Thermal Operations in Food Process Engineering: Theory and Applications
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Lecture - 08
One Dimensional Conduction Heat Transfer in Cartesian Coordinate (Contd.)

So, we come back again to One Dimensional Conduction Heat Transfer. We were developing the generalized equation. And we have made generalized equation in the form of $\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n k\frac{\partial T}{\partial r}\right)+g=\rho C_p\frac{\partial T}{\partial t}$ that was equal to 1 by alpha in this equation. So, now we go to that lecture 8 and that one dimensional conduction heat transfer continuation.

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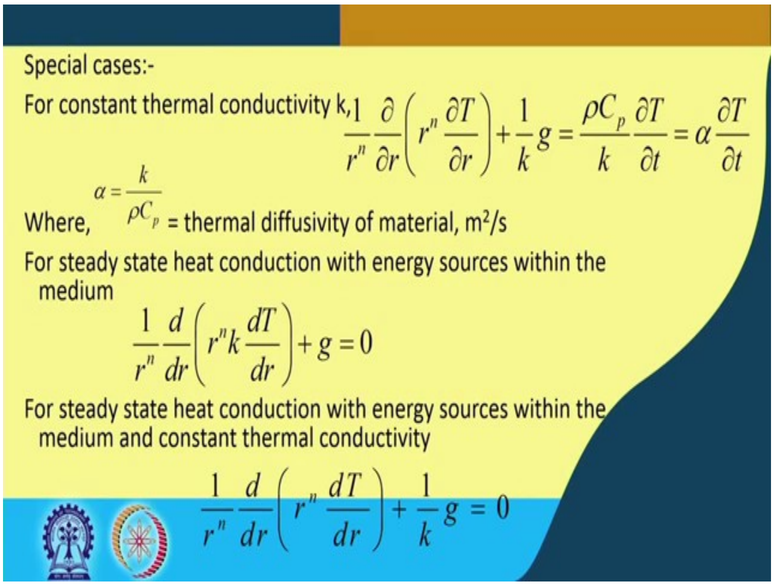

So, we were in this special cases 'right'. So, for constant thermal conductivity when we have taken k constant $\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial T}{\partial r}\right)+\frac{1}{k}g=\frac{\rho C_p}{k}\frac{\partial T}{\partial t}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$ see this is 1 by alpha not alpha, because k by rho Cp is alpha.

So, it is 1 by alpha del del T of t, 'right' where alpha is that thermal diffusivity in terms of $\alpha=\frac{k}{\rho C_p}$. So, thermal diffusivity of the material in the unit of meter square per second; so, from here we can see that had it been not constant thermal conductivity, then the equation generalized equation would have been $\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial T}{\partial r}\right)+\frac{1}{k}g=\frac{\rho C_p}{k}\frac{\partial T}{\partial t}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$ this would have been the generalized equation. And if we have taken conductivity to be constant that is k is constant then it has come to this form, 'right'.

For steady state conduction with energy sources within the medium, then we can write it is a steady state. So, del T del r this goes up since this del T del r the rather del T del t this goes off, because it is steady state; that means, it is independent of time 'right' is no longer time dependent this was unsteady till this point this was unsteady, 'right' because it is time dependent. But, now if it is a steady state conduction heat transfer within the material there is a energy generation then we can write that equation $\frac{1}{r^n}\frac{d}{dr}\left(r^n k\frac{dT}{dr}\right)+g=0$ So, this is for steady state without constant thermal conductivity.

Now, for steady state conduction heat transfer with the energy generation within the body, but thermal conductivity is constant if that be true then $\frac{1}{r^n}\frac{d}{dr}\left(r^n\frac{dT}{dr}\right)+\frac{1}{k}g=0$ this is a another special case.

