Course on Momentum Transfer in Process Engineering
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Mod 02 Lecture 06
Equation of motion (Part-2) or Navier Stokes equations

A small recapitulation of the previous class that we have done that is the momentum transfer, how that is being done for a particular component that is a velocity component that is x component of the velocity, Vx acting on the x, y and z directions by the bulk transport as well by the molecular transport, you remember we said that the molecular transport is being done by the molecules of the layers, which are vibrating in staying in the layer, but they are giving the energy or they are exchanging the energy with the adjacent layer molecules, right. This way the two ways of transport of momentum occurring we have done for the one component that is x component. Now we will do for y and z and then sum up all of them and derive the basic equation of motion that is the NAVIER stokes, okay.

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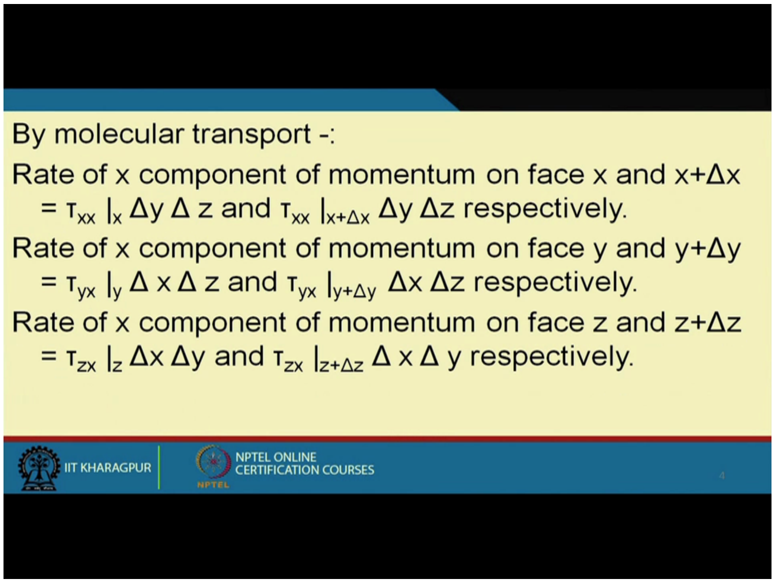


$\therefore$ Sum of the convective and molecular transport terms:

$(\rho v_x v_x|_x - \rho v_x v_x|_{x+\Delta x})\Delta y \Delta z + (\rho v_y v_x|_y - \rho v_y v_x|_{y+\Delta y})\Delta x \Delta z + (\rho v_z v_x|_z - \rho v_z v_x|_{z+\Delta z})\Delta y \Delta x - (\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x})\Delta y \Delta z + (\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y})\Delta x \Delta z + (\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z})\Delta x \Delta y$

Other terms,

Pressure force:- $(p|_x - p|_{x+\Delta x})\Delta y \Delta z$

Gravity force- $\rho g_x \Delta x \Delta y \Delta z$

Accumulation -:

Rate of accumulation of x component of momentum

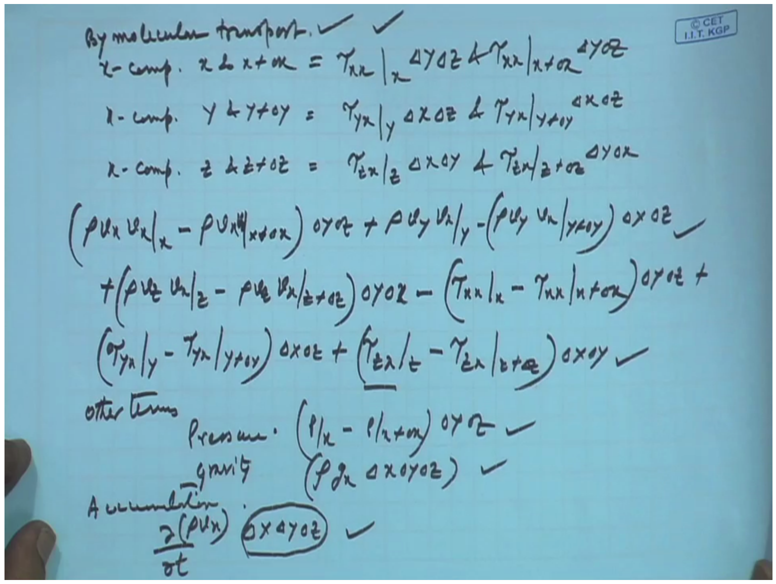
$= (\partial(\rho v_x)/\partial t)\Delta x \Delta y \Delta z$

