Course Name: INTELLIGENT FEEDBACK AND CONTROL

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Hi. In this video, we'll look into large-scale MIMO systems. What do we mean by large-scale? Large-scale means my number of states are more than 100, at times thousands, ten thousands, millions, and so on and so forth. Some of the good examples I can give here is nuclear power plant or electrical grids where you have the number of users which are in millions and so on.

The grid transmission, the electrical transmission system where the nodes are so many in terms of 10,000 and so on and so forth. So how do I control such a large scale system Our approach here has been to simplify the system dynamics in order to make use of the PID in all the cases. In this case, in this way, what we have previously seen is simplifying the techniques for MIMO systems with the help of variable pairing, decoupling, or if it's not possible to design multivariable SISO system, then we go for the centralized control system. But at the same time, when we are talking about large-scale system,

Now, this large-scale system, one can consider controlling with the help of multiple subsystems. Now, each of these subsystems may be represented as a multivariable SISO system or a small MIMO system itself. When we are talking about these 10,000 states or whatnot, first of all, we have to consider that all the states are important for us or not. Or can I consider lesser number of states and still represent my system uniquely? Of course, the system cannot be represented uniquely, but completely, sorry.

Now, when I'm designing such a system, and I have represented with say 10,000 state variables, how do I know that I need all the 10,000 variables to be controlled or observed? So in that case, the idea is to consider finding the minimal system. Now what is minimal system? That minimal system is the one that says that okay, these are the states which definitely needed for representing a system the rest others are purely some combinations, or one can always derive those states from the from the minimal states and so and so forth. Minimal system, what is the dimensions of this minimal system is what we will have to find out when we are dealing with large scale systems.

First effort is to get the minimal system and then this minimal system can then be divided into subsystems in order to consider these simplified techniques for the control of the MIMO systems. So in this video, we will cover various realizations and to understand what would be the minimal system for presenting a particular system. For the state-space realization, we'll take help of state-space realization in this case because we are looking into various different realizations. We can, as soon as we think about state-space realization, we can always switch to transfer function and so on and so forth. Since we are talking of number of states, number of variables to represent a particular system, we are resorting to the state-space analysis in this video and the next video.

So our transfer function in the SISO case is the ratio of numerator and denominator of the polynomials in S. For the state-space realization, we can arrange no more than n states, which is nothing but the degree of the characteristic equation or the denominator of the transfer function g of s, which is given by a of s over here. Now, similar to the SISO case, there are different ways of realization. For example, now for the transfer function G of S, which is representing two input-two output system here, because this is my matrix form of the transfer function now. Now, if I consider first input and the first output, then it's a second-order system because my characteristic equation is given by this.

If I consider, first input and the second output, then we what we have is the characteristic equation having three degrees. So then how do I realize this such that this is more or less looking like a MIMO system instead of individual one input to one output and one input to the second output and so on and so forth. All right. So this realization, one of the ways we can consider here, as I was explaining, is to consider these individual blocks for transfer functions between first input and the first output and the first input and the second output. So, then we have this kind of a block transfer function block form when I am realizing into the state-space form.

My system matrix turns out to be representing individual blocks and of course, these first two rows for the B matrix is corresponding to the conversion of the first input first output transfer function G11 over here. Similarly, next two rows are corresponding to the G12 element of the matrix transfer function. The next three rows correspond to G21 transfer function and this corresponds to giving you the A matrix of this form. Rest of the elements of the A are all zeros here. All right.

So this is one of the realization. Now, if I look into this realization, we have nine number of states here. OK, now let's consider another way of forming the state space realization, which is describing the system matrix A, input matrix B and the output matrix C in terms of the block controller form. So if I have to consider block controller form, then individual transfer functions need to be clubbed in some way. So, one way to do here is considering the least common multiple of the denominators, which is given by D of S.

Now, we had, for example, in the previous case, we had four transfer functions. And from the four transfer functions, I consider least common multiple LCM of the denominators of all the entries here. So, that will give me D of S. Now, as soon as I have D of S, this N of S is nothing but the polynomial with individual coefficients as the matrix in the matrix form. And then the block controller form I can write is for the M input can be returned in the form of this similar to our SISO case, which is minus D1, minus D2, minus DR, and all the other entries are in terms of the diagonal ones.

Instead of ones, now I have this IM, which is M cross M identity matrix. Similarly, input matrix B is in the form of IM, which is M cross M identity matrix. And C is nothing but all the coefficients in the matrix form here. Let's take an example here for the block controller form, which will clarify further what is my least common multiple denominator and how do I form these N1, N2, and N3 coefficients. So, here I took the same example of G of S.

