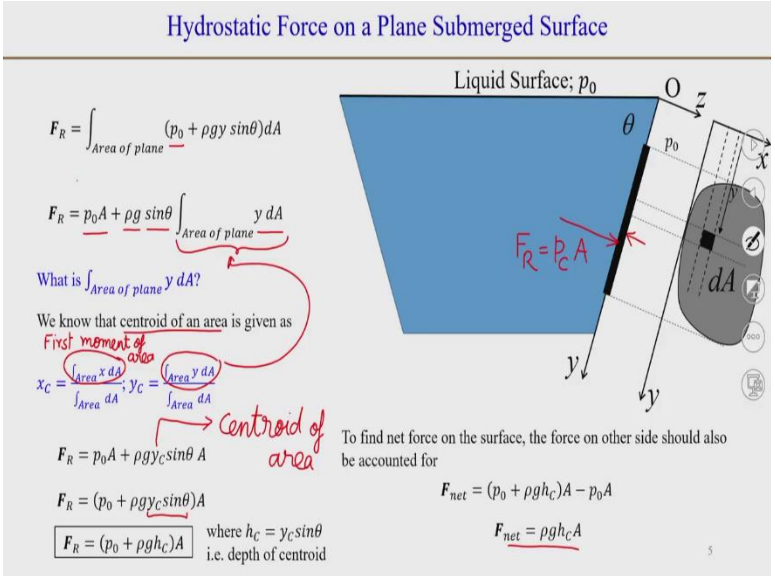

Now, we can see that p^o is a constant with respect to area. So, the first term, when you open the bracket and write down, you will get integral of p^o dA and integral of dA will be A. So, the first term becomes p^o A. Now, if the density of fluid is a constant, it does not vary with y and g is also constant. Then, we can take ρ and g out of the integral. And of course the angle θ is a constant. So, we have p^o A + ρ g sinθ, within this integral, what we are left with is integral y dA , because y varies with the surface.

Now, if we remember a bit of solid mechanics or engineering mechanics that we have studied earlier, so what is integral y dA? You might remember that the centroid of an area is defined as the x coordinate integral xdA of area divided by integral dA. And this integral, this is also called, first moment of area. So, x_c, or the centroid x coordinate of the centroid of an area, which is in the xy plane will be integral x dA divided by integral area, or divided by the area.

Similarly, y coordinated integral y dA over the area divided by area. So, we can see that this integral ydA is what we are looking for in this. So, we can substitute this integral y dA = y_c × area so F_R = p^o a + ρ g y_c sinθ dA, so where y_c is y coordinate of the centroid of area. So, from this, we know we can see from this formulas that y_c or centroid is a geometrical property of the area, and knowing the centroid of area, we can calculate the resultant force, which is p^o A + ρ g y_c sinθ A.

We can take A out and put the two terms in bracket, so that will be $p^o + \rho\ g\ y_c \sin\theta \times A$. So, the net force will be $p^o + \rho\ g\ y_c \sin\theta \times A$ or if we can substitute this $y_c \sin\theta = h_c$, where h_c is depth of the centroid, or $h_c = y_c \sin\theta$. So, we can write resultant force $= p^o + \rho\ g\ h_c \sin\theta$, which is the force that acts on this surface. So, $F_R = P_c \times A$, where Pc is the pressure at this centroid point. So, the magnitude of the force is $P_c \times A$.

Now, if we want to find out the net force in this surface, so if we also consider the force from the other side on this surface. And the pressure on the other side is atmospheric pressure so, the net force on this surface will be equal to $P^o + \rho\ g\ h_c\ A$, which is acting from the liquid on the surface $-\ p^o\ A$, which is the force from the other side. So, the net force will be $\rho\ g\ h_c\ A$.

Now on the other side, it might be open to atmosphere, so that is air. So, because there is a large difference in the densities, so variation in the hydrostatic pressure on the air side can be neglected. So, that is why we assume P^o to be uniform over this entire distance of the surface or for the entire height of the surface. So, that is the net force, where we consider the force from the liquid side and force from the other side of the surface also.

If you have two liquids, liquid 1 on the left side and liquid 2 on the right side, then, you will need to consider the variations in both the fluids and then it will not be simply P^o, you might need to consider the variation in height. But you can use the same concept to find the force on the surface. So, what we have been able to obtain the resultant force.

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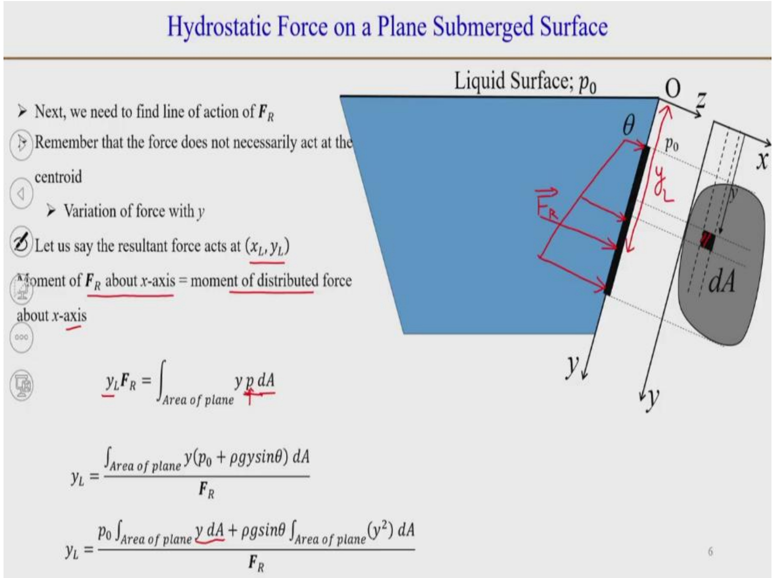

The next thing we need to obtain is the line of action of this force. So, we might need to remember here that this will not be the centroid, though we found that $F_R = P_c \times A$, where P_c is the pressure at centroid. So, $p^o + \rho\, g\, h_c \times$ area. So, one might be, he might want to use, or he might want to think, or he might get confused that it is the force is acting at the centroid of this surface.

But if we remember that the force is not constant, it is varying linearly on this surface, minimum being at the lowest height and maximum being at the maximum height. So, in that case, we can, by looking at this figure itself, we can think that the pressure will not act at the centroid of this surface. So, what we need to do to find the line of action of this force or the point of action of this force? We will need to consider the moment of the resultant force. Let us say, this is the point of action. So, we say that this is the resultant force F_R, and we will consider the moment of force F_R about the axis x and equate it with the moment of the distributed force. That will give us the point at which the resultant force is acting.

So, let us find out the y coordinate first. So, let us say that the coordinates at which the resultant force act or the point at which the resultant force acts has the coordinate x_L and y_L. So, to find y, we need to consider moment of resultant force about at x-axis and equate this with the moment of this distributed force about axis. By distributed force, I mean that we need to again consider the elemental area and consider the moment at this elemental area and then sum it over the entire surface.

So, let us do this. So, when we take moment of resultant force, which we have assumed that acts at point x_L y_L. So, the distance between the x-axis, so the distance between the x-axis and the point where this force is acting is y_L. So, y_L multiplied by F_R is equal to, if we take the elemental area dA, the force acting is p dA and the distance of this is distance y. So, the moment on an elemental area will be moment about x-axis will be yp dA. And when we integrate over the surface, we will get integral of yp dA over the entire surface of the area. So, we basically get the Y coordinate integral. We can substitute p here. So, $p = p^o + \rho\, gy \sin\theta$, we have already seen, and simplify it.

So, the first term, as we saw earlier, this = p^o integral y dA, and the second term is ρ g $\sin\theta$, assuming that ρ and g are constant, integral y^2 dA divided by F_R. So, now, we have in the integral y dA the first term. And we remember this integral y dA = y_c × area we just discussed. Now, we have another integral, which is integral y^2 dA. So, let us see that, do we know such terms from solid mechanics or from the properties of or area of a surface?

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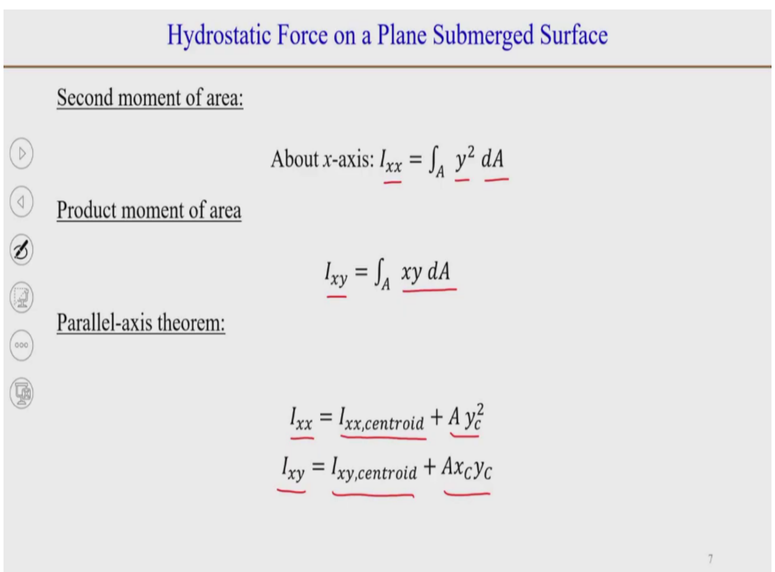

So, if we recall that the second moment of area is defined about an x-axis, this is defined as moment of area or moment of inertia I_{xx} = integral over the area y^2 dA, so about x-axis, I_{xx} = integral y^2d dA. Then, there is another moment of area, which is called product moment of area, which takes into account the distance from both the axis. So, I_{xy} which is called product moment of area is integral over the area xy dA, where x is the distance from y-axis and y is the distance from x-axis.

Now, we can use parallel axis theorem, because we might not know in the textbooks or in the literature, you will find the second moment of area or product moment of area is easily about the centroid. But it is not necessary that your origin in the problem and the centroid they coincide. So, we can use parallel access theorem to relate that what will be the second moment of area with respect to the x axis, when you know the second moment of area with respect to centroid.

So, this is $I_{xx} = I_{xx}$ centroid + A y_c^2, where y_c is the coordinate of the centroid. Similarly, the second moment of area we can use or we can obtain from parallel axis theorem so $I_{xy} = I_{xy}$ at centroid + A x_c y_c. So, this is just to recall that what we already know about the area, or second moment of area, or product moment of area. And some of these integrals we are going to see, or we have already seen in our analysis of line of action. So, we will use these results directly.

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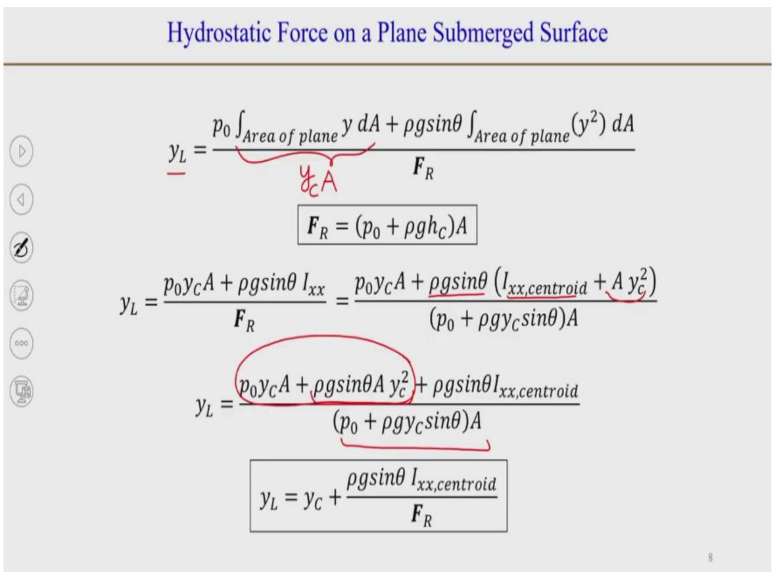

So, let us get back to the formula for y_L, where y_L is the y coordinate of the point at which the resultant force acts. So, we have p^o integral y dA + ρ g sinθ integral y^2 dA divided by the resultant force. And we know that the resultant force is p^o + ρ g h_c sinθ or h_c is c sinθ, c is the centroid.

So, we can substitute that p^o and this value = y_c × area. so, the first term becomes p^o y_c A. The second term ρ g sinθ × I_{xx}, integral y dA, we saw in the previous slide that it = I_{xx}. So we represent this in terms of commonly known quantity, which is the second moment of area with respect to centroid. Using the parallel axis theorem, we write that so I_{xx} + A y_c^2d.

So, now we simplify this further and we will get first term as it is p^o y_cA + ρ g sinθ multiplied by A y_c^2. So, ρ gy sinθ multiplied by A y_c^2 in + ρ g sinθ I_{xx} at centroid. Now, if we look at this term, this is nothing but the resultant force, which is p^o + ρ g y_c sinθ × A multiplied by y_c.

So, we can simplify this further that y_L = y_c + ρ g sinθ I_{xx} or second moment of area at the centroid divided by the resultant force F_R. so, this gives us the y coordinate of the point at which the resultant force acts. Now, remember that this was, the plane is an xy plane. We know the y coordinate. The other thing we need to know is the x coordinate.

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Hydrostatic Force on a Plane Submerged Surface

To find x_L:

Moment of $\boldsymbol{F}_R$ about y-axis = Moment of distributed force about y-axis

$$x_L \boldsymbol{F}_R = \int_{Area\ of\ plane} x\, p\, dA$$

$$x_L \boldsymbol{F}_R = \int_{Area\ of\ plane} x(p_0 + \rho g y \sin\theta)\, dA$$

$$= p_0 \int_A x\, dA + \rho g \sin\theta \int_A xy\, dA$$

$$= p_0 x_C A + \rho g \sin\theta I_{xy}$$

$$= p_0 x_C A + \rho g \sin\theta (I_{xy,centroid} + A x_C y_C)$$

$$\boxed{x_L = x_C + \frac{\rho g \sin\theta I_{xy,centroid}}{\boldsymbol{F}_R}}$$

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So, to find x coordinate, we do the same thing, but this time about y-axis. So, we find the moment of the resultant force about y-axis. And we equate it the moment of the force at an elemental area, dA and integrate it over the entire surface. So, this will be equal to x_L, where x_L is the coordinate of the point at which the net force acts, which is, this is the x coordinate of the point at which the resultant force acts. This = integral of p dA is the force on an elementary area and x is the distance from the y-axis.

So, you will get x_L F_R = integral x_p dA. Now, we substitute again what is p? p = p^o + ρ gy sinθ dA and the first term we can see that it will be p^o integral xdA. So, integral xdA from the definition of centroid, it will be x_c × area. So, p^o x_c A + ρ g sinθ, the other integral here comes integral xy dA. and just now we discussed that integral xy dA is product moment of area. So, this is what we write

I_{xy}. Now we can write this in terms of, again, the known quantities, which we can find easily in the literature. So, p^o x_c A + ρ g sinθ I_{xy} centroid + A x_c y_c. So, x_L is equal to, we simplify this.

We can combine the two terms this term and this term, and we will see that x_L will be equal to x_c + ρ g sinθ I_{xy} centroid divided by F_R. So, now we are able to obtain the x coordinate of the point at which the resultant force acts. So, our plane is an xy plane. Our surface is in the xy plane. And now we know the net force, which is p_c × A. We know the x and y coordinate.

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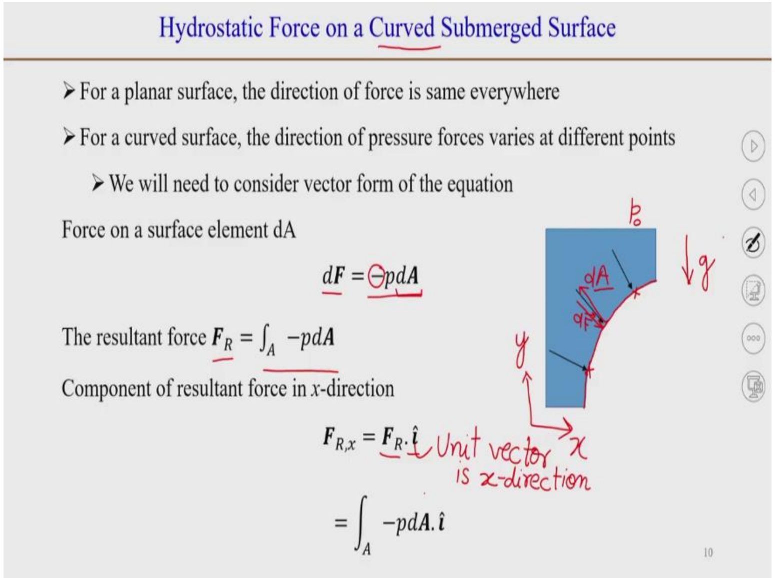

So, we have obtained the net force that acts on a surface and its coordinates. Now, we will come to look at what is the force on a curved submerged surface. Now, if we just think about what we have done a few moments ago for a plane surface and we will tend to think, or to do the same thing. So, what is the difference between the plane surface and a curved surface? The easier thing in the plane surface was that even though the magnitude of the force was varying across the surface, the direction was always normal to the surface.

And we chose our coordinate system such that the force acts along one direction only. Now that simplified the problem. And in place of working with vectors, we just worked with the magnitude of the force. For a curved surface, this is not so, because if you look at, let us say, a typical curved surface. So, let us say, this is our curved surface. And there is liquid above it. This is the liquid level above which the pressure is atmospheric.

Now, on this curved surface, if we take three different points, we will see that the force on three different points has different directions. So, the direction of the force is not same in all the cases. So, we will not be able to work with just magnitude. We will need to consider the vector form of the question. So, because the direction of pressure force varies at different points, so we will consider the vector form of the equation. So, we will again take an elemental area dA and the force on this will be dF = - p dA.

So, just to remind ourselves that this - sign is coming, because the direction of area vector dA, which is outward normal. So, on, if we take a point, this is the direction of area vector. So, on the outer surface, this will be the direction of area vector, whereas on the surface at which we are considering the force, the area vector will be pointing out in this direction. So, let us not confuse ourselves. And we will remove this. So, the area vector acts normal to the surface, but outward for the surface, whereas, the force will be acting on the surface, towards the surface as a compressive force. So, that is the direction of the force. They are acting in the opposite direction.

To take that into account, we have a - sign here. Why did we not take for a planar surface? Because we knew what is the force and we did not consider the vector form of the equation. So, if we take dF = - p dA, and, then the resultant force will be, integrate this over the entire area, of course, considering that dA is area vector. So, it will be easier if we break this into components, so we can find the x component of the force, y component of the force and z component of the force.

And let us say that the resultant force in the x direction, where our coordinate systems are x acting horizontally in this plane, and y is vertical and let us say gravity acts vertically downward. So, which is negative y direction. The z surface is or z coordinate is normal to the screen. So, the resultant force x component F_Rx will be equal to the dot product of $\hat{\imath}$, which is unit vector in x direction. This we can see simply if we write $F_R = F_R\ x_i, + F_R\ y_j + F_R\ z_k$ and take its dot product with $\hat{\imath}$, why $\hat{\imath}$ because we are looking for the component in the x direction.

So, we can see that $\hat{\imath}.\hat{\imath}$ will be 1. So, that will be F_Rx, whereas $\hat{\jmath}\ .\ \hat{\imath}$ will be 0 because $\hat{\jmath}.\hat{\imath}$ will be 1 × 1 × cos 90, cos 90 is 0. Similarly k.$\hat{\imath}$ will also be zero. So, F_Rx is $F_R.\hat{\imath}$. So, with this, we can find that F_Rx is equal to, we substitute F_R - p dA.$\hat{\imath}$, where dA is the area vector and dot product of dA. $\hat{\imath}$. So, from the same argument, dA.$\hat{\imath}$ will be dAx, which is the x component of area. So, let us see what that means.

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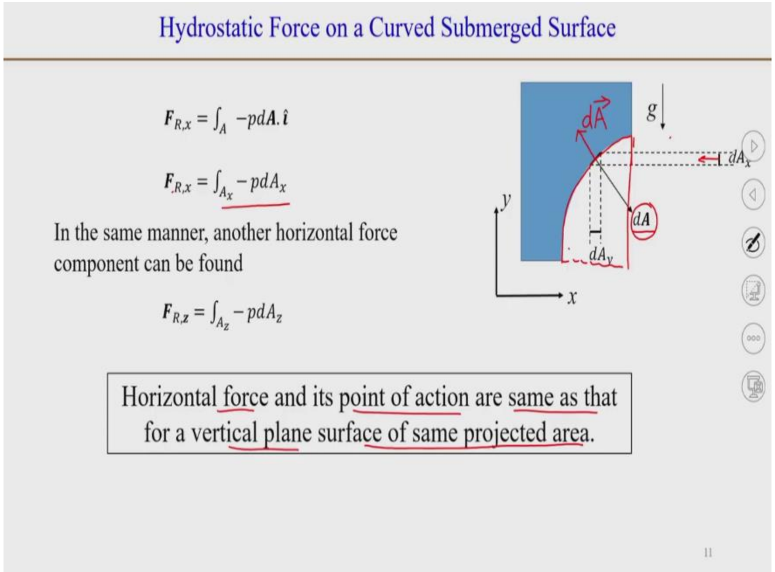

So, if our area vector on this surface, which is on this surface, the area vector is dA, but we are finding force on this. So, the area vector will be pointing out in this direction. So, even though I have drawn it wrongly, we need to consider the area vector on the surface, which is in the liquid. This will be the direction of the area vector on the surface, which is exposed to the atmosphere, or which is exposed to the surroundings.

But that does not change the components of the force, so the components of the area vector. So, dA, the component of area dA.$\hat{\imath}$ and dA. $\hat{\jmath}$. So, we have - p dA.$\hat{\imath}$ = - p dA x. So, this will be the area vector. The direction will be in this direction and dAx. So, the normal to this plane is x, and this will be the area. So, the projection of this. If we take here, let us say this is the area vector dAx. So, F_Rx will be integral over Ax - pdAx. Now, if you integrate it over the entire surface, what you are going to get is this complete surface.

So, the projection of the curved surface on yz plane will give you dAx. So, we will get F_Rx = - pdAx. So, that means that F_Rx = force on this curved surface, the x component that will be equal to the force on a plane surface, which has the area same as the projected area of the curved surface on yz surface or on x plane. Similarly, we have another horizontal surface. So, on the surface, you will have F_Rx and F_Rz. So, similarly, one can find F_Rz = - pdAz. So, we can obtain the two horizontal components of force.

So, that says that the horizontal force and its point of action are same as that for a vertical plane surface of same projected area. So, the problem has been simplified that if you have a planar surface, and if you want to find its component, find the projected area along that surface on a vertical surface, and then you will be able to find the horizontal force. You can have, if it is just in one direction, you can find F_Rx. And if there are two directions, you can have F_Rx and F_Rz so you will have two horizontal components of the force. Now let us focus on how to find the vertical.

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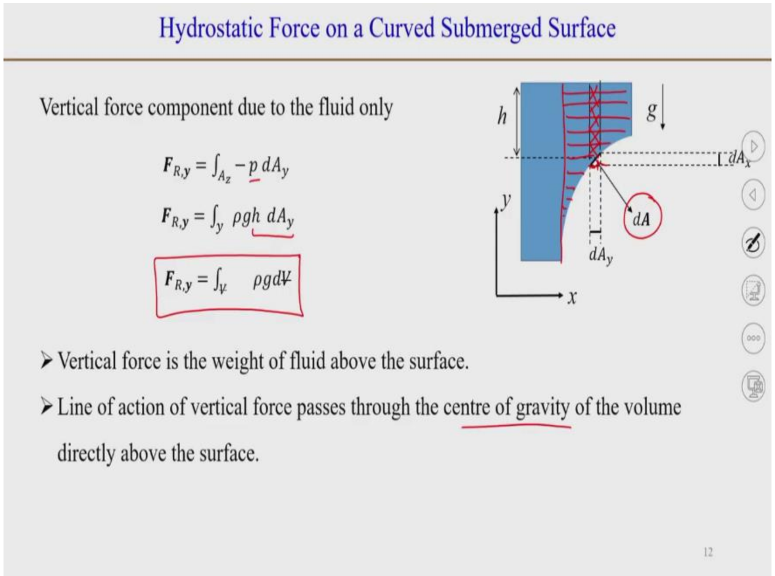

So, in this again, pardon me, this, the direction vector will be opposite of this, the area vector. So, the vertical force component due to the fluid only that we will have, so we are not considering P^o here. Now that will be equal to integral - pdAy. So, fair enough as we had for the F_Rx, F_Rz. And now we need to see what is F_Ry or what is p here? p is ρ gh. So, if we take this elemental area and the height is h, so p = ρ gh × dAy. And this area is dAy and multiplied by h. So, h × dAy is the volume of this liquid column. So, we can say that this is dV.

So, F_Ry = integral ρ gdV. So, this basically simplifies to the vertical force on this surface or the vertical component of the force on this curved surface = weight of the fluid. And when we integrated, we basically end-up finding out the weight of this liquid column that will give us the vertical force.

So, it becomes a simpler problem. We just need to find using geometry principles, what is the weight of this column that is above this surface. And the line of action, as we will see, or as we can see, by doing a bit of analysis that this will pass through the center of gravity of the volume directly above the surface.

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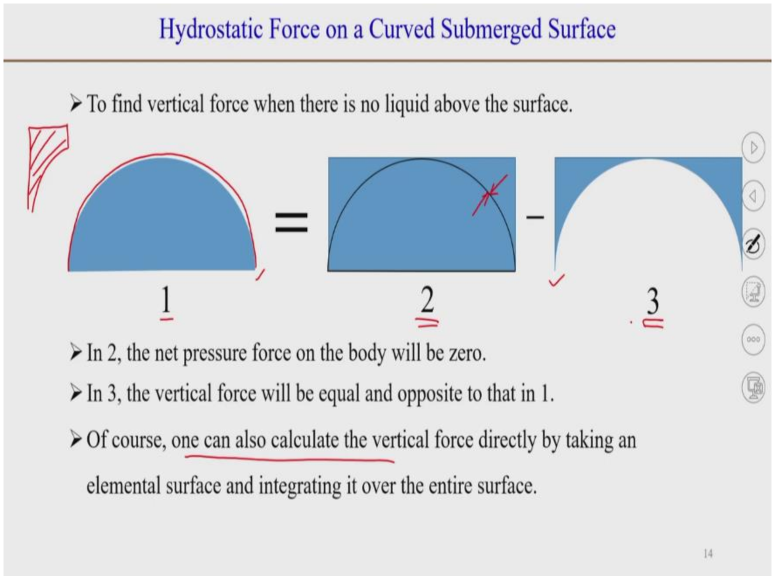

So, what we have done, that for a curved surface, we can divide it into vertical and horizontal components and the vertical force and the horizontal force that will be equal to, so the vertical force will be equal to the weight of the liquid column that is above this surface, whereas the horizontal force will be equal to the force horizontal force on a vertical plane, which has the same area as the projected area of this surface.

So, if we project the area of the surface along this, and we can find the horizontal force. So, we have been, even though when we started with the analysis, it looked complicated, but it has become, we just, we are able to break it down to two simpler problems, that finding out the force on a vertical plane surface and finding out the weight of the liquid column. So, using this, we can find the horizontal component of the force and vertical component of the force.

Now, we plotted in the previous figure, where there was liquid above the surface, and we were talking about the weight. So, we had figure something like this, where the liquid is just above the surface. We may also encounter situations, where there is no liquid above the surface. We can of course find the elemental area and find the vertical component of the force integrated over the area, and find out the net vertical force.

But using the principle we just discussed or using the result that we just found, how we can find the vertical force on such a surface, where there is no liquid above it. So, we will consider this. If we want to find the force on the surface as in condition one. Now, we can consider this as condition two and subtract it from condition three. Why, because in condition two, there is no net force on the surface. The force on the two sides on the surface will cancel each other. So, net force will be zero.

So, that means the force in condition one and condition three, they will be equal and opposite. So we can consider the force on condition three, obtain the weight of the liquid column on this surface and the equal and opposite of this will be the force in condition one. So, that will be easier to find. So, in summary, what we have been able to do is consider a plane surface and find the net force on this plane surface, the line of action or the point of action, the coordinates x_L and y_L of this plane surface.

We are also able to find the force on a curved submerged surface, and have been able to break it down the problem of force on a curved submerged surface in the problem of force on a vertical plane submerged surface to find the horizontal force and calculating weight of the liquid above the surface to find the vertical component of the force. So, we can analyze a number of problems, where we need to find a force on the surfaces. Thank you.