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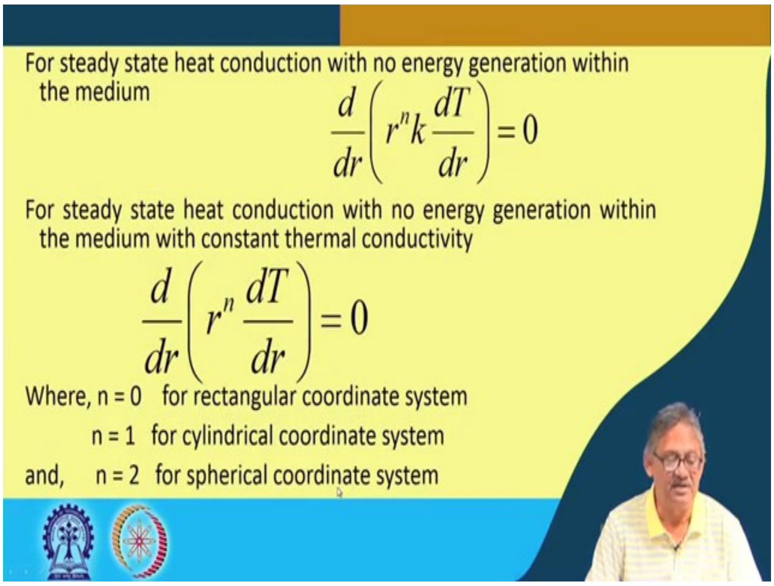

Third special case could be for steady state heat conduction with no energy generation that is g is 0 and steady state. So, 'right' side is also 0. So, $\frac{d}{dr}\left(r^n k\frac{dT}{dr}\right)=0$ and for steady state heat conduction with no energy generation within the medium, with constant thermal conductivity then you can write $\frac{d}{dr}\left(r^n\frac{dT}{dr}\right)=0$. In most of the cases you deal with this equation where if n is 0, then it becomes d/dx instead of r d/dx and this becomes $\frac{d}{dx}(\frac{dT}{dx})$ that is in turn we can write $\frac{d}{dr}(r\frac{dT}{dr})$ and that is where no energy generation constant his thermal conductivity and steady state and geometry is rectangular coordinate system, 'right', then it becomes like that, but if it is n is equal to 1.

Then it becomes d/dr of rdT/dr is equal to 0 again there is no heat generation and the steady state and constant conductivity and the geometry is rectangular coordinate system. If it is spherical coordinate system then your n becomes 2 then it is d d r of r square d T d r is equal to 0, where there is no internal energy generation steady state thermal conductivity is constant and the body is spherical in system spherical coordinate system, 'right'.

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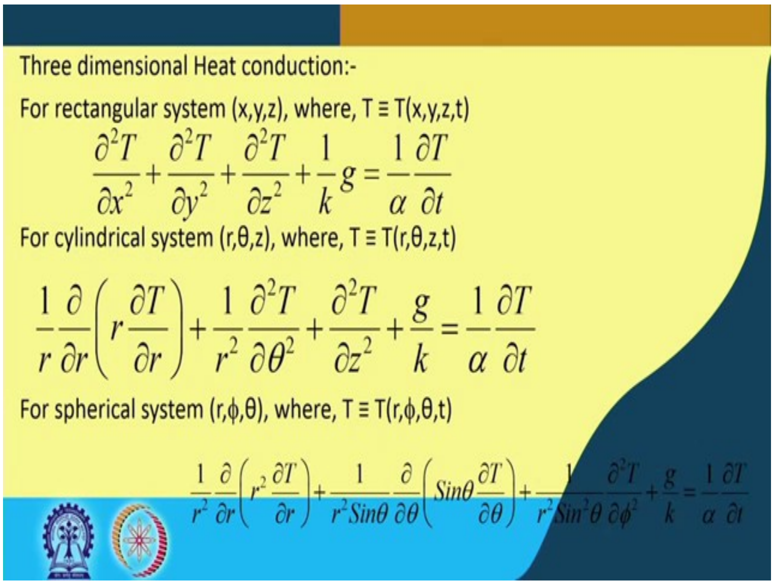

So, this if this be true then if we go into this three dimensional one, we are not developing it, but three dimensional equations we can write for rectangular system where the coordinate is x y z and T is a function of x y z, then $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{1}{k}g=\frac{1}{\alpha}\frac{\partial T}{\partial t}$ You remember in the previous slide we had written wrongly by typographical mistake it was alpha del T del t, but it is 1 by alpha del T del t, I said it to be corrected. So, you please note it.