Now, this has least common multiple of the characteristic equation from each of the transfer functions over here. Entries into the matrix G of S is S minus 1 whole square S plus 3 whole square. You see S minus 1 whole square over here and S plus 3 whole square over here. And so that gives you the least common multiple denominator which gives you a polynomial in s which we say is that is d of s and now my n of s becomes n1 s cube and n2 s cube where each n1 n2 n3 and n4 are matrices form, so if I consider this as the D of S, what turns out to be N1, N2, N3, N4? When we modify this particular block over here, we will get to introduce these terms over here.

Because now I am dividing it by S plus 3 whole square. All right. OK. So now when I get this and I write it in the form of the block controller form, which is nothing but having. If I go to the previous slide now, I have the number of states given by R times M. All right.

Where R is the degree of the common denominator over here. Now in this case, in this particular example, the common denominator has the degree four and it is m is two, which is number of inputs. So here we get number of states equals eight. For the realization one, which was very straightforward, we got the number of states nine. Here for the block controller form, I'm getting the number of states as eight.

It means the previous realization was definitely not minimal and I am still not clear if the block controller form is also a minimal form, which is can I reduce the number of states further when I am realizing using a state-space form. Let us take the block observer form similar to the block controller form for the P output case. So for the P outputs, my system matrix now becomes D1 times IP and DR times IP. And my input matrix now has the coefficients of N1, N2, N3, NR, whereas the output matrix has the form of the identity matrix P. Now, in this case, number of states are going to be r cross p.

So, in this case, for my example here, again, my r was 4. Since this is two input, two output case that we are considering this particular example that we picked up from the start, we will have the number of states as 4. But we still do not know whether what is going to be the minimal realization or realization which has the minimum number of states for a MIMO system. Just to understand, minimal system representation is also not unique. But the number of states in the minimal realization is going to be minimum, is what we are looking for.

Okay. Now, let us consider another case. Another case when we have the denominator polynomial which has the distinct roots. If that is the case, then the common denominator that I took out, I can represent in the form of a product of multiple roots, distinct roots that we have. So then I will be able to write the transfer function, which was in the form of N of S by D of S, I will be able to write in the form of a residual form.

Now, this residual form is a summation of R I by S minus lambda I, and this is something similar to you can relate it to a SISO system, which helps in the division in terms of the residual factors and so on. But here RIs are going to be in the matrix form as compared to my SISO case. All right. And RIs can be found similar to the residual, finding the residual cases and so on. We multiplied by S minus lambda A with G of S and we say limit as S tends to lambda i.

Now we consider individual Ri matrices. We calculate its rank. Let this rank is rho I. So now Ri can be represented, can be formed. Since these are distinct roots, I can write it in Ri as Ci times Bi.

This is not going to be, the diagonal matrix is going to be identity. I can decompose in terms of CIN and BI such that my dimensions of CI is going to be P cross rho I and dimensions of BI will be rho I cross M. All right. So now in this case, I will be able to write the realization as a block diagonal form. Now, these block diagonal form can be represented in terms of lambda I, I multiplied by identity matrix with that particular Ith residual rank, rank of the residual matrix of Ith element. And B and C matrices can be found with the help of this BICI form.

All right. So in this case, if I look at the system matrix form, what I have is the number of states equals I equals 1 to R rho I. So summation of all the ranks that I got, rho 1 plus rho 2 plus rho 3 up to rho R. All right. Fine. So now, given this particular transfer function, if I want to find out the diagonal states, because here again I have the distinct roots.

In my previous case, I had the common denominator as s minus 1 whole square plus s plus 3 whole square. So that was not having the distinct roots. So now I'm taking another example where I have the distinct roots as s plus 1, s plus 2. So, R is equal to 2 only, but now I have to find out the rank of the system. So, with that, I should be able to find the realization, a new kind of realization.

So, up to this, what we have seen, three kinds of realizations or rather four kinds of realization. One was straightforward. The second was block controller form. The third was a block observable form. And the fourth is the diagonal form.

It turns out that the diagonal form is the minimal form. Anyway, we are finding out ranks and so on, but the diagonal form always is not possible to get because I may or may not have the distinct roots. At the same time, a wonderful property that we get from LTI SISO system is minimal realization implies irreducible transfer function and vice versa. Similarly, irreducible transfer function implies joint controllable and observable system and vice versa. So what does this mean?

Let me spend a little more time here. I'm not proving it for SISO system, but for the SISO system will convey that, OK, similar properties are possible with MIMO systems or not. Irreducible transfer function means I have, for example, the transfer function given by B of S by A of S, and we know that there is a possibility that there are poles and zeros which are not cancelled. When I am representing the transfer function with all the poles and zeros cancelled, that is something is my irreducible transfer function. And that transfer function, when I represent in the form of a system space form, it gives me a minimal realization, which means I get the state-space form, which has the minimum number of states.