Equating and dividing by $\Delta x \Delta y \Delta z$

$(\rho v_x v_x|_x - \rho v_x v_x|_{x+\Delta x})/\Delta x + (\rho v_y v_x|_y - \rho v_y v_x|_{y+\Delta y})/\Delta y + (\rho v_z v_x|_z - \rho v_z v_x|_{z+\Delta z})/\Delta z + (\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x})/\Delta x + (\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y})/\Delta y + (\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z})/\Delta z + (p|_x - p|_{x+\Delta x})/\Delta x + \rho g_x = \partial(\rho v_x)/\partial t$

or,

$\partial(\rho v_x)/\partial t = -[\partial(\rho v_x v_x)/\partial x + \partial(\rho v_y v_x)/\partial y + \partial(\rho v_z v_x)/\partial z] - [\partial\tau_{xx}/\partial x + \partial\tau_{yx}/\partial y + \partial\tau_{zx}/\partial z - \partial p/\partial x + \rho g_x$ …………………. (1)



So let us follow. So then we had come if you remember we had come up to this that accumulation, right where the volume element was there del x del y del z and we also had taken that by molecular transport by the bulk transport this was for molecular transport, this was for bulk transport we had done, right and then pressure term and the gravity term that is sum of the other forces, right.

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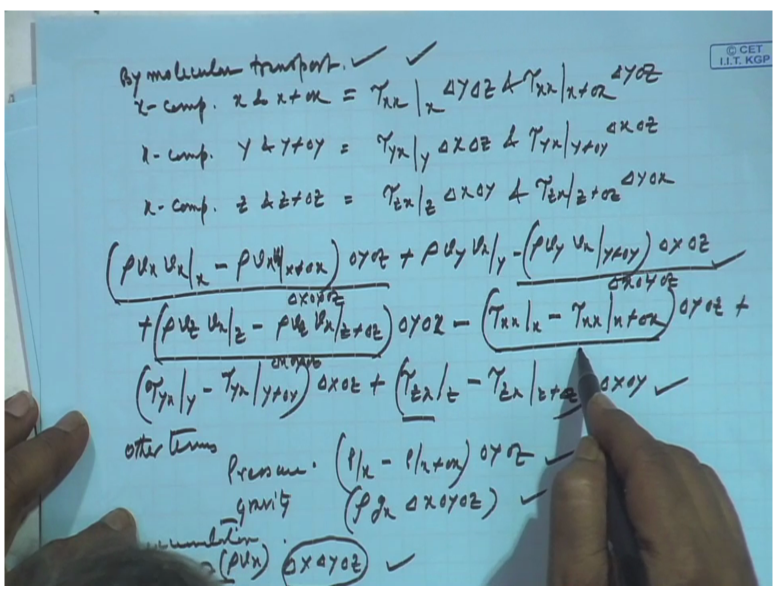