For cylindrical system where r theta z, 'right' and T is a function of r theta z and also time T, then $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$ And for spherical system where the coordinate is r theta phi or r phi theta and T is a function of r phi theta and time t, then it is $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2 Sin\theta}\frac{\partial}{\partial \theta}\left(Sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2 Sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$

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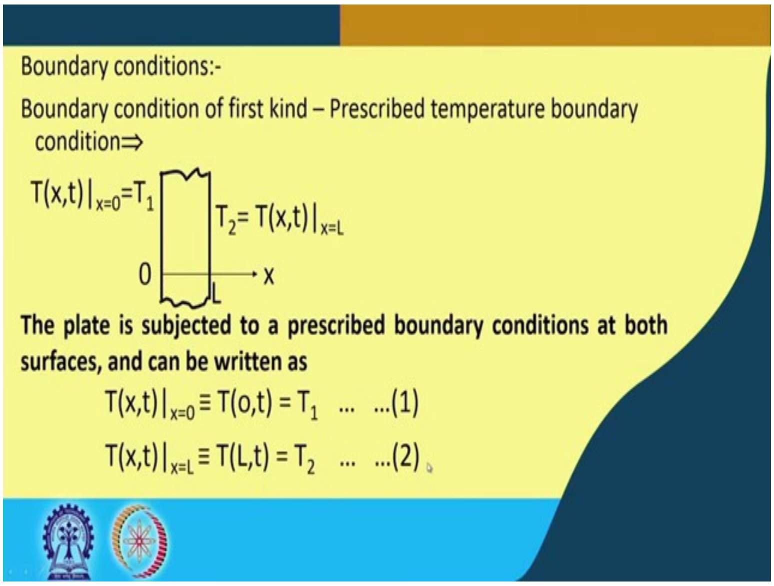

So, we have seen that one dimensional conduction heat transfer basic equation how we can develop it? And from there we have come to the three dimensional what is the 3 coordinate system equations, 'right'. Three dimensional equations also we have not derived, but we have shown how the equations look like, 'right'.

Then comes when you are solving it, 'right', it is a differential equation. So, when you are solving it you need to have boundary condition, if it is $\partial^2T/\partial x^2$; that means, you can or you can get T on double integration, 'right' on double integration you can get T. So, to get T you need to have boundary condition.

And this boundary conditions are to be also understood and they are to be, rightly specified, 'right'. So, boundary conditions there are three types, 'right'; one is called prescribed boundary condition or boundary condition of first kind, then prescribed boundary condition of second type and condition that is boundary condition of second kind and the third one is boundary condition or prescribed boundary condition of third type or boundary condition of the third type, 'right'.

So, if that be true then, let us take a body as it is shown here in the picture, 'right' we have drawn it that we have two this is two sides. So, our x direction is this. So, other two dimensions are much bigger than this dimension. So, that is why it is one dimensional, 'right'.

So, bound this is called boundary condition of the first kind or prescribed temperature boundary condition, ‘right’, prescribed temperature boundary condition; that means, boundary condition is defined in terms of temperature, ‘right’. If it is that if the both boundary means both the sides of the body, if the body is like this.

So, this side of the body is one temperature boundary condition and this side of the body is another boundary condition and the body is like this, ‘right’. If this be true which we have drawn here we can say that $T(x,t)|_{x=0}=T_1$, that is one boundary and the body is two sides, other two sides infinite or very large compared to the one through which heat is being conducted and that is the x direction, ‘right’. And this is 0 to L the two sides, one side x is 0 and the other side x is L, ‘right’.