Similarly, if I start with a minimal realization of a state space, for example, it is nth, n is the for a given system, n is the minimum number of states possible. Then when I convert this particular state space form in the transfer function form, I am going to get the characteristic equation with the degree n. And that transfer function is definitely not going to have any pole and zero sitting at the same place. Right. Because finally, I started with minimal realization.

I will land up into getting the irreducible transfer function. Now, when I take the connection with the last property, which is joint controllable and observable. Wonderful part here is that as soon as I consider G of S, which is irreducible, it is also when I take its state-space representation, which is definitely a minimal realization, and when I consider a minimal realization, all the states are controllable as well as observable. Means controllability rank is full as well as observability matrix rank is full. A wonderful property.

Similarly, if I start from a state-space realization, which is jointly controllable and observable, then it is certainly a minimal realization. We already know that we can do similarity transformations such as to come to a realization which is controllable. I can do similarity realization to get an observable system If the system is controllable as well as observable, it is certainly a minimal realization, which means I can figure out what is the irreducible transfer function it has. We're talking about large scale systems.

Large scale systems where my system state is of the order of 100, 10,000 and so on. If my characteristic equation has the degree of 1,000, I'm looking for the pole zero cancellations and whatnot. The best way is to follow the the properties over here in order to get an irreducible transfer function when we get the minimal realization we are very certain that it is an irreducible transfer function and now I will be able to design its controller or the observers and so on so forth because the irreducible transfer function is certainly jointly controllable as well as observable. Now these are the properties of the SISO system

Can we expect similar properties in MIMO system? So in this video, I will give pointers to say that yes, it follows the similar properties and there's a whole lot of mathematics behind it, but I will give you only the gist of, or the flow of the proof that says that, okay, this particular minimal realization implying irreducible transfer function and vice versa. Irreducible transfer function implies joint controllable and observable and vice versa is applicable on MIMO systems as well. Let's see now. Let's start with a simplistic, as we move from SISO to MIMO,

The very next step, I will complicate the system by saying, okay, it's single input and multiple output system. Let's consider, so since it is single input, it is going to have m equal to 1. So, it has single column and p rows over here. So, the transfer functions will be g11, g21, and gp1. All right, fine.

Now, the least common multiple of the denominator is say d of s, so now, individual n i of s is going to be b i of s by a i of s times d of s, all right. So, then I will be able to write this g of s as n 1, n 2, n m divided by d of s, all right. b i of s by a i of S is strictly proper, then does it mean that NI of S times E of S is also strictly proper? Yes. What is strictly proper that we are saying?

That the degree of BI of S is less than degree of AI of S. This is what we have been always following when we do the transfer function analysis. So my transfer function is strictly proper. Otherwise if it's not in strictly proper, then it's there's going to be a normal input-output relationship also. In all the control system forms, we are focusing on analysis which are related to the dynamical properties of the system.

And that's the reason we always consider strictly proper case for the transfer function. If I consider B I of S by A I of S is strictly proper, then I when I multiply it by D of D of S, it is also going to be. So N I of S by D of D of S is also going to be strictly proper because N I is derived by this. Finally, I'm dividing it by D of S again. So it lands up into D.

when I calculate NIE of S by D of S, NIE is again PI by AI, and that's why it is strictly proper. All right. Since it is strictly proper, I will be able to write the controller form realization. And as soon as I write the controller form realization, I get the state space controller form as minus D1, minus D2, minus D3. Remember, it is single input case.

And that is why the identity matrix is nothing but just the one identity. All right. So that's why my system matrix turns out to be minus d1, minus d2, minus 3 and the controller form, whereas my output matrix turns out to be n1, n2. It's coming from my ni transfer functions, and its realization looks like something very straightforward in the form of Y3. For example, I consider Y1.

It considers outputs N11, N12, and N13 being added now, whereas the inputs are directly going as a single block over here. All right, now state space controller form, if I look at it, it is definitely controllable, but it is observable if and only if polynomials D of S, N1, N2, NP have no common factors. It's a wonderful property because now we are talking in terms of controllable as well as observable form. And this turns out that the system state-space realization, which we have said in the controller form, if it is also observable, then this needs to be true. All right.

Fine. If I consider controllability and observability matrices, I have the controllability matrix defined for the SISO system as B, A, B, A power of n minus 1 B. Take a recall when the system has N states, we have the dimension of the controllability matrix turns out to be B N cross N for a SISO system, right? For SISO system. But in case of a MIMO system, what will happen is this B matrix is also what P cross M, P cross N,

Right. So what happens is this is adding P rows. This is adding N times P rows. And then I will have something else coming out here. So its dimension is much more than N cross N.