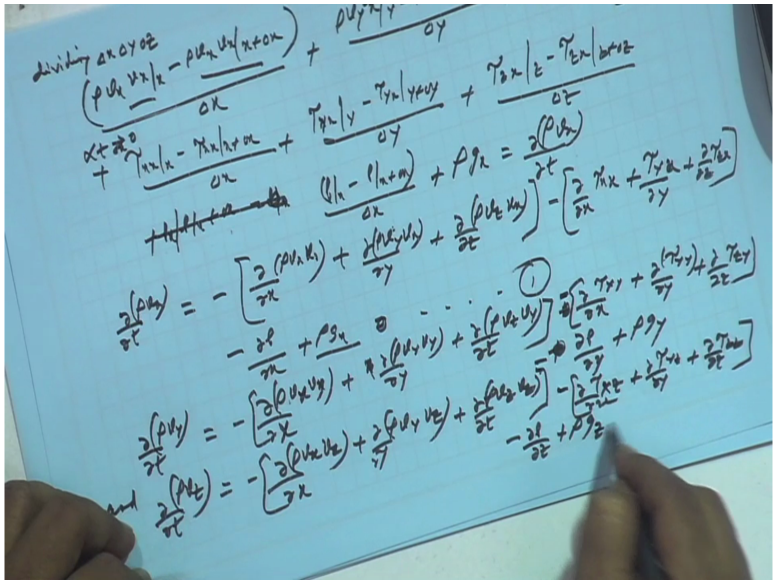

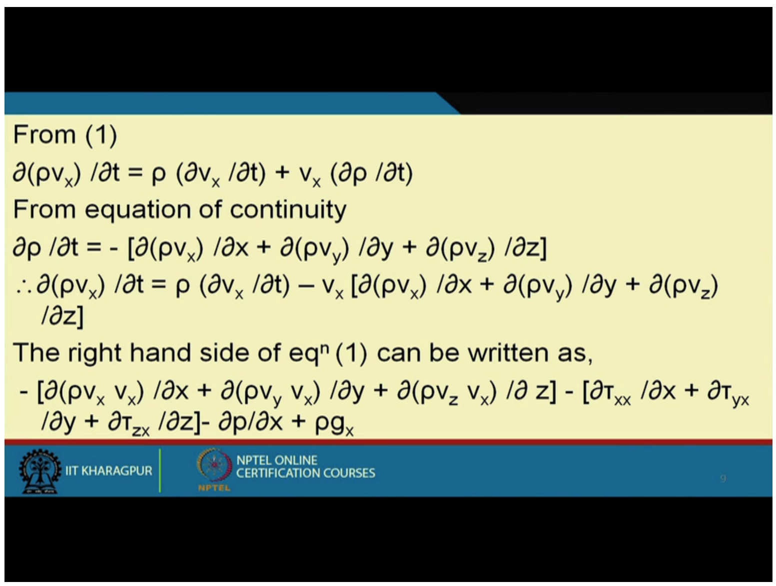

So now if we divide in both the sides with del x del y del z, right. If we divide with del x del y del z dividing both the sides, we can write that Rho Vx into Vx at x minus Rho Vx into Vx at x plus delta x, right. This divided by del divided by del x del y del z and we had the on the top this if you remember that this was del y del z. So dividing it to with del x del y del z we take of del y del z it remains del x, right.

Similarly here also del x del y del z we take of del y del z or rather del x del z del y remains and in this case also del x del y del z. So we take of del y del x del z remains, right. Similarly, here also, so dividing all with del x del y del z we get respective terms like that divide by del x right plus Rho Vy Vx at the face y minus Rho Vy Vx at the face y plus delta y over del y plus Rho Vz at Vx at the face z minus Rho Vx Rho Vz into Vx at the face z plus delta z over del z, right plus Tau xx at the face x minus Tau xx at the face x plus delta x over del x plus Tau yx, right Tau yx at the face y minus Tau yx at the face y plus delta y over del y plus Tau zx at the face z minus Tau zx at the face z plus delta z over del z, right.

So this is equals to and the pressure terms that is p at x plus delta x minus V at x plus delta x minus p at rather sorry, p at x minus p at x plus delta x, it should be p at the written (())(5:45) p at x minus p at x plus delta x into or divided by hat was also, we had del x del y del z, so here it is del x. So this plus in Rho g rather we had del x del y del z if you remember so that goes off, so Rho gx remains and this is equal to nothing but del del t of Rho Vx that is the accumulation term, right.

So this applying the theory of limit x del x tends to zero and from the definition of applying this limit x del x del y del z tends to zero and applying the definition of derivative, right you know derivative that is out minus in over the del. So this is a derivative, so we can write then this is nothing, but del Rho Vx del t right. This is equals to minus del del x of Rho Vx into Vx, right plus del del y of Rho Vy into Vx, right plus del del z of Rho Vz into Vx, right. This minus del del x of Tau xx plus del del y of Tau yx plus del del z of Tau zx, right. This minus del p del x plus Rho gx over this Rho gx this is equals to rather already we have made it equal. So this we can say to be equation number 1, right.

In this case, here you remember that we have applied this concept that limit del x del y del z that tends to zero number 1and number 2 when limit tends to zero and by the definition of the derivative this out minus in over delta that is nothing but the derivative that is del del x of Rho Vx Vx del del y of Rho Vy Vx del del z of Rho Vz Vx, right. Similarly del del x of Tau xx del del y of Tau yx del del z of Tau zx, right. Here in this case also minus this that is why negative is coming, so del p del x and here, we had Rho gx all x, y z was out. So only Rho gx remaining, right.

So this we can say that equation number 1 and subsequently we can write for the other two components that is for the y component and z component we can also write that in similar to this del del t of Rho Vy this is equals to minus del del y of rho v Rho Vy, right Rho Vy Vy or rather Rho Vx Vy, right and this plus this is del del t, so del del x of Rho Vx Vy plus del del y of Rho

Vy Vy, right plus del del z of Rho Vz Vy, right. This is for the bulk transport and for molecular transport we can write Tau del del x of Tau xy, right plus del del y of Tau yy right plus del del z of Tau zy, right. This plus that del p or rather here would to be minus, because by definition.

So minus del p del y plus Rho gy and in the same way we can write the z one right. We are not writing then we will okay, it will take only just a little time del del t of Rho Vz, this is equals to minus del del x of Rho Vx Vz plus del del y of Rho Vy Vz plus del del z of Rho Vz Vz, right. So this was like this plus or this plus of course that will come minus ultimately, right. Here you should have been minus; here also it is minus, right. So del del x and one this term so del del x of right Tau xy or in this case it will be xz, right plus del del y of Tau yz plus del del z of Tau zz, right this minus del p del z plus Rho gz, these are the three equations for the three co-ordinates systems, right.

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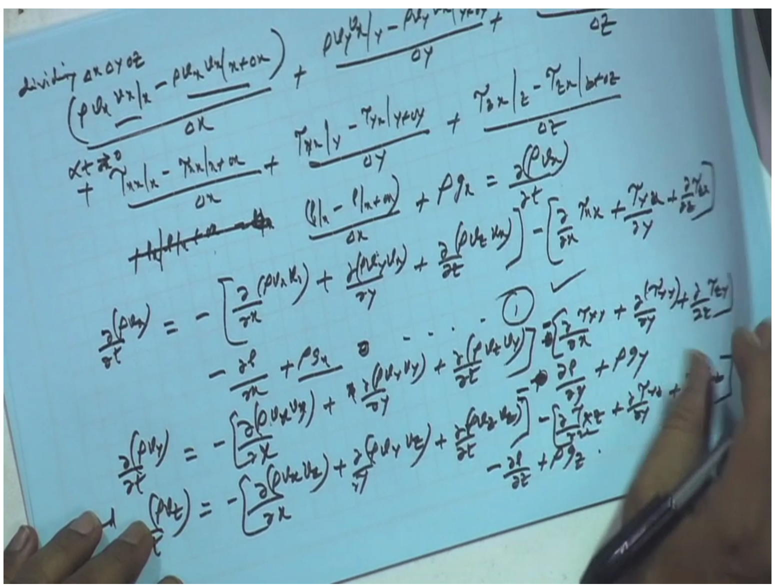

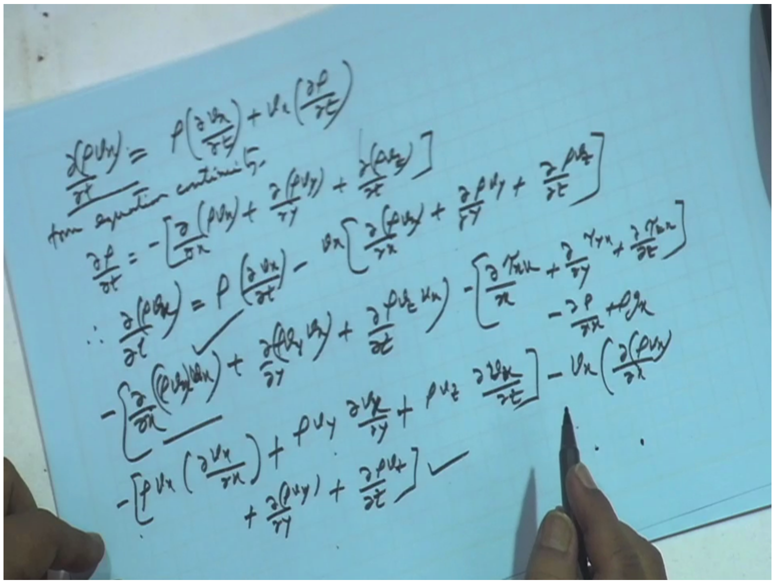

Now if we remember we had given this was the equation 1, right. So keep in mind that this equation then from that equation we can write del del t of Rho Vx, right. This is again nothing but Rho times del Vx del t or del del t of Rho Vx plus Vx del Rho del t, right again this is on x function uv we can write, right. So this from equation of continuity we can write that del Rho del t, this is equals to minus del del x of Rho Vx, right plus del del y of Rho Vy plus del del z of Rho Vz, right. This was taken in the equation of continuity and we can write it here.

So we can write then del del t of Rho Vz or rather Rho Vx is equal to Rho times del Vx del t minus Vx into del del x of Rho Vx, right plus del del y of Rho Vy plus del del z of Rho Vz, right. So this we can write and if you remember the right of the equation which was this was equation 1 and the right side was this big minus del del x of Rho Vx Vx, right.

So one term if we take then we can write minus del del x of Rho Vx Vx, right del del x of Rho Vx Vx, right plus del del y of Rho Vy Vx plus del del z of Rho Vz Vx, right. This was like that minus del del x of Tau xx plus del del y of this will be under bracket. So del del y of Tau yx plus del del z of Tau zx, right. This minus we also had del p del x plus Rho gx, right. So this on further expansion we can write, so this first term can be expanded like this that minus Rho Vx if we take this as u and this as v the minus Rho Vx into del Vx del x, right minus Rho Vy del Vy del y plus minus rather minus since, we have not given in a bracket if you were have given a bracket then we could have written like this, then Rho Vz del Vz del z rather del Vx del x, this is del Vx del y del Vx del z, right. This we could have written and this minus Vx times del del x of Rho Vx plus del del y of Rho Vy plus Del del z of Rho Vz, right. This term on expansion we can write like this, right as U and V.

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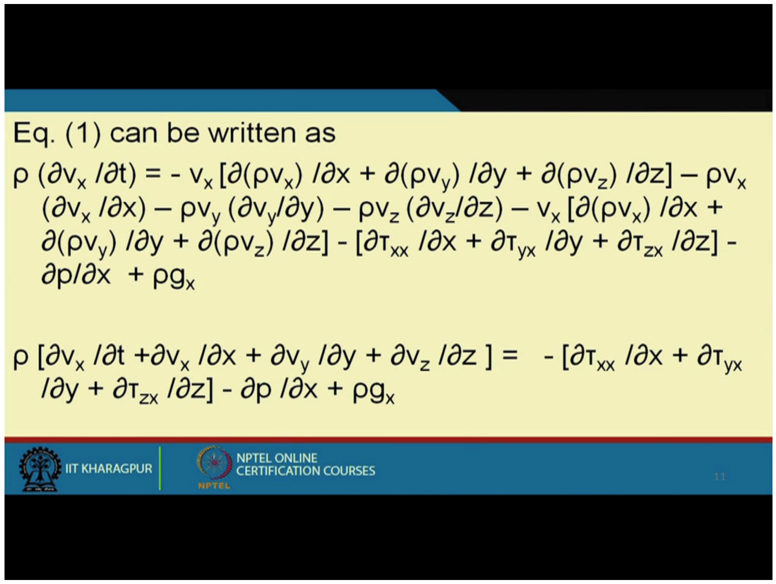

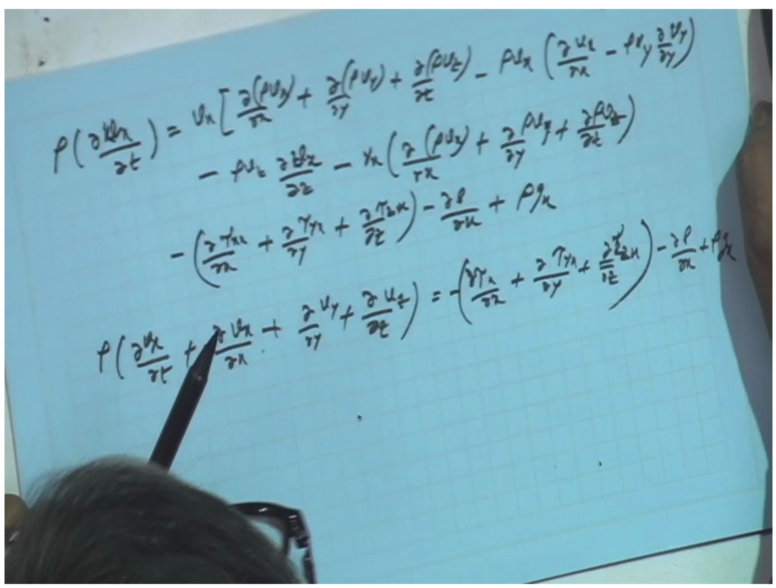

So if we do like that then we can rewrite this equation one which we have shown you earlier as Rho del Vx del t, this is equals to Vx into del del x of Rho Vx, right plus del del y of Rho Vy plus del del z of Rho Vz minus Rho Vx times del del x of Vx, right minus Rho Vy del del y of Rho Vy, right and also minus Rho Vz del del z of Rho del del z of Vz, right. So this minus Vx del del x of Rho Vx plus del del y of Rho Vy plus del del z of Rho Vz Rho Vx okay Rho Vx. So this also will be Rho Vx. So we can write, so del del Rho Vx del del x, sorry (())(19:30). This is y and this is z, right Rho Vz. This if we write then minus del del x of Tau xx plus del del y of Tau yx plus del del z of Tau zx, right. So this minus del p del x plus Rho gx, right. So this we can write further that we can say that Rho del Vx del t plus del del x of Vx plus del del y of Vy plus del del z of Vz, this is nothing but Tau del del x of Tau y xx del del x of Tau xx plus del del y of Tau yx plus del del z of Tau zx, right. So this we can write and of course here, we can with a if we remember that earlier we had taken this and of course (())(21:09) minus del p del x plus Rho gx was also there, right.

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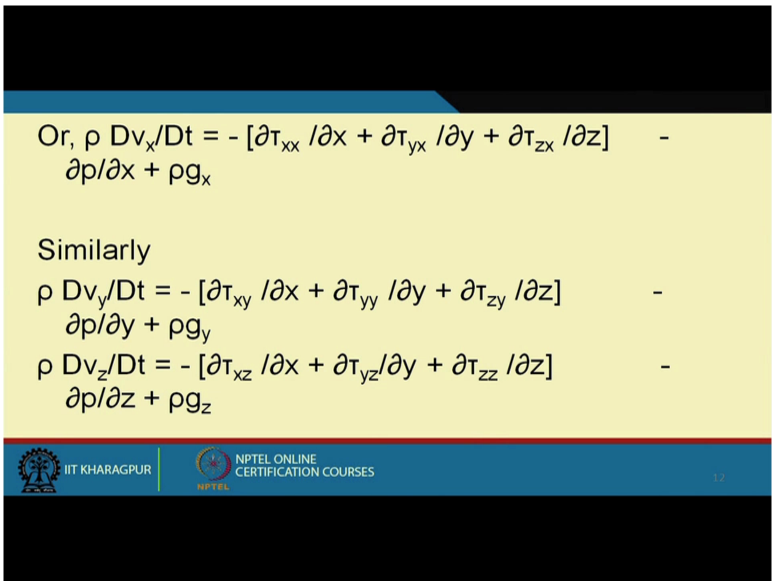

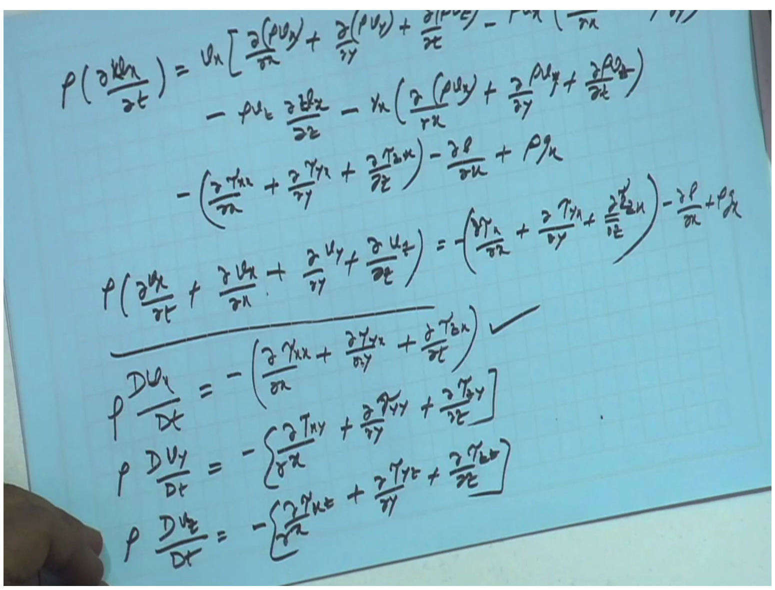

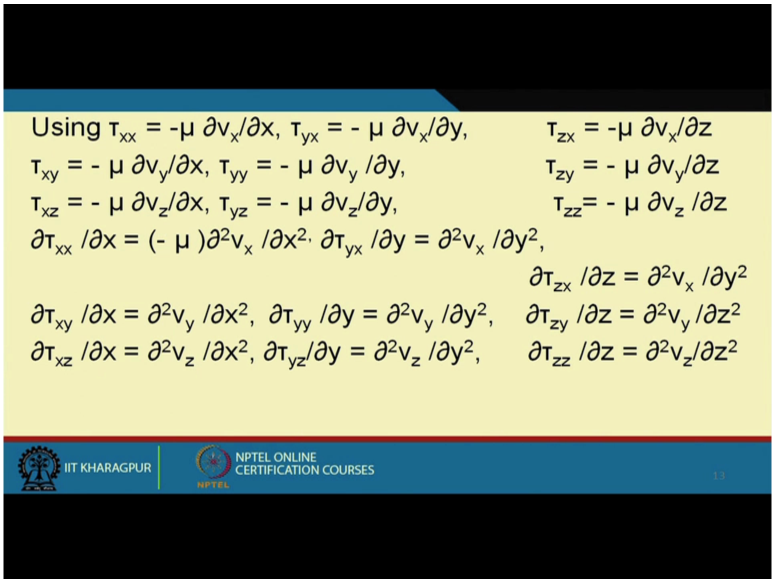

Then we can say that this to be nothing g but the operant (())(21:24) capital D. So we can write that capital DVx the into D t times Rho this is nothing but minus del del x of Tau xx plus del del y of Tau yx plus del del z of Tau zx, right. So this is so this is for the x component. Similarly for the y and z component we can write Rho capital D operand del Vy del t, right. This is equals to minus del del x of Tau xy plus del del y of Tau yy plus del del z of Tau zy, right. So this and similarly for the z component Rho D of V D Vz Dt this is equals to again minus del del x of Tau zx or in this case xz, right plus del del y of Tau yz plus del del z of Tau zz, right. So this three can be written easily, right.

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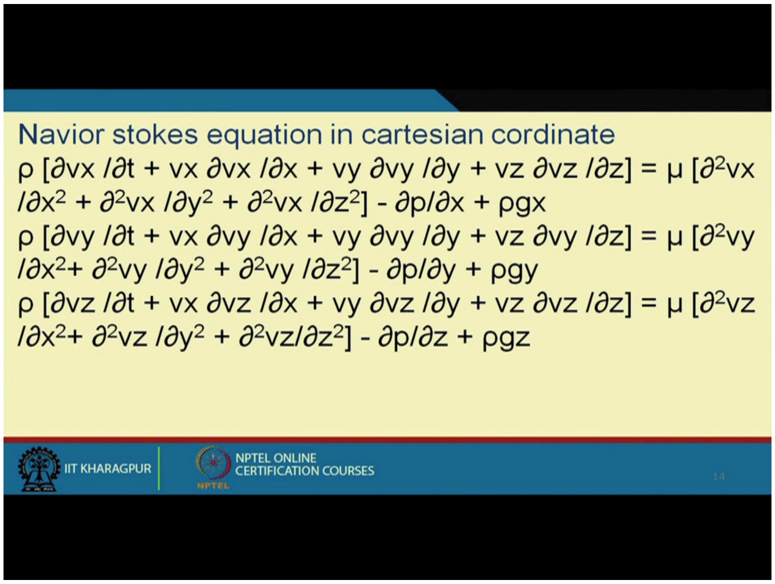

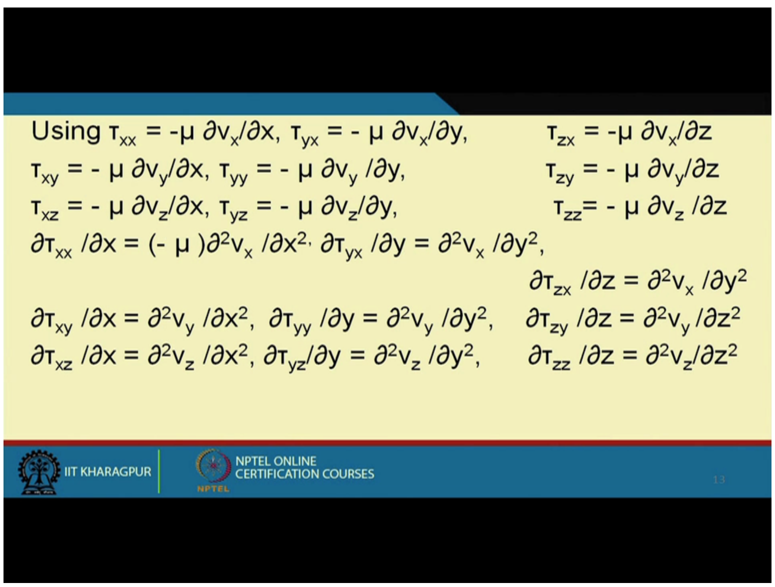

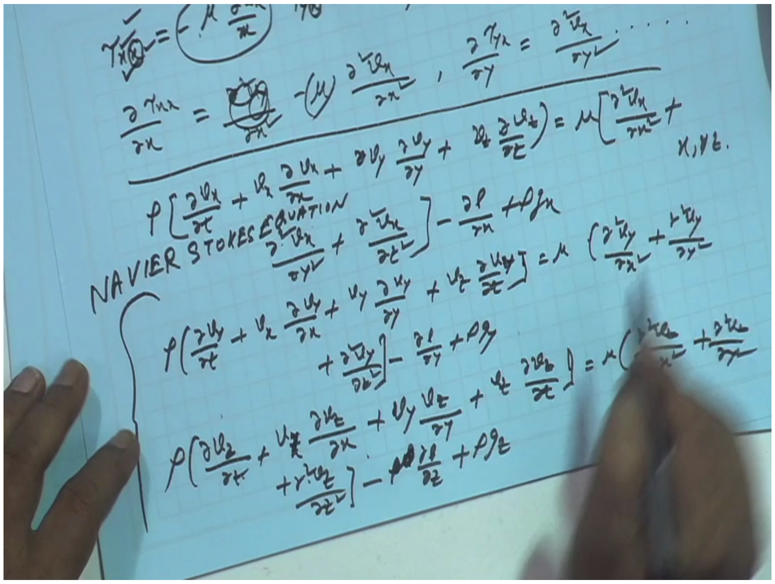

Now if we substitute that by using the value Tau xx is nothing but minus Mu del del x of Vx is Tau yx. Similarly we can also write Tau yx is equals to minus Mu del del y of Vx, right and using the similar way all others they (())(23:51), right you remember that this one and this one, okay I have given two, okay let me also give the third one Tau zx yz Tau zx, this is equals to minus Mu del Vx del z, right. So this is for the velocity component, this is for the direction, this is for the velocity component this is for the direction this is for the velocity component this is for the direction, if we keep in mind, this simple thing then we will be able to expand it in terms of V, right.

All this Tau terms that is your shear stress terms we can expend in terms of the velocity terms if there in Tau xx, first x is the direction, second x is the velocity component, Tau yx, y is the direction x is the velocity component. So that means (())(25:04) will be del Vy del x right. Del Vx del x then similarly, Tau yz will be del Vz del y, right. So keeping all these in term in mind if we substitute then in this equation, because okay some more if we show that, for example this we can write okay, Tau this minus Mu this, therefore we can write del del x of Tau xx Tau xx is defined as this, right. So del del x of Tau xx we can write, this is nothing but del 2 Vy del xx square, right, del 2 v this is Tau not xx Tau xx will be del del x of Tau xx will be minus, okay minus Mu right minus Mu into del 2Vx del xx square, right del 2Vx del xx square**.**

So keeping all minus Mu similarly we can write del del y of Tau yx right is equals to again del 2 Vx del y square like that all the terms we can write with of course minus mu(26:48), right and in that case if we substitute them in the equation what we have shown as equation 1 if we substitute all these Tau values in terms of velocity and viscosity then we can say that Rho times del del t of Vx plus Vx del del x of del Vx del del x of Vx plus del Vy del del y of Vy plus Vz del del z of Vz, this is equals to that Mu now it has come to the right side.

So negative has gone out Mu del 2 Vx del X square, right plus del 2 Vx del y square plus del 2 Vx del z square, right minus del p del x plus Rho gx, right. This is for x component for y component we can write Rho del del t of Vy, right plus Vx del del x of Vy plus Vy del del y of Vy plus Vz del del z of Vz, right or rather this will be y Vy , this is equals to Mu del 2 Vy del x square plus del 2 Vy del y square plus del 2 Vy del z square, right minus del p del y plus Rho gy and for the z component similarly, del Vz del t plus Vz del Vz del x plus Vz plus, this is Vx then Vy sorry, Vx del Vz del y plus Vz del Vz del z right. This is equals to Mu del 2 Vz del x square plus del 2 Vz del y square plus del 2 Vz del z square minus Rho no rather del p del z minus del p del z plus Rho gz.

So these are known an NAVIER stokes equation. When NAVIER stokes equation. So when and this is for the Cartesian co-ordinate x, y, z right. Now when you will be doing it you have to identify from the problem given or from the situation which term corresponds to what.

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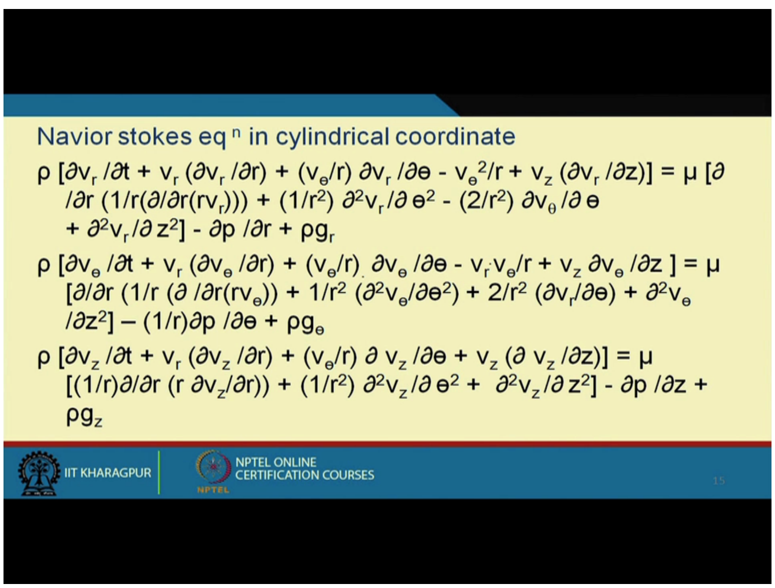


R-component

$$\rho\left[\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right]$$

$$=\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rv_r\right)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]$$

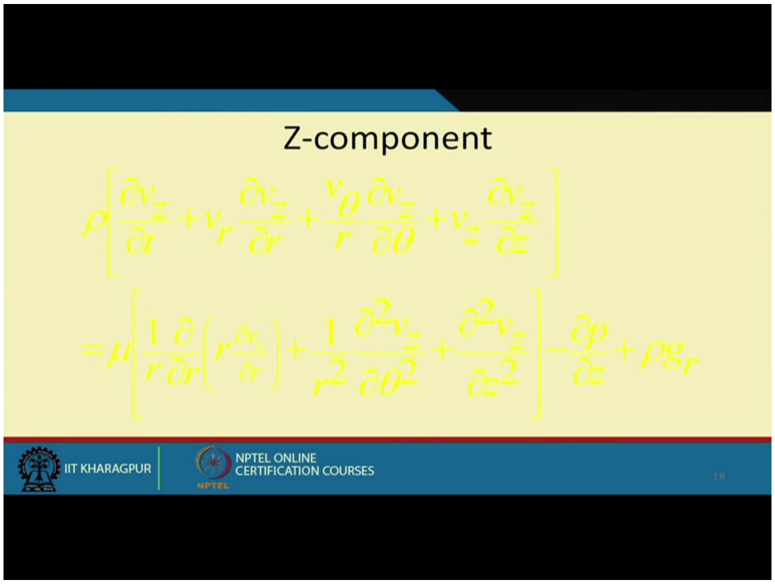
$$-\frac{\partial p}{\partial r}+\rho g_r$$

θ-component

$$\rho\left[\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right]$$

$$=\mu\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\left(rv_\theta\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho g_r$$



So if you can identify you can solve any problem on the momentum transfer with the help of this NAVIER stokes equation, right we will do some problems next time and then identify and understand this individual terms, okay. So that is what here we do, so NAVIER stokes equation in a cylindrical co-ordinate I have given, in the slide you can check and also this of course not coming proper way in there R component, theta component like that. So thank you.