Mathematics III
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Lecture - 9
Laplace Transformation (Contd.)

Dear viewers, my today's lecture is in continuation to my previous lecture on Laplace Transformation, where in we studied the existence theorem for the Laplace transforms and some of it is properties. We will continue that discussion with some more properties of the Laplace transforms.

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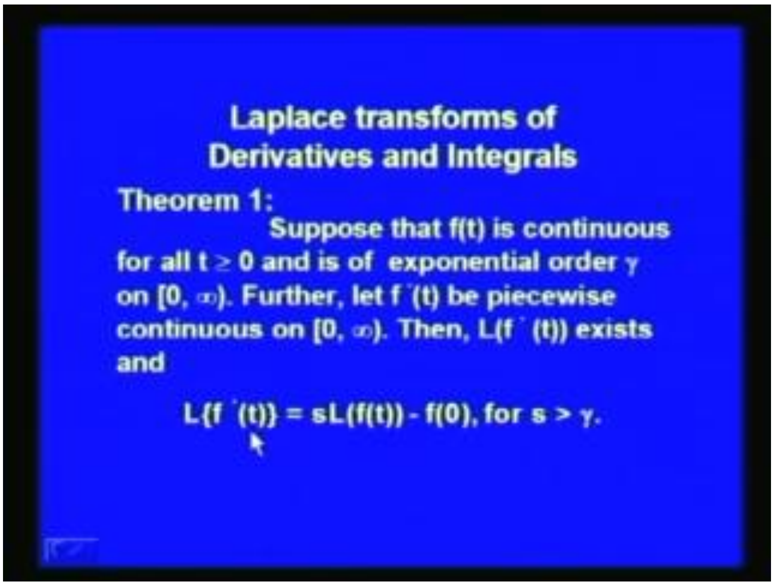

Let us first discuss the Laplace transform of derivative of a function, that the theorem 1 tells us that when we take the Laplace transform of derivative of a function what we get is the Laplace transform of the function multiplied by s that is s into L f t minus f 0. So, this means that when the Laplace transform x on the derivative Function, you get the function transform the function multiplied by s. That means the operation of calculus by acting on when we act by the Laplace transform on the operation of the calculus, what we get is the algebraic operation on the transforms of the function Laplace transforms of the function. So, the operation of calculus at transformed into the algebraic operations on the transforms using the Laplace transformation.

So theorem 1 tells us that, suppose f t is continuous function for all t greater than or equal to 0 and is of exponential order gamma on the interval 0 infinity. Let us recall the definition of exponential order gamma, we say that the function f t is exponential of exponential order gamma provided. We can find constant m greater than 0 such that mod of f t is less than or equal to m times e to the power gamma t for all t greater than or equal to sum t naught. So, Further let f dash t be piecewise continuous on the interval 0 infinity, then the Laplace transform of f dash t exists and we have L of f dash t equal to s times L f t minus f 0 for all s greater than gamma.

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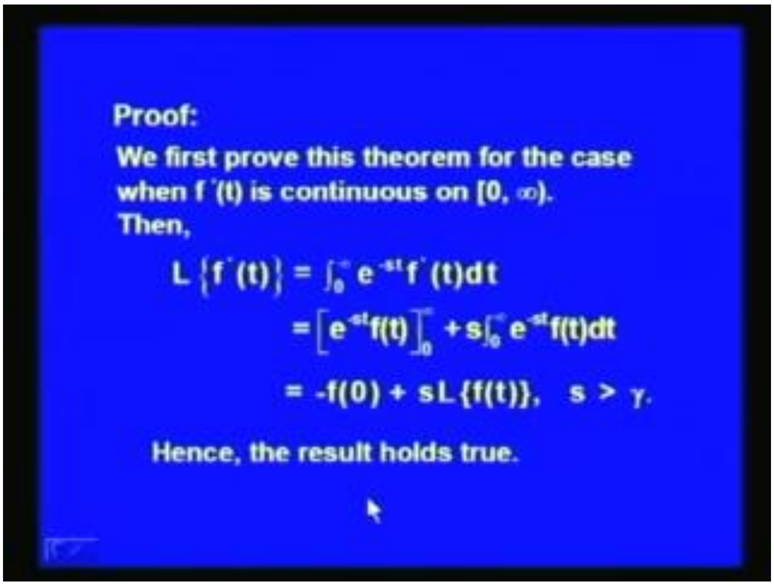

Let us proof this theorem, we first will prove this theorem for the case when f dash t is a continuous Function on the interval 0 to infinity. Let us say we call that every piecewise continuous Function need not be continuous, while a continuous function is always piecewise continuous. So let us assume, let us first discuss the proof of the theorem for the case when f dash t is continuous on 0 infinity.

Then, the Laplace transform of f dash t which is equal to by definition of Laplace transform integral 0 to infinity e to the power minus s t into f dash t d t is equal to when we integrate by parts it will be equal to e to the power minus s t into f t evaluated at 0 in infinity plus s times integral 0 to infinity e to the power minus s t into f t d t.

Now, since we have assumed that f t is of exponential order gamma, so mod of f t is less than or equal to m times e to the power gamma t for large values of t. And therefore, s t goes to infinity e to the power minus s t into f t will 10 to 0 as t goes to infinity, because f t mod of f t less than or equal to m times e to the power gamma t.

So, mod of e to the power minus s t into f t will be less than or equal to m times e to the power minus s minus gamma into t. So, when s will be greater than gamma and t will 10 to infinity this quantity will 10 to 0 and at the lower limit its value is f 0. So, we get minus f 0 here, plus s times and this is nothing but the Laplace transform of the function f t. So, we get L of f dash t equal to L f t minus f 0, so this proves the theorem for the case, when f dash t is continuous on the interval s 0 infinity.

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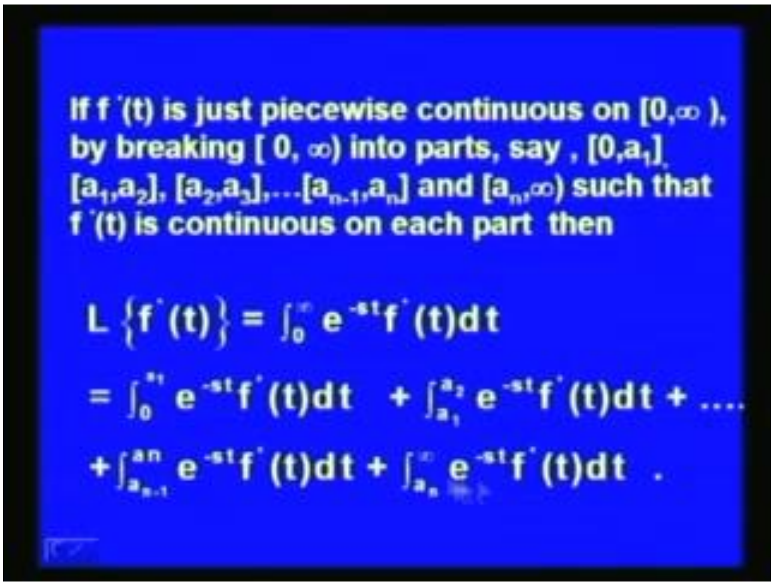