So, one boundary can be $T(x,t)|_{x=0} \equiv T(o,t) = T_1$ and the other side is T_2 is equal to $T(x,t)|_{x=L} \equiv T(L,t) = T_2$. So, this is called prescribed boundary condition of first kind or the prescribed temperature boundary condition.

The plate is subjected to a prescribed boundary condition at both surfaces and can be written as $T(x,t)|_{x=0} \equiv T(o,t) = T_1$, ‘right’. And this is equation number 1 and $T(x,t)|_{x=L} \equiv T(L,t) = T_2$

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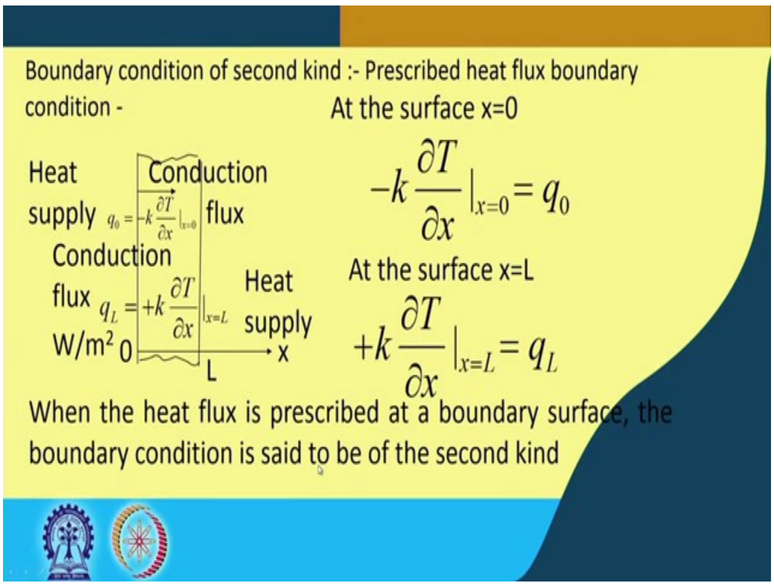

This is these are the two boundary equations we got in terms of temperature T_1 and T_2, ‘right’. So, second type is boundary condition of second kind or called proscribed heat flux boundary, 'right', prescribed heat flux boundary. So, heat flux is what watt per meter square, 'right', or any flux is that thing per unit time unit area per unit area is that thing or here it is watt per meter square. So, any flux is that thing it is energy so, Joules per meter square per second per unit time is the flux.

So, watt that is joules per second per meter square, 'right'. So, that is the flux. So, here we are providing prescribed boundary condition of the second kind or prescribed heat flux boundary condition, 'right'. If, it is heat flux it is q_0 that is watt per meter square or Joules per second per meter square, 'right'. Similarly, for the other energy terms maybe for fluid flow or for mass transfer so, you will see that everywhere flux is there and this flux is that thing per unit area per unit time is the flux ok.

So, here it is q_0 or watt per meter square that is the heat flux which you are using. And by the definition of q we have we are we can write that $-k\frac{\partial T}{\partial x}|_{x=0} = q_0$, this is the L and at the other surface at x equal to L, if it is minus flux and the other side has to be plus flux. It cannot be the both side is having this side is minus $-k\frac{\partial T}{\partial x}|_{x=0} = q_0$ and this day side also minus $-k\frac{\partial T}{\partial x}|_{x=0} = q_0$, then what will happen it will transfer from both the sides and the temperature will go on increasing and it is unsolvable, 'right'. Whereas, if heat flux is coming from one side, it has to disappear to the other side with whatever be the flux, 'right'.