Now, in case of MIMO system, it turns out that it can have n columns and n matrix even before reaching to a power of n minus 1. Fine. Now, what happens to the rank? The rank of the controllability matrix should be full, which is the rank of in case of SISO system, since this is n cross n itself, the rank should be always n. So, there should be n independent columns or n independent rows that we are talking about.

For a MIMO system, since this is N cross NP, right, still we are talking about N, the rank N for the controllability matrix. All right. Now, why do we need to count up to A bar of N minus 1? In case of SISO, we had to go till up to a power of n minus one in order to get n independent rows and n independent columns. Right.

But at the same time for the MIMO system, since this has multiple columns, then I can stop at calculating even before N minus one because I may get N independent columns even before calculations of a n minus 1 which is fairly okay but I why do I stop at a power of n minus 1 b because a power of n b with the help of the Cayley Hamilton's rule Cayley-Hamilton's theorem says that a power of n b is going to be nothing but linear combination of these anyways So I'm not going to get any independent rows or columns beyond this. And that's why I stop at a power of n minus 1 times b. But for a MIMO system, I can stop even before this as soon as I get the n independent columns here. Right.

So that's the difference between SISO and MIMO in this when we calculate the controllability matrix for the MIMO systems. All right. Now, at the same time, if I have the controllability and observability indices given, for example, I have a controllability matrix of A and B has a rank R, which is less than N. It's a wonderful theorem, even in SISO, which is applicable in MIMO as well. I'm not talking about fully controllable system as of now.

I start with a system which is non-controllable, which has the rank of the controllability matrix given by R, which is less than N. N is the number of states in the state-space realization that we are considering. So then we can find a similarity transformation T such that I have the transform system given by A bar T inverse A T, B bar T inverse B, which is a simple, straightforward similarity transformation way of getting this. And I will be able to find a similarity transformation which gives me A bar, B bar, C bar, interestingly in this particular form. Now, A bar is such that this AC bar is a controllable system.

So, it gives me a system which is AC bar, BC bar and CC bar. All right. Corresponding to these states and these states. So, this is there are R rows here and N minus R rows over here because the full rank was the rank of the controllability matrix was R. So, what this says is I am able to decompose the system states in terms of controllable states and non-controllable states.

R is the rank of the controllability matrix, then number of R states are going to be controllable and N minus R as non-controllable. So, I need to find a transformation T, which is giving me this formula, which is irrespective of whether a SISO system or a MIMO system is applicable. All right. This is more or less a Kalman filter decomposition method. Now, this way, the thing is now I have a controllable system.

I will be able to get a system realization which I am able to say in terms of block controllable form. Block controllable form is definitely controllable, but I know when is it observable as well. And once I get the observable form, controllable as well as observable system I will have joint controllability and observability said and this is going to result in the minimal realization because further decomposition is not possible all right so that's where That is one way of looking at it. The second way of looking at it is now I have the controllable system, AC bar, BC bar, CC bar, which I formed from the nth order system, state-space realization.

Now I have rth order system, all right, because AC bar has r number of states and so on. I further decompose it in terms of an observable system. So further, I will have a similarity transformation that gives me an observable system as an analogy here. I am not covering that part. But the controllable system, then I can have a similarity transformation that decomposes into observable and non-observable systems.

Now, controllable and observable system gives me the minimal realization. All right. So what we have here is a wonderful result in a way. You will be able to prove it as well. And that's why I'm not covering the proof of it.

But this similarity transformation that we talked about earlier, right? This similarity transformation that gives a decomposition in the form of controllable and non-controllable subsystems. This particular transformation is of this form actually. Remember our controllability matrix was not having the full rank, was having only the R rank and that is why we have to figure out a similarity transformation which is invertible as well. So, this turns out to be an invertible similarity transformation, and when I apply this particular similarity transformation, I get A2, B2 and C2 which is a minimal realization.

If I start from a minimal realization itself and I come up with this similarity transformation, this is also giving a minimal realization because which is going to be saying that I start from a controllable system, when I apply a similarity transformation of this form, I will definitely get a controllable system. All right. And that's that's how the things are. The arguments are related. Similarly, I will be able to show that the diagonal form that we said in the earlier case is controllable as well as observable.

Now, since that is controllable and observable, it is a minimal realization. A wonderful result. Now, I have at least one technique which says that, okay, I have similarity transformation given by this form. I get controllable system. And similarly, I'll be able to get an observable system.

Controllable as well as observable system, that gives me a minimal realization, which is having the minimum number of states. We started the discussion with large scale systems. Large scale systems of the order of 10,000, first of all, we have to check whether it's a minimal realization or not. This is a wonderful way of checking whether the system is in the form of minimal realization. That's what I ended here.

I'll cover the rest of the part on irreducible transfer function connecting with the minimal realization for further reduction of the system in the form of the transfer function in my next video. Thank you.