Now, if f dash t is jus t piecewise continuous on the interval 0 infinity, then what we do is, we break the interval 0 infinity into finitely many parts say 0 a 1 a 1 a 2 a 2 a 3 n minus L n and n infinity such that f dash t is continuous on each part. Then by definition L of f dash t is integral 0 to infinity e to the power minus s t, f dash t d t it can be broken up into these n plus 1 integrals integral 0 to a 1, e to the power minus s t f dash t d t plus a 1 to a 2 e to the power minus s t f dash t d t and so on, n minus 1 a n e to the power minus s t f dash t d t plus a n integral over n to infinity e to the power minus s t f dash t d t.

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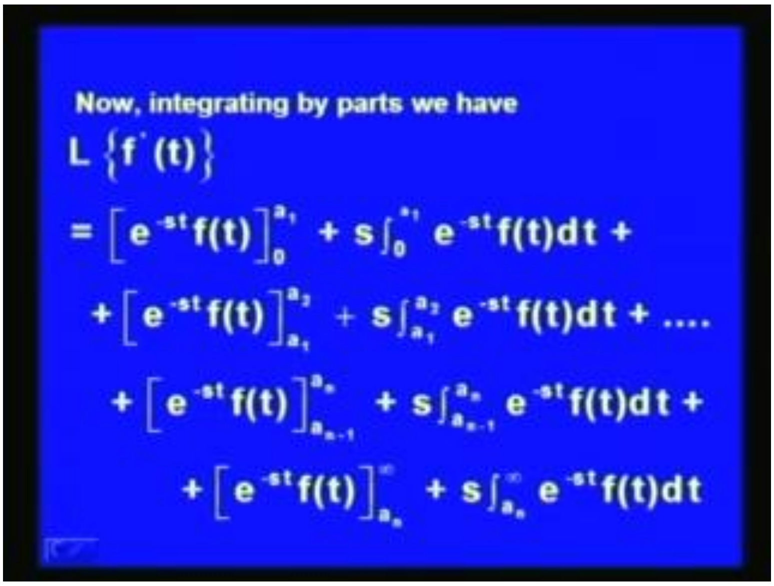

Now, let us integrate each integral let us evaluate each integral by integrating by parts, then the first integral on the right gives e to the power minus s t f t 0 a 1 plus s times integral 0 to a 1 e to the power minus s t f t d t plus second integral on the right gives e to the power minus s t f t a 1 a 2 plus s times a 1 to a 2 e to the power minus s t f t d t and so on.

The nth integral gives e to the power minus s t f t n minus 1 an plus s times integral over n minus 1 to n e to the power minus s t f t d t and the last integral gives on integration by parts e to the power minus s t f t a n to infinity, and then plus s times integral over a n to infinity e to the power minus s t f t d t.

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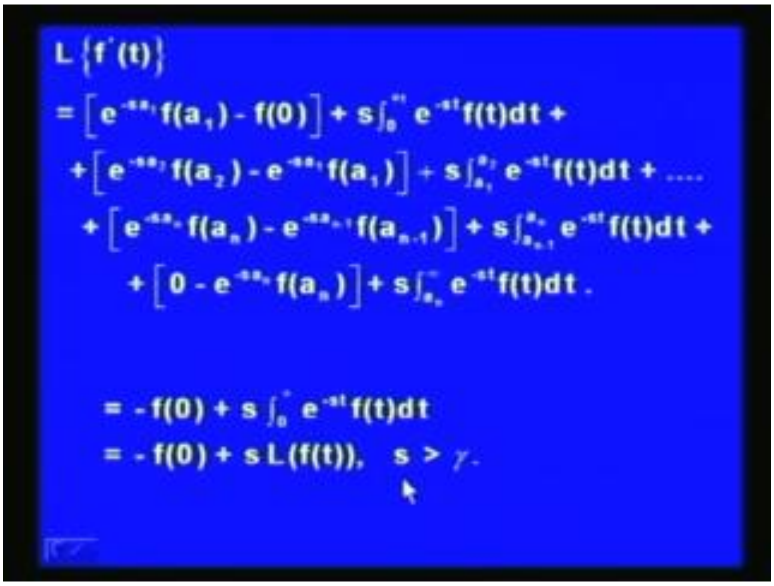

And hence, L of f dash t equal to e to the power minus s a 1 into f a 1 minus f 0 plus s times integral over 0 to a 1 e to the power minus s t into f t d t plus e to the power minus s a 2 f a 2 minus e to the power minus s a 1 into f a 1 plus s times integral over a 1 to a 2 e to the power minus s t into f t d t plus and so on, e to the power minus s a n into f a n minus e to the power minus s a n minus 1 into f a n minus 1 plus s times integral over n minus 1 into n e to the power minus s t into f t d t.

And then we get the last integral is equal to 0 minus e to the power minus s a n f a n plus s times integral over n to infinity e to the power minus s t into f t d t. This 0 comes when we take the limit of e to the power of minus s t into f t s t goes to infinity, because f t is of exponential order. And then these terms will all cancel out e to the power minus s a 1 f a 1 will cancel with these e to the power minus s a 2 f a 2 will cancel with the next term coming in the next term, and then e to the power minus s n f n will also cancel with this. So, ultimately on simplification the right hand side becomes minus f 0 plus s times integral over 0 to infinity e to the power minus s t into f t d t. Because, integral over 0 to a 1 integral over a 1 to a 2 integral over n minus 1 into n and integral over n to infinity can be combined together and we get the integral 0 to infinity e to the power minus s t into f t d t.

So, we get this Further equal to minus f 0 plus s times Laplace transform of f t and thus in the case when f dash t is jus t piecewise continuous, again we get the Laplace transform of f dash t s s times Laplace transform of the function t minus f 0 whenever s is greater than gamma.

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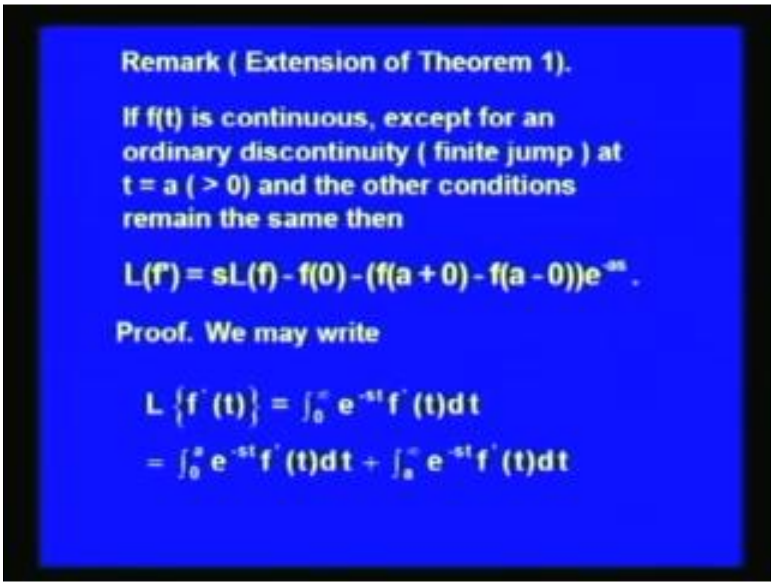

Now, let us study an extension of theorem 1, suppose that the Function f t is not continuous, then it has the discontinuity at the point t equal to a, and the discontinuity is the ordinary discontinuity. So, let us assume that the function f t is continuous for all t greater than or equal to 0 except for an ordinary discontinuity, ordinary discontinuity means when t times 2 a minus 0 that is t times 2 a from the left side or t times 2 a from the right side, we the function f t tends to a finite limit. So, it has got finite jumps on either side of t equal to a and all the other conditions of the theorem 1 remain the same.

Then, the Laplace transform of f dash t is equal to s times Laplace transform of f t minus f 0 minus f of a plus 0 which is the right hand limit of f t as t times 2 a minus f of a minus 0 which is the left hand limit of f t as t times 2 a into e to the power minus a s.

Let us prove this result, by definition we may write Laplace transform of f dash t as integral 0 to infinity e to the power minus s t into f dash t d t. Now, let us break this integral on the right side into two parts integral over 0 to a e to the power minus s t into f dash t into d t and the other integral over a to infinity e to the power minus s t into f dash t d t, and then we integrate by parts.

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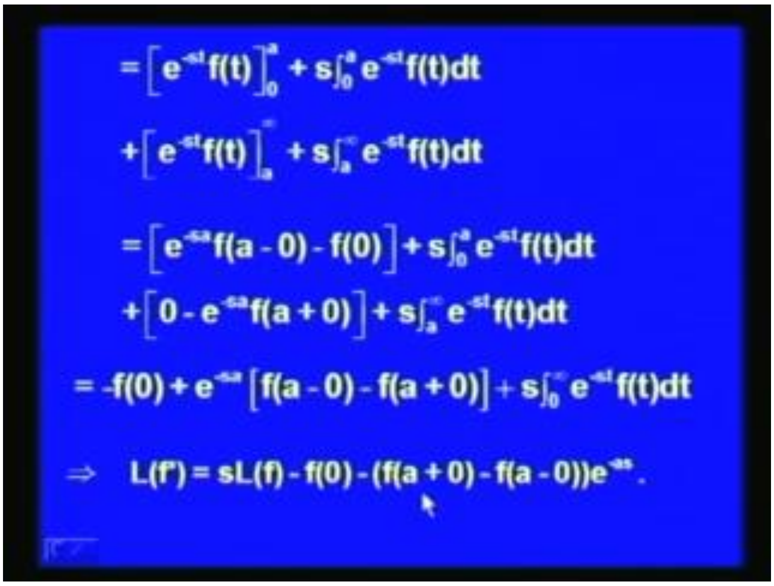

So, then the firs t integral on the right gives e to the power minus s t into f t evaluated at 0 and a plus s times integral over 0 to a e to the power minus s t into f t d t. The second integral on integration by part gives e to the power minus s t into f t evaluated at 0 a to infinity plus s times integral over a to infinity e to the power minus s t into f t d t.

So, when t times to a here, then t will tend to a from the left side, so e to the power minus s t into f t will tend to e to the power minus s a into f a minus 0 f a minus 0 is the limit of f t s t tends to a from the left. And then minus f 0 we get after we put the lower limit here, t equal to 0 and then plus s times integral 0 to a e to the power minus s t into f t d t and here, when t times to infinity e to the power minus s t into f t goes to 0, because f t is of exponential order.

And, when we try to put the lower limit t will tend to a, and it will tend to a from the right, because the interval is from a to infinity. So, when t tends to a from the right e to the power minus s t into f t will tend to e to the power minus s a into f of a plus 0 f of a plus 0 is the right hand limit of f t s t times 2 a plus s times integral over a to infinity e to the power minus s t into f t d t.

And, when we combine these two integrals, we get integral over 0 to infinity e to the power minus s t into f t d t. So, on simplification this right hand side of the equation gives minus f 0 plus e to the power minus s a into f a minus 0 minus f plus s times integral 0 to infinity e to the power minus s t into f t d t, which is same as L of f dash equal to s into L f s L f minus f 0 minus f of a plus 0 minus f of a minus 0 into e to the power minus a s which gives the required result.

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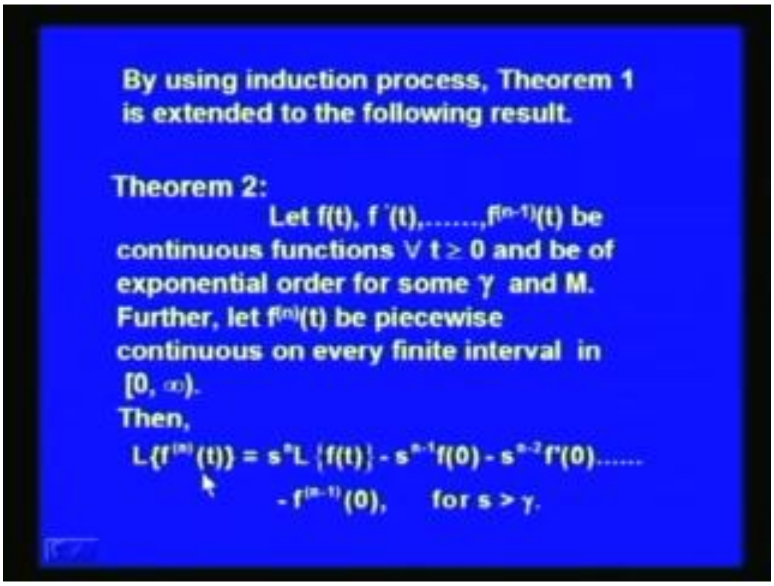

Now, let us use the induction process on theorem 1, so on using induction we have the following result. Let f t f dash t and so on f n minus 1 t be continuous functions for all t greater than or equal to 0 and be of exponential order for some gamma and M. Further, let the nth derivative of f t by piecewise continuous on every finite interval in the range t greater than or equal to 0 that is the interval 0 infinity.

Then, the Laplace transform of n the derivative of f t that is f n t is equal to s to the power n into L f t minus s to the power n minus 1 into f 0 minus s to the power n minus 2 into f dash 0 and so on minus f n minus 1 0 for all s greater than gamma. Now, this theorem is easy to prove by induction on n for n equal to 1 it is it holds by theorem 1 and assuming it for n minus 1, we can easily show it for n by induction.

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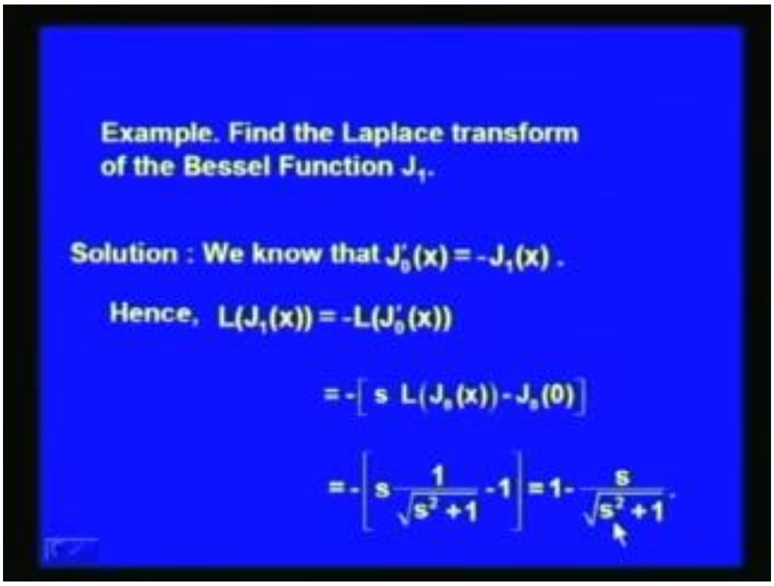

Now, let us take an example based on the Laplace transform of the derivative of a Function, let us find the Laplace transform of the Bessel function of order one that is J 1. We know that by the properties of Bessel functions of firs t time, we know that Laplace transform, we know that the derivative of j naught x is equal to minus J 1 x. So, if we want to find the Laplace transform of J 1 x we can write by using the linearity of the operator L Laplace transform of J 1 x equal to minus l of J dash x and which will be equal to minus l of J naught x minus J naught at evaluated at 0 using the theorem 1.

So, this is equal to minus s times upon Laplace transform of J naught x is equal to 1 over square root s square plus 1 which we had shown in our previous lecture on the Laplace transformation. So, s times 1 over the square root s square plus 1 minus J naught x if you put x equal to 0 in the series expansion of J naught x, then you get the value J naught 0 s 1. So, we get the Laplace transform of J 1 x s 1 minus s over a square root of s square plus 1.

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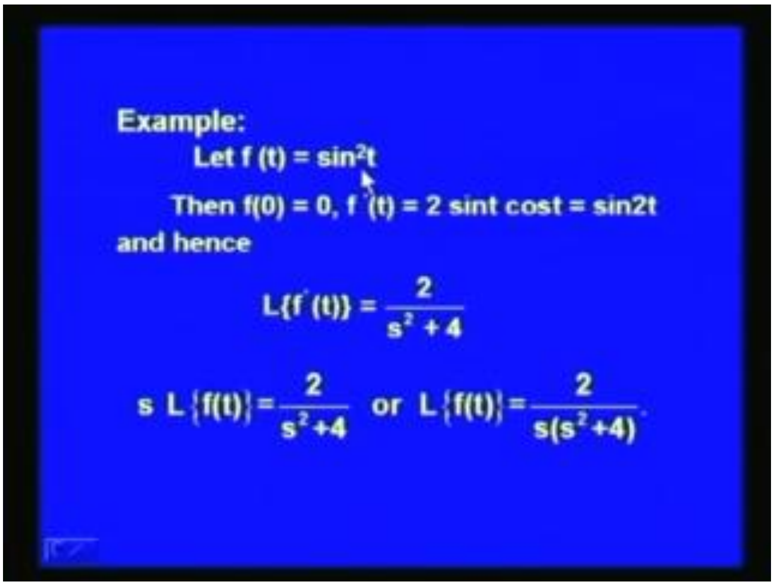

Let us now take another example, let us s ay f t equal to sin square t we want to find the Laplace transform of f t. And we cannot find this Laplace transform of f t by directly from the known results on the Laplace transform of elementary function which we have studied in the previous lecture of Laplace transformation. But, we can do it either by writing sin square t as 1 minus cos 2 t by 2 and using the linearity property of L and the Laplace transform of cos 2 t, we can also find the Laplace transform of f t equal to sin square t using theorem1, as we are going to see now.

So, f 0 equal to 0 f dash t is equal to 2 sin t cos t that is it is equal to sin 2 t, and we can see than f t is the continuous function for all t greater than or equal to 0 f dash f t of exponential order. Because f t is the bounded function and then its derivative f dash t is sin 2 t which is also a continuous function for all t greater than or equal to 0. So, we can apply theorem 1 and by theorem 1, then Laplace transform of f dash t.

Now, Laplace transform of f dash t we can rightly write using the known results we know that Laplace transform of sin a t is a upon s square plus a square. So, Laplace transform of f dash t will be equal to 2 over s square plus 4. Now, Laplace transform of f dash t is equal to s into Laplace transform of f t minus f 0, but f 0 is 0.

So, L f t L f dash t becomes s into L f t which is equal to 2 over s square plus 4 dividing by s, we get Laplace transform of the function f t that is Laplace transform of sin square t s 2 over s into s square plus 4. So, by using theorem 1 we can find the Laplace transform of f t equal to sin square t.

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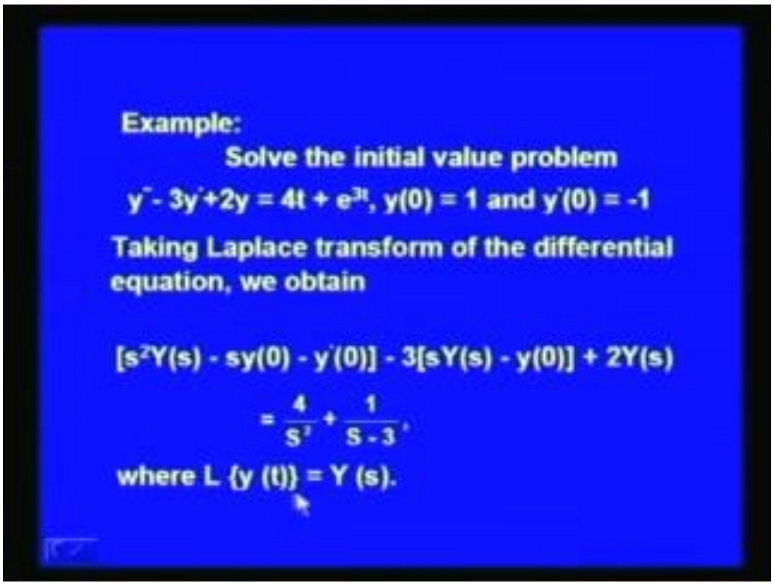

Let us, now study how we can find the solution of a differential equation with the given initial conditions the conditions are y 0 equal to 1 and y dash 0 equal to minus 1. The differential equation is y double dash minus 3 y dash plus 2 y equal to 4 t into e to plus e to the power 3 t. Now, we will take the Laplace transform of the given differential equation on both sides when we take the Laplace transform of y double dash what we get is s square Y s minus s y 0 minus y dash 0 using the theorem 2. And then minus 3 times Laplace transform of Y dash will become s into y s minus y 0 plus 2 times Laplace transform of Y will become y s.

And this is equal to 4 times Laplace transform of t that is 1 by s square plus Laplace transform e to the power 3 t that is 1 over s minus 3. So, we are making use of theorem 2 here, and the linearity property of the Laplace operator, where we are assuming that L of y t that is Laplace transform of y t is equal to Y s.

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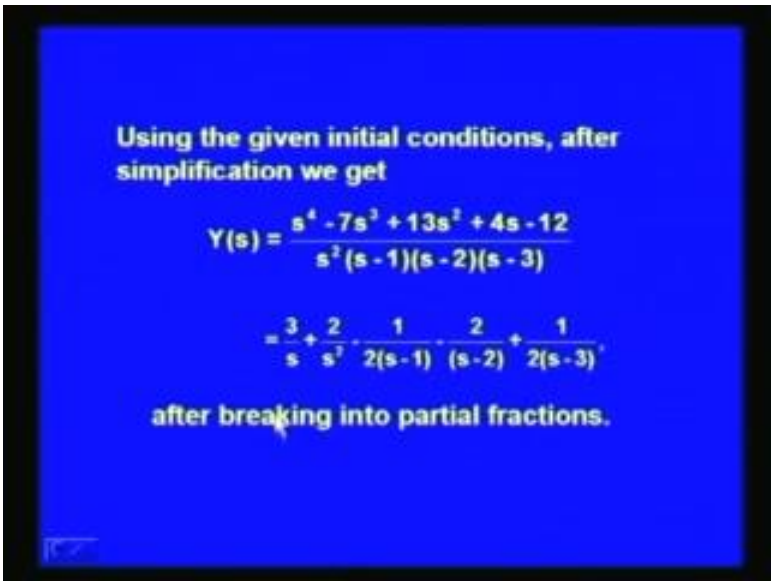

Now, let us use the given initial conditions that is y 0 equal to 1 and the initial condition for y dash 0 will get that Y s after simplification, the value of Y s comes out to be equal to s to the power 4 minus 7 s cube plus 13 s square plus 4 s minus 12 over s square into s minus 1 s minus 2 s minus 3 s minus 3 which is the rational Function of s and is a proper fraction. So, we can break it into its partial fractions and after breaking it into partial fractions we get that Y s becomes equal to 3 over s plus 2 over s square minus 1 over 2 n 2 into s minus 1 minus 2 over s minus 2 plus 1 over 2 times s minus 3.

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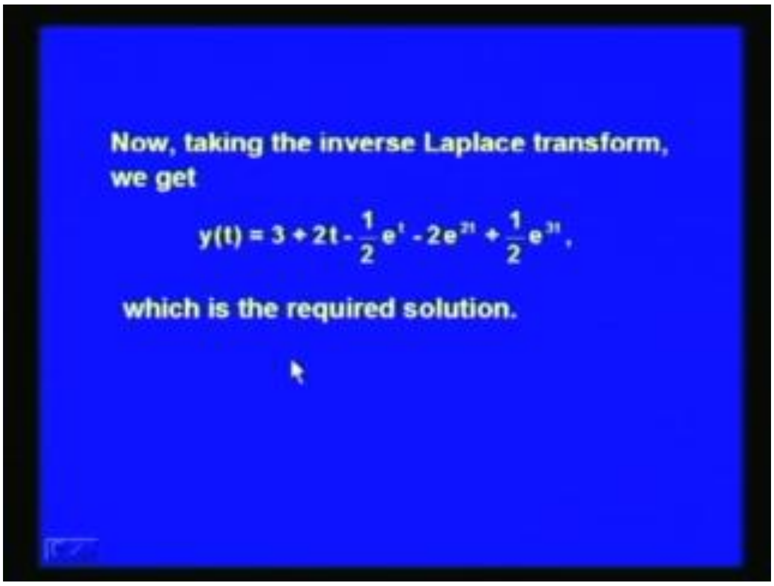

Now in order to find the desired function by t we will take the inverse Laplace transform on both sides of the last equation. So, we get Laplace inverse Laplace transform of y s equal to y t, y t is the equal to then 3 plus 2 t minus half e to the power t minus 2 times e to the power 2 t plus 1 by 2 e to the power 3 t which is the required solution of the given initial value problem.

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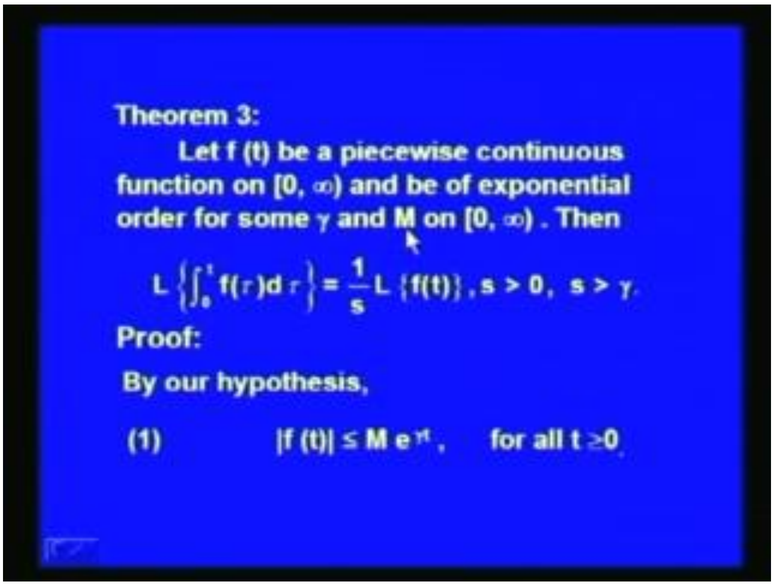

Now, let us study next the theorem which is based on the Laplace transform of integral of a Function and we see that it again transforms into an algebraic operation Laplace transform of f t is divided by s. So, let f t be a piecewise continuous function on the interval 0 infinity and be of exponential order for some gamma and M on the interval 0 infinity. Then the Laplace transform of integral 0 to t f tau d tau is equal to 1 by s Laplace transform of f t, whenever s is greater than 0 and s is greater than gamma.

So, by our hypothesis that is f t is of exponential order for some gamma and M on 0 infinity and that f t is piecewise continuous function on the interval 0 infinity, it follows that mod of f t is less than or equal to some constant m into e to the power gamma t for all t greater than or equal to 0. This M may not be the same as the m here in the theorem.

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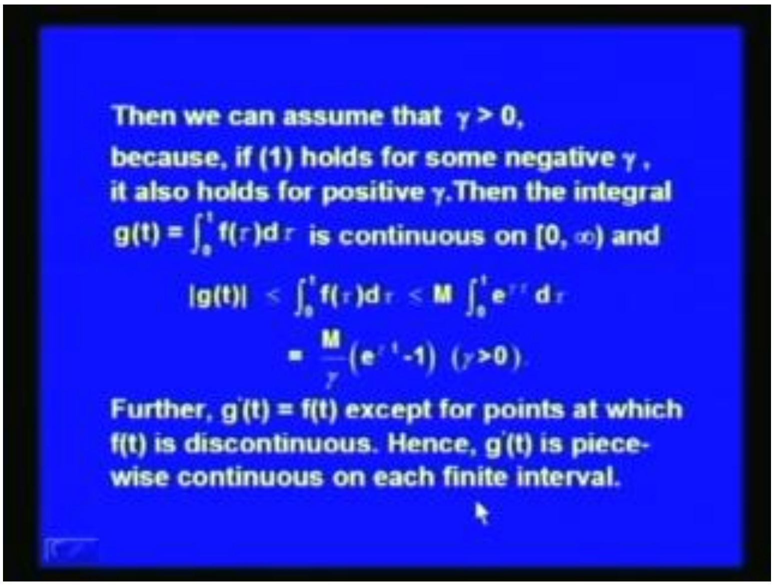

Now, we can assume here that gamma is greater than 0, because if the equation 1 that is mod of f t is less than or equal to M times e to the power gamma t holds for some negative gamma then clearly it also holds for a positive gamma. And hence, gamma can be assumed to be positive.

Now, then the integral 0 g t equal to integral 0 to t f tau d tau it is a continuous function on the interval 0 infinity and mod of g t is less than or equal to integral 0 to t f tau d tau less than or equal to M times integral 0 to t e to the power gamma t into d tau e to the gamma tau into d tau and when we integrate e to the power gamma tau and put the limits we get the right hand we get this value of the integral as e to the power gamma tau t e to the power gamma t minus 1 over gamma, where gamma is positive.

Further, we found we find that g t is equal to integral 0 to t f tau d tau is g dash t equal to f t except for those points where the Function f t is discontinuous. Hence, g dash t is a piecewise continuous Function on each finite interval.

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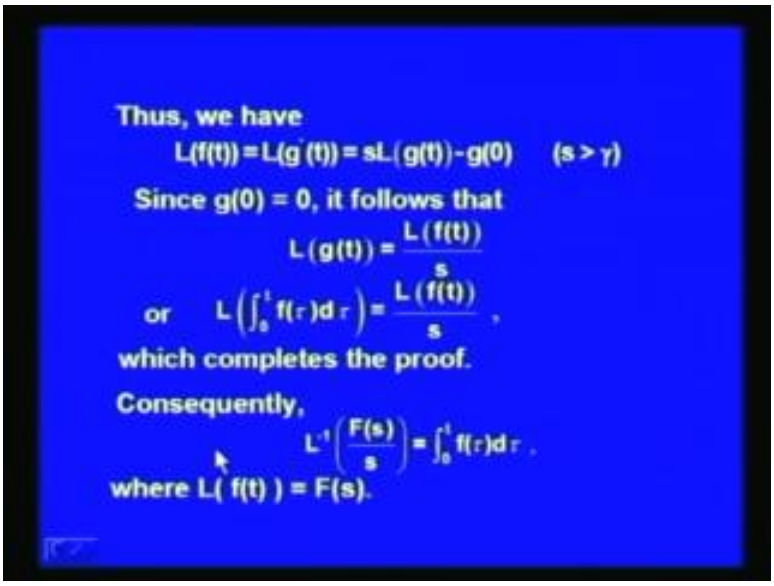

And thus, we get L of f t equal to L of g dash t and L of g dash t is equal to s times L of g t minus g 0 by the theorem 1. So, this will give us since g 0 equal to 0, it follows that L of g t is equal to L of f t over s or we may say L of integral 0 to t f tau d 2 is equal to l of f t over s which completes the proof of the theorem. Consequently L inverse of F s over s L of f t is equal to F s, so L inverse of F s over s is equal to integral 0 to t f tau d tau, that is the inverse Laplace transform of F s over s gives integral over 0 to t f tau d tau.

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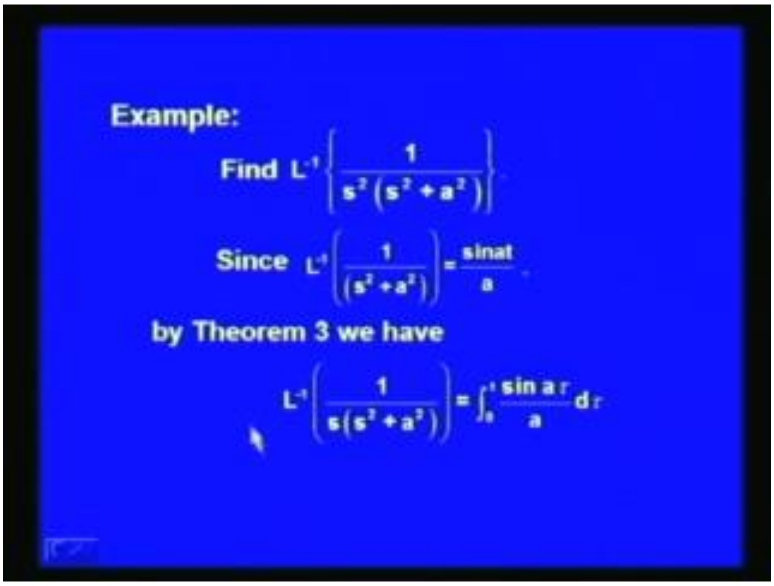

Now, let us apply this theorem to find the inverse Laplace transform of L over s square into s square plus a square. We know that the inverse Laplace transform of L over s square plus a square is sin a t over a. So, by the previous theorem that is theorem 3 we have inverse Laplace transform of F s over s F s we take as 1 over s square plus a square. So, l inverse of f s over s that is 1 over s into s square plus a square will be equal to integral of sin a tau over a d tau when integral over 0 to t sin a tau over a d tau.

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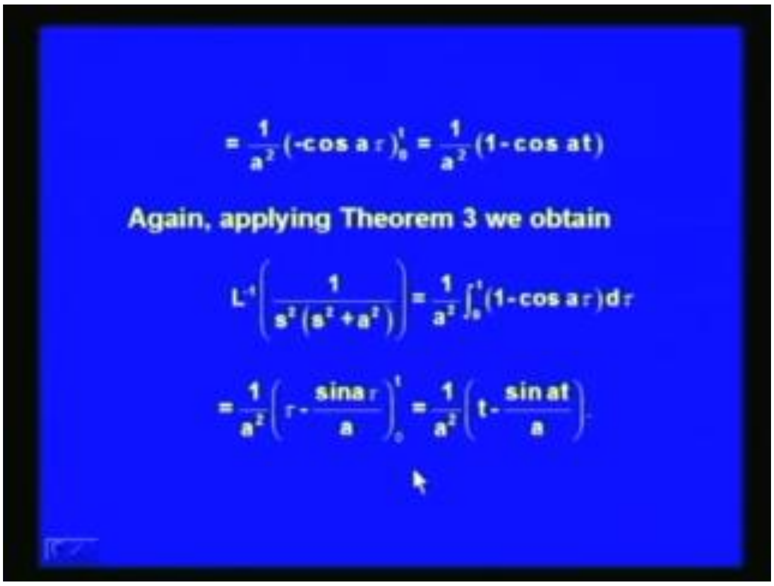

And, when we evaluate the integral and substitute the limit the right hand side becomes 1 by a square into minus cos a tau evaluated at 0 and t, so this is equal to 1 by a square 1 minus cos a t. Now, we again apply theorem 3 to the Function F s equal to 1 over s into s square plus a square. So, then L inverse of F s over s that is 1 over s square into s square plus a square will be equal to 1 by a square integral 0 to t 1 minus cos a tau d tau and when we integrate 1 minus cos a tau with respect to tau what we get is tau minus sin a tau over a and the when we evaluate it at 0 and t and substitute and multiply by 1 by square 1 by a square we get 1 by a square into t minus sin a t over a. So, this theorem helps us in finding the inverse Laplace transform of many new Functions once the Laplace transform of some other Functions are known.

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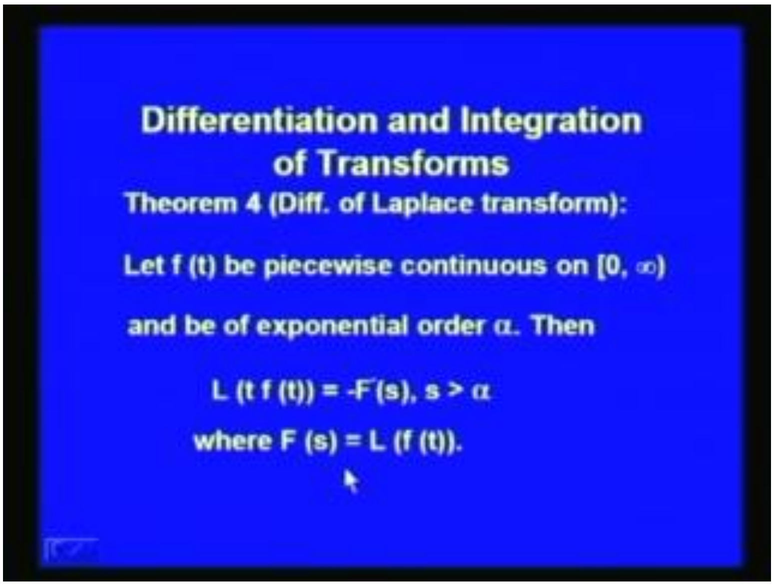

Now, let us study differentiation and integration of transforms, so first theorem on differentiation of Laplace transform is the following. Let f t be piecewise continuous on the interval 0 to infinity and be of exponential order alpha. Then L of t f t is equal to minus F dash s whenever s is greater than alpha, so this theorem tells us that whenever we differentiate the Laplace transform of a certain Function f t with respect to s. Then the function f t gets multiplied by t that is Laplace transform t f t becomes minus Laplace whenever we differentiate the Laplace transform by function f t, the original function f t gets multiplied by minus t. So, we get l of f t equal to minus f dash s whenever s is greater than alpha, F s here has been we know already is L of f t.

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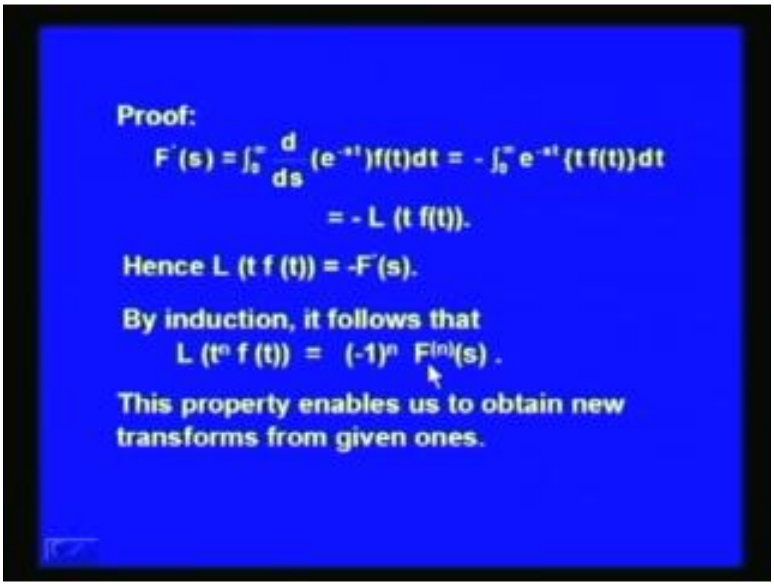

Let us prove this theorem, we can write f dash s equal to d over d s of F s that is d over d s of integral 0 to infinity e to the power minus s t into f t d t. So, d over d s can be brought inside the integral 0 to infinity d over d s of e to the power minus s t into f t d t because s n t are independent of each other and this will be equal to when we differentiate with respect to s. So, this will be equal to e to the power minus s t into minus t. So, minus t this minus comes outside the integral and t will combine with the function f t, and then this is nothing but minus Laplace transform of t f t, so minus L of t f t. Hence, L of t f t is equal to minus f dash s.

Now, again we can use induction on n and can show that Laplace transform of t to the power n into f t is equal to minus 1 to the power n into f n s. When you take n equal to 1 here, then the result follows from this theorem f t equal to minus f dash s, so assuming that the result holds for n we can easily show it for n plus 1.

Now, this property of the Laplace transform enables us to obtain new transforms from the given ones, that is if we know certain Laplace transform F s by differentiating it. We can get new transforms that is we can we can get the transforms new transforms f dash s f double dash s we jus t have to multiply f t by minus 1 into t to the power n.

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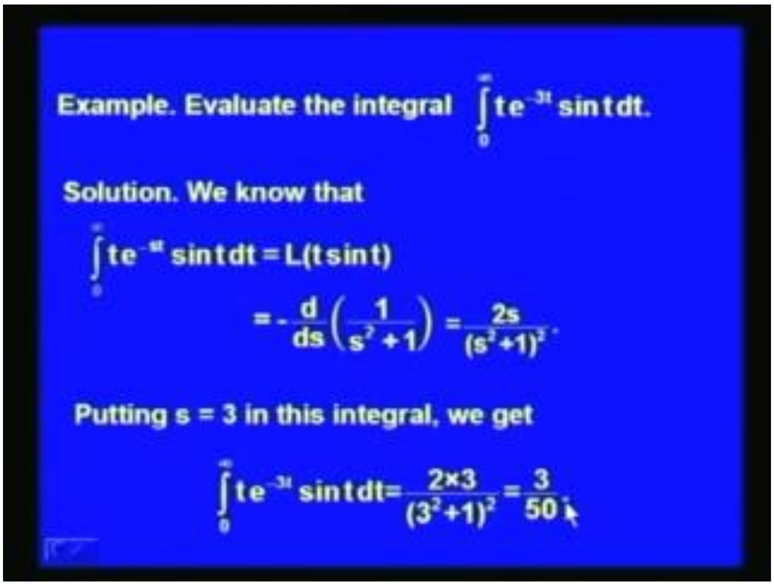

So, let us take an example on this theorem, let us evaluate the integral of integral 0 to infinity t e to the power minus 3 t into sin t d t. If we identify this integral with the definition of the Laplace transform then we note that integral 0 to infinity t into e to the power minus s t sin t d t, this is nothing but the Laplace transform of t sin t, because e to the power minus s t into t sin t it is and then it the integral 0 to infinity.

So, this is Laplace transform of t sin t and Laplace transform of t sin t by the previous theorem and differentiation of Laplace transform is minus d over d s of Laplace transform of f t equal to sin t. And Laplace transform sin t we know it is 1 over s square plus 1, when we differentiated with respect to s and multiplied by minus 1 we get t s over s square plus 1 whole square.

So, now, let us put we want the value of this integral. So, let us put s equal to 3 here in order to get the value of the desired integral. So, putting s equal to 3 here, what we get is integral 0 to infinity t e to the power minus 3 t sin t d t equal to 2 into 3 over 3 square plus 1 whole square which is equal to 3 over 50.

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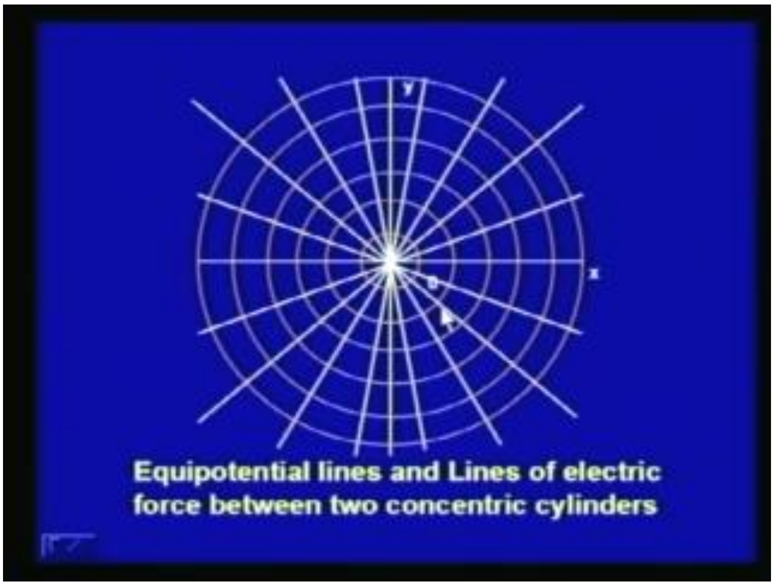

So, they are orthogonal trajectories of each other.

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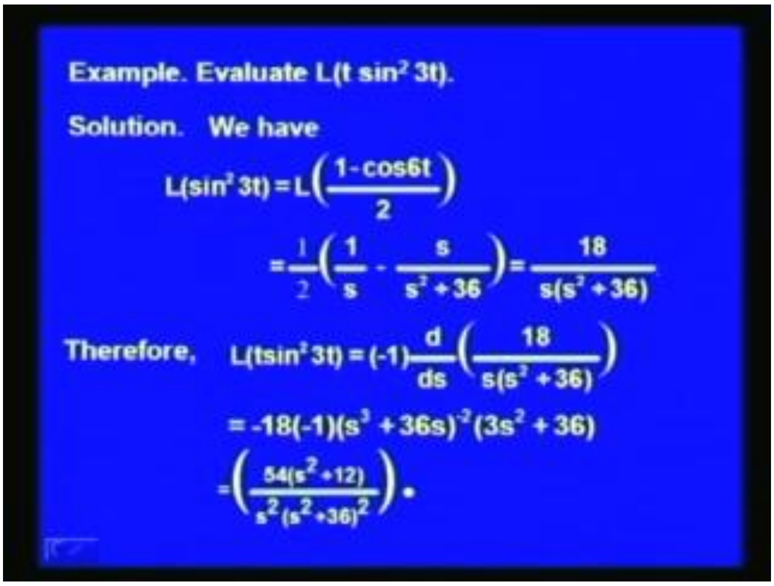

Likewise we will study example based on the last theorem let us s ay evaluate the Laplace transform of t into sin square 3 t. Let us first note that the Laplace transform of sin square 3 t is same as Laplace transform of 1 minus cos 6 t by 2 and using the linearity property of the Laplace operator L we can write it as half of Laplace transform of 1 which is 1 by s minus Laplace transform of cos 6 t which is s over s square plus 36. So, when we simplify this we get the right hand side equal to 18 over s into s square plus 36.

And now, if we apply the theorem on derivative differentiation of Laplace transform then we get Laplace transform of t into sin square 3 t equal to minus of d over d s of 18 over s times s square plus 36, which is equal to minus 18 into minus 1 s cube plus 36 raise to the power minus 2 into 3 s square plus 36. Simplification this becomes 54 into s square plus 12 over s square into s square plus 3 6 whole square.

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