So, one is positive if it is positive, then the other side has to be negative or if it is negative the other side has to be positive. So, that the heat can enter and also exit otherwise there will be accumulation and if the accumulation is there; that means, inside temperature will go on increasing, which is beyond solution or unsolvable by normal this way yes it can be, but in all other different techniques. So, we can write $-k\frac{\partial T}{\partial x}|_{x=0} = q_0$ at the surface x equal to 0 and at the surface x equal to L, we can write $+k\frac{\partial T}{\partial x}|_{x=L} = q_L$, which we have written here, 'right'. So, heat supplied by conduction or by flux is (Refer Time:

17:31) $-k\frac{\partial T}{\partial x}|_{x=0} = q_0$ and conditions this is called conduction heat flux. Another one heat supply where it is watt per meter square $+k\frac{\partial T}{\partial x}|_{x=L} = q_L$, it is between 0 to L and in the x direction, 'right'.

So, when the heat flux is prescribed at a boundary surface, the boundary condition is said to be of the second kind, 'right' the boundary condition is said to be of the second kind.

So, here we have $-k\frac{\partial T}{\partial x}|_{x=0} = q_0$ and $+k\frac{\partial T}{\partial x}|_{x=L} = q_L$,'right'.

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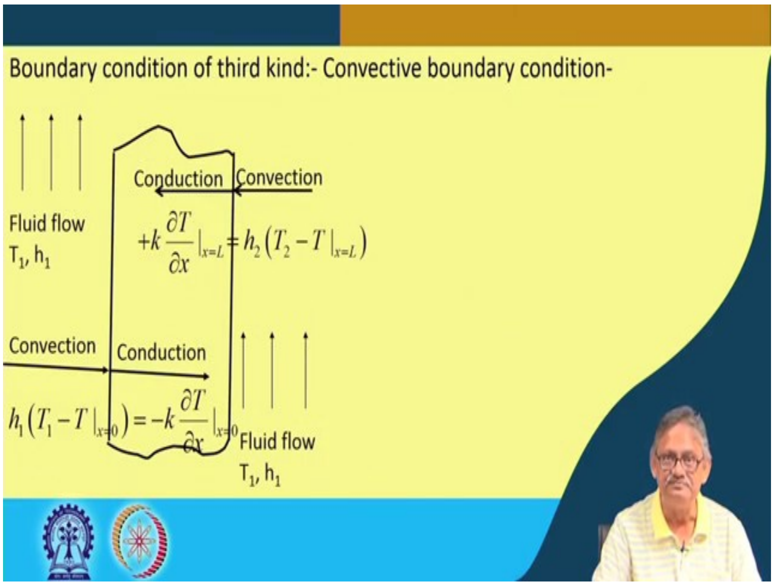

Then the boundary condition of the third kind, i.e., called convective boundary condition, 'right'. Here also you see we have taken one object which has two boundaries defined and the other two sides are much bigger compared to this one. So, that is why we can consider it to be one dimensional, 'right'.

So, here there is a convective fluid in one side flowing at the temperature T_1 with a heat transfer coefficient of h_1. So, this is under convective condition, 'right'. So, this convection is coming to the surface and then this surface is attending your temperature corresponding to this fluid flow and fluid properties, 'right' and which is equivalent to.

$h_1(T_1 - T|_{x=0}) = -k\frac{\partial T}{\partial x}|_x$ equal to 0, 'right'

And this convection heat is then conducted through the body, 'right' which is $+k\frac{\partial T}{\partial x}|_{x=L}=q_l$ at x equal to 0 or $-k\frac{\partial T}{\partial x}|_{x=0}=q_0$,'right'. So, this convection or whatever has come is conducted and here also this conducted $+k\frac{\partial T}{\partial x}|_{x=L}=q_l$ at x is equal to 0 is being going out through the fluid, which is at T_1 or rather which is at T_2 here it is of course, it should be T_2, this was T_1 h_1 this should be T_2 h_2, because it would be because of cut and paste that error was there. So, it should be T_2 h_2, 'right' fluid flow we need fluids flowing on the other side T_2 h_2, which we have corrected here of course, $+k\frac{\partial T}{\partial x}|_{x=L}=h_2\left(T_2-T|_{x=L}\right)$

So, this is called boundary condition of the second third kind or convective boundary condition. Now, the situation is that you may have that all these three are different, 'right'; one is prescribed temperature boundary condition, second is prescribed heat flux boundary condition, and third one is prescribed convective boundary condition. And all these three we have shown in both the sides, 'right', but it may happen that you have in one side prescribed temperature boundary condition and another side it could be prescribed heat flux boundary condition or prescribed convective boundary condition, that may be one situation, two situations or you may also have one side prescribed heat flux boundary condition and the other side prescribed convective boundary condition.

So, any combination can happen and but these are the generalized boundary conditions. So, when you will face the actual problem, it may be a combination, it may be individually like that or a combination of any three, any two of the three, 'right'. So, that is further we will look at, 'right'.

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So, this is what at the surface x equal to L which we have written here at the surface x is equal to L, we have written that it is equal to $+k\frac{\partial T}{\partial x}|_{x=L} = h_2\left(T_2 - T|_{x=L}\right)$.

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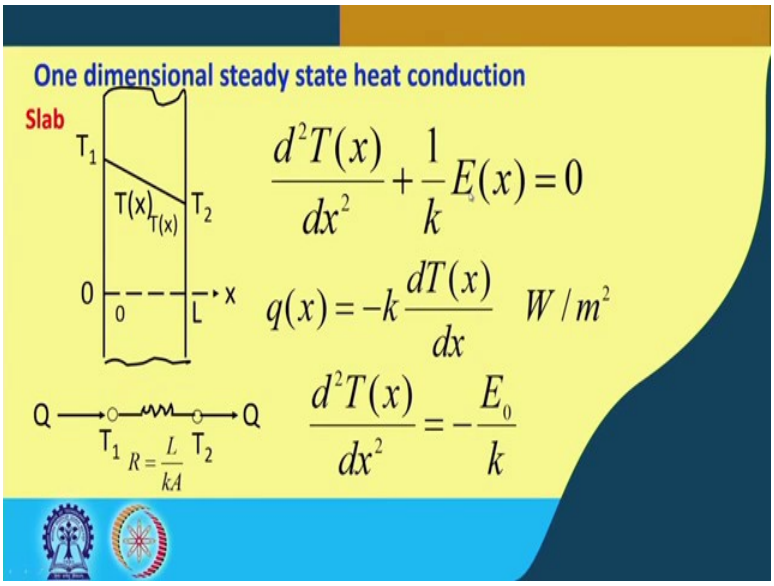

So, one dimensional steady state heat conduction, 'right'. So, we were discussing we are taking a typical case for slab, 'right'. So, if it is for slab, then we have taken this one dimension, other two dimensions are infinite or much bigger and this is a slab, where one side is subjected to a prescribed temperature boundary at T_1, other side is also at the prescribed temperature boundary condition of T_2, 'right' and we have inside T_x, 'right'.

So, this inside these a function of T is a function of x and it is 0 to L, 'right'. This we can write in 2 ways, one in the one which we have written in terms of methodics or it is at one dimensional heat conduction or theoretically we can write it this way $\frac{d^2T(x)}{dx^2} + \frac{1}{k}E(x) = 0$ This is that is equal to 0 this is one equation which we have seen earlier and the other equation is k x is equal to $q(x) = -k\frac{dT(x)}{dx}$ W/m^2 So, we can rewrite this one in the form of $\frac{d^2T(x)}{dx^2} = -\frac{E_0}{k}$ Here, we also can write in the form of this Q heat is coming to this surface where the temperature is T_1 and because of this body there is a resistance equivalent to R equal to L, that is the length over k A where k is the conductivity of the material and A is the area through which the heat is getting transferred and it is coming to the other surface at temperature T_2 and the same Q is going out, 'right'.

Because, Q cannot be this cannot be created there or when it is a steady state because it is a steady state. So, same Q will come in and go out of the body 'right'. So, the temperature was here T_1 and the other face the temperature is T_2 'right'. So, we are taking this solution of this equation that $\frac{d^2T(x)}{dx^2} + \frac{1}{k}E(x) = 0$,'right' it is not saying E_0 it is E at any x. So, it is 0 and then q x can be written as minus

$$q(x) = -k\frac{dT(x)}{dx} \quad W/m^2 \qquad \frac{d^2T(x)}{dx^2} = -\frac{E_0}{k}$$

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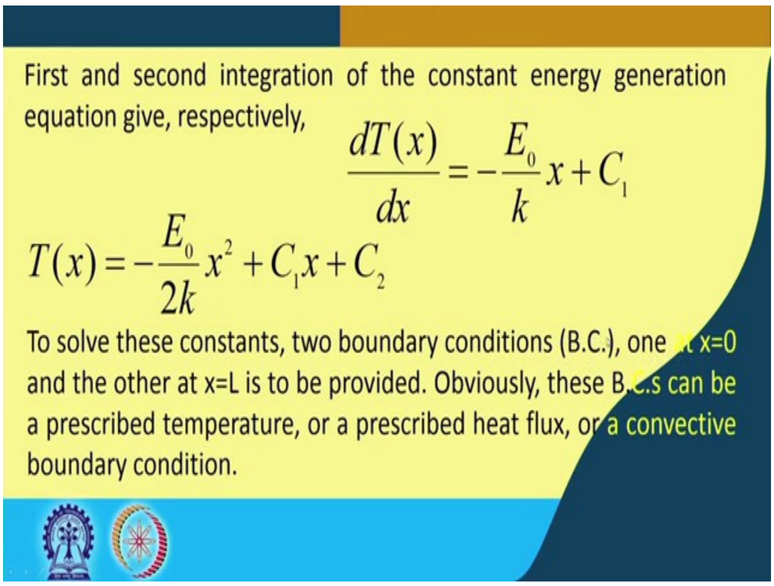

Then first and second integration of this can be written as $\frac{dT(x)}{dx} = -\frac{E_0}{k}x + C_1$ this is the first integration constant. And on second integration we get $T(x) = -\frac{E_0}{2k}x^2 + C_1x + C_2$ So, to solve it we need two boundaries, 'right', two boundary conditions we need one at x is equal to 0 and the other at x equal to L this is to be provided. And obviously, these boundary conditions can be prescribed temperature, of prescribed heat flux or a prescribed convective boundary condition, 'right'.

So, this is one example we have shown, how we can solve this kind of problem that if you have a slab and if in the slab that q quantity of heat is coming in where the surface is becoming T u surface temperature is T_1. And then the heat is getting conducted through the material going to the other side at T_2 and you are then getting it dissipated that q to the other side. Corresponding to the resistance of the material, it is R which is equivalent to L over k and A. So, k is the conductivity of the material and A is the area through which it is getting transferred, 'right'.

So, we started with you look at we started with that generalized equation, but the conditions are a steady state and constant conductivity. So, if it is steady state, 'right' side is 0 and internal energy generation is E. So, if conductivity is constant k then our governing equation becomes $\frac{d^2T(x)}{dx^2} + \frac{1}{k}E(x) = 0$ that is equal to $q(x) = -k\frac{dT(x)}{dx}$ W/m^2 So, we integrated it and our we reformulated that $\frac{d^2T(x)}{dx^2}$ ~~and if it is~~ $-\frac{E_0}{k}$ a constant E, then E_0 over k and we solved it by integrating first integration we got $\frac{dT(x)}{dx} = -\frac{E_0}{k}x + C_1$ and in the second integration you got $T(x) = -\frac{E_0}{2k}x^2 + C_1x + C_2$ So, we need two boundaries to solve it and that can be boundary condition of first kind, or boundary condition of second kind or boundary condition of third kind, 'right'. So, next we will solve it or similar problems, ok.

Thank you.