Course Name: INTELLIGENT FEEDBACK AND CONTROL

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Week - 03

Lecture - 19

Hello, this video is about understanding irreducible transfer function. In the previous video, we were looking into large scale systems and specifically for these large scale systems, we are looking into giving a minimal realization so that we are able to minimize the state-space for the control actions to be taken. We also associated this concept of minimal realization with joint controllability and observability. If the system is jointly controllable as well as observable, it is certainly a minimal realization. Today, we will be looking into what is irreducible transfer function specifically in the MIMO system case.

At the same time, for the minimal realization, we have been talking about converting into some other state-space realization with the help of similarity transformations. So when we consider these theoretical concepts and we want to apply these realizations to a controllable form or controller form or observable form or an observer form for a specific case and so on, How we are going to apply is something very straightforward. If I have given I have been given a system with a particular system representation of nth order. All right.

Now, if this is a minimal realization, that is okay. But even if it is not a minimal realization, I want to convert this particular realization or this particular representation in terms of a minimal realization. Then what we can have is, since there is an input and output associated here, we have this X dot equals AX plus BU form. And Y is equal to CX form. So in this case, my system is represented with state X, input vector U and output vector Y.

But when I am applying a similarity transformation and converting the system in terms of X bar, B bar, X bar and for which the system representation is given by A bar, B bar and C bar. And let us say this A bar, B bar, C bar has been obtained by applying similarity transformation TA on (A,B,C) system which was my original system. So, then what we are looking into giving here is the similarity transformation being applied at the input and then at the output we are doing some kind of a inverse similarity transformation to obtain a minimal realization of a particular form right. So, now with this transformation and the original system and the inverse transformation to certain extent we can more or less represent the input and output of this of this transformed system in a controller form or controllable form and so on and so forth. So, that is how we apply these transformation methodology for getting some particular form of a realization in practice.

Now, something similar we will try to look into getting the transfer function which is irreducible. Once this is irreducible, we know that this is what the minimum requirement of system representation is. With this minimum requirement, now I should be looking into designing the controllers and whatnot. Otherwise, we'll land up into working with very large system, trying to control this control and it's not working. I mean, we will not be able to work on all kinds of state variables to be controlled and at the same time transfer function to be controlled.

So in this case, let's try understanding what my MIMO system transfer function representation is. So, my transfer function representation is H of S given by the by numerator polynomial and the denominator polynomial. Numerator polynomial of course, we said is going to have coefficients which are in the matrix form and D of S is a polynomial in S, which has scalar coefficients. So, now, let us say this d of s is having degree equal to r. There are p number of outputs and m number of inputs and say we have the order of realization is s n.

So, we will have something like this. Now, for SISO systems n of s is scalar and n will can be larger than r. But for a block controller or a block controllability form, the number of states N is equal to R times N. Similarly, for block observer or observability form, the number of states N will be equal to R times P. So, what is my minimal order realization?

And we have also said this has some relationship or exact relationship with joint controllability and observability. Let's look into talking about these matrix transfer functions. These now, if I try associating with the SISO case, to certain extent, we are talking about pole zero cancellations. Now, pole zero cancellations in the matrix form, how it happens is what need to be understood. Now, these pole zero cancellations are the cancellations that we are talking in terms of the numerator and denominator in order to get the irreducible transfer function, which is going to have the minimum order representation or minimum degree in the characteristic equation is what we are looking for.

So, there is going to be some kind of matrix coming into picture and let us try understanding it from the concept of right matrix fraction now. So, this n of s by d of s now I will be able to write in the form of nr of s and dr inverse of s. What is my dr inverse? This d of s I am more or less writing in terms of dr of s. So since there are matrix multiplications, I will have to represent this 1 over D of S as a matrix inverse.

So, this is a matrix inverse. So, I will be considering this as D of S times I m, which is identity matrix. I m is identity matrix of m cross m. Nr of s is same as n of s. So, one straightforward is nothing but what we have this right matrix representation as n of s d of s I n inverse.

All right. There could be other right matrix fractions that we can take it out and can represent. So this particular form is not a unique form of representing the transfer function in the right matrix fraction type form. Right matrix forms in terms of inverse denominator matrix. All right.

So, in this case, what we have is degree of D of S is equal to the degree of determinant of D R of S. All right. And which is equal to R times M. If that is the case, the block controller form can be associated. Something similar, if I consider this as the left description, left matrix fraction description. So, now I am considering representing the characteristic denominator on the left-hand side multiplication part here.

So, this particular DL of S is now going to be multiplied with number of inputs. You recall we had P as the number of outputs and M is the number of inputs. So, this considers number of inputs, m is my number of inputs and whereas when I consider left matrix fraction we consider number of outputs to be equal to p cross p P is my number of outputs and IP becomes my P cross P identity matrix. And now I will be able to associate the block observer form.

Whenever there are output parts to be considered, remember that we are associating observer part or our observability form. Whereas when it is inputs to be considered, that time we have controller or a controllability form associated. All right. But at the same time, the right matrix fraction form is associated with inputs as well as now we'll consider left matrix fraction forms associated with observers or observability forms. Let's take one example.

Now, when I'm considering one such way of representing matrix, right matrix fraction for the previous case, In this illustration, we will see there are many ways by which we can write the right matrix fraction. Let us consider getting this right matrix form as simple way since I am talking about matrix. We considered the S plus 1 square S plus 2 square is the least minimum characteristic equation from all the elements of the matrix H of S. So, this is what gives you D of S.

So, this is exactly what we considered DS IM form. All right. But there is another way which where we can reduce the degree of the determinant of dr of s by considering this particular complete least minimum form polynomial. But I can always consider that this particular row what is least minimum this particular row what is least minimum and that is forming my right matrix description. The other way is like, okay, I can further reduce by considering non-zero of diagonal elements in dr of s.

By using this, my determinant is reducing, right? The degree of the determinant is reducing. In this case, my degree of determinant was 4 plus 4, 8. In this case, my degree of determinant is 6, whereas in this case, my degree of determinant is 5, all right? Further, now I will be able to write, for example, in this particular case, I will be able to write inverse form.

In the fourth case, I can again consider that in the diagonal form, there are least number of degrees available. And anyway, I'm doing the inverse, so I will be able to figure out this. Now, since this is going to give me a different degree of determinants of this, it means I do not know how far I can represent this and what should be my correct representation of dr of s in order to call it as an irreducible transfer function that I cannot further reduce it. All right. Let's consider cases where claims for getting the minimal order for the reducible transfer function.

Now for example, I have a right MFD or matrix fraction descriptor for H of S given by N R of S, D R inverse of S, which is, we know that it is not unique, but let us consider one such case where we have a right matrix descriptor. Now, we can obtain a controllable state space realization (A,B,C) of order n, where n is given by degree of determinant of Dr of S, which is nothing but defined as the degree of the MFD. Now, controllable state-space realization we have seen we will be able to get it by considering D of S IM as the DR of S. Now, okay, I can do that and I can get the controllable state-space realization. Something similar I can do it by getting the left descriptor, left matrix descriptor as a representation and in that case now degree of determinant of DL of S is equal to the degree of the MFD and I will get the observable state space realization as we have said this.

When this minimal determinant degree of the denominator we get, whether it is left or right descriptor matrix fraction of H of S, then we get the minimal order of the any state-space representation. So, this minimal part is important. We start by saying that my degree of determinant of the, whether we consider left MFD or the right MFD, If it is a minimal order, this will result in the minimal order of any state space realization of H of s. Fair enough? But now what is my minimal order is what to be understood. Now let us consider a right divisor.

We will focus on the right divisor, understanding this particular in terms of the right divisors. This can be copied for the left divisors as well. Let's consider that this D of S is a non-singular matrix fraction descriptor. Let's consider we have a non-singular D of S, W of S, which is given here. And I'm writing this N bar of S as N of S W inverse because W of S we have considered as non-singular.

Something similar, I can write W bar of S as D of S W inverse. Now, what is going to be H of S? H of S was given by N of S by D of S. So, now, since I have W inverse common, I can write this as N bar of S and D bar of S as well. But since I have considered, so this you relate to pole zero cancellation.

I am not doing pole zero cancellations here. I am doing the cancellations of considering the entire matrix because now I have W inverse associated. When I take the inverse of W inverse, I get W of S and W inverse of S times W of S is nothing but my identity matrix. And that is why I am getting N bar of S by D bar. Always remember, we are doing matrix manipulations and just not cancellations of terms over here.

Now, if W of S will be called as the right divisor of N of S and D of S, if degree of D of S is greater than or equal to degree of D bar of S. What does this mean? It means that this W of S that I am considering is nothing but some polynomial in S. So, it is a polynomial in S. Now, this particular polynomial has a degree, of course, this is in terms of matrix polynomial.

But the degree of this particular D of S is equal to d bar of s means the degree of W of s was equal to 1. But if it is greater it means the degree of W of S was more which means I was able to reduce the transfer function and I was able to represent in terms of N bar of S by D bar of S when degree of D of S is strictly greater than degree of D bar of S, which means W of S has degree more than 1, all right. So, this way if we have consider the degree of MFD for reducing especially the right divisors for reducing the degree of D of S, which is helping in reducing the transfer function in this with the help of the right divisors. So, here we consider removing right divisors of common right divisors of N of S and D of S. But how do I obtain the minimum degree in this case?

This is found by greatest common right divisor. As long as this W of S is not greatest common, I can still keep on reducing by removing the right divisor, common right divisor. But if this W of S or W inverse of S in this case is the greatest common right divisor, then the option that I get in terms of N bar of S by D bar of S is going to be the irreducible transfer function. All right. Now, this is what exactly it is explaining is when the GCRD is extracted, then degree of D of S will be equal to degree of D bar of S.

And for all non-singular right divisors of W of S of N of S and D of S, this is going to be true. If this is what the minimal order has been formed, then W of S that you are getting is called the unimodular matrix. Now, what is this unimodular? We have determinant of W of S is a non-zero constant as well as degree of determinant of W of S is equal to 0. So, that is the reason this is nothing but simply a constant matrix.

But constant matrix as in all the elements will not have any polynomial in S or it is a polynomial in S with S to the power 0 term only. All right. Earlier I said degree of determinant of W of S is equal to 1, but please make a correction here. Degree of determinant of W of S is equal to 0 because now I have only S to the power 0 terms here, constants over here. All right.

So if my GCRD itself is a unimodular matrix, then I have got the irreducible transfer function. Now, question is, how do I get the GCRD? Now, how do I know that GCRD, whatever I'm finding out is the greatest common right divisor, is the greatest common, first of all, right? And then, of course, if I get the GCRD, which is unimodular, I'm done. I have arrived at the irreducible transfer function form.

All right. Let us see what these GCRDs are. For N of S and D of S with same number of columns, we have only have unimodular common right divisor, which means it is relatively right co-prime. All right. Right co-prime means now it is irreducible, H of s is irreducible for us.

This is what we have established so far. Now, is the irreducible MFD unique? No, because I can always be adding, I can always have W of S which is unimodular and keep getting N of S by D of S in the form of N bar of S by D bar of S by removing always the unimodular matrices W of S, right? Remember, unimodular has degree of determinant of W of s is equal to 0. All right.

So, this is what we are arriving that GCRDs are not unique because I can get unimodular or non-unimodular made W of s in this case. Even if it is an irreducible transfer function, I can remove a unimodular right matrix in order to get another form of the transfer function, which is fairly enough with our concept that the transfer functions are not unique anyway. So, the irreducible transfer function is also not unique. So, in conclusion, we have minimal order of any state-space realization, which is equal to the degree of any irreducible right or left MFD of the transfer function. All right.

Now, the big question that is to be solved is how to get a GCRD. And there's a very simple method to get this GCRD. Let's consider this method of finding the GCRD. There are many, but this is what I found as the easiest to explain and at the same time easiest to get, especially in terms of the transfer function case. Let's consider the GCRD of N of S and D of S as R of S.

For our case, N of S and D of S will have same number of components because we are representing in the form of transfer function. Now, properties of GCRD is what? Since it is right divisor of N of S and D of S, we can write N of S is equal to N bar of S times R of S and D of S is equal to D bar of S times R of S because it is a right divisor. If R1 is another right divisor of N of S, then we can write R of S is equal to W of S times R1 of S and W of S is a polynomial matrix. All right.

Fine. Now, how do I get this particular GCRD with the concept that we just said in terms of R of S and R1 of S? Let's consider N of S and D of S. Of course, N of S is P cross M. P is number of outputs and M is number of inputs.

We will form D of S and N of S in a row vector form in this way, D of S and N of S. If we get this and we perform elementary operations on it such that we get R of S and 0 here. Elementary operations, just to make sure that you are making the row operations most of the times in order to make sure that these number of rows are coming out to be 0, all right. Now, what happens? Though it is a very simple method, it turns out that R of S is a GCRD.

Simple method, right? We packed D of S and N of S in a row matrix form. We performed row operations and recorded what row operations we are forming in terms of U11, U21, U22 form here. As soon as we have done this such that these number of rows are zeros, we get R of S and this R of S is GCRD. Now, I will have to convince you why is this GCRD, right?

Now, let us consider this case. I am not showing the complete proof of it, but giving you the flow of the proof in this case. Now, we have this U11, U12, U21, U22, D of S, N of S. We did elementary operations to get this particular form. All right.

We got the GCRD as R of S with this particular method. Perfectly fine, but we want to convince ourselves that this is a GCRD and that is the reason I am considering U11, U12, U21, U22 kind of elements in explaining the simple elementary operations that we did. Now, in this case I have to show that R of S is a right divisor. How do I show? I can say that okay U11 D of S plus U12 N of S is equal to R of S and U21 D of S plus U22 N of S is equal to 0.

When we just substitute on each other we will be able to find that R of S is a right divisor. All right. Now, the second flow we have is we have to show that R of S is equal to W of S times R1 of S and R1 of S is another right divisor. Fair enough. I can always do a new kind of a simple operations in order to get R1 of S and I can say that R of S is W of S times R1 of S.

At the same time, the third hint, third point in the flow when we prove it is that d of s n of s which is what our initial point of matrix formulation has full column rank. Then all non-singular right divisor can differ only by unimodular MFD. So, if this has a full rank then R of S that we are getting is nothing but our unimodular matrix that we will get. Now, if that is unimodular, then what we get as R of S is nothing but the W of S in this case with our terminology over here. So, since we are getting all the right divisors and this right divisors are nothing but turning out to be unimodular if this has a column rank this is what is my GCRD.

So, to certain extent we establish the fact that okay in order to get the irreducible transfer function we will use this simple method to get the GCRD. Once we remove the GCRD, this becomes my irreducible transfer function because now my D of S and N of S form which is after removing the right divisor over here, GCRD over here. When we form this D of S and N of S has a full rank and now since it has a full rank, any further reduction that I would like to do will not be possible. And only what I get as GCRD is a unimodular MFD. Wonderful concept.

So as a summary, in order to get the irreducible transfer function, what we are doing is we are getting the GCRD. Once GCRD is removed, we get the irreducible transfer function. And the degree of the denominator now becomes the minimal order for the realization. So our main result now, this is actually applicable for all MIMO systems. We can apply mainly for the large scale systems to reduce the order by considering these concepts of knowing that the minimal realization follows from joint controllability and observability.

It follows from the irreducible transfer function and vice versa. So any of these concepts can be used to get the minimal order of the representation which then can be solved since that is a minimal transfer function form this is definitely controllable as well as observable, we don't need to check it further and we can straight away design the controller for the minimal form of the realization or the transfer function that is irreducible The concepts over here have been taken from this book of linear systems by T. Kailath. One can refer for the proofs and further understanding of these concepts. The application of these concepts are going to be enormous because to certain extent we are saying that we'll use the realization ways, ways of realization in order to transform the system and apply for getting a proper controllable farm or a controller farm here.

That's it, thank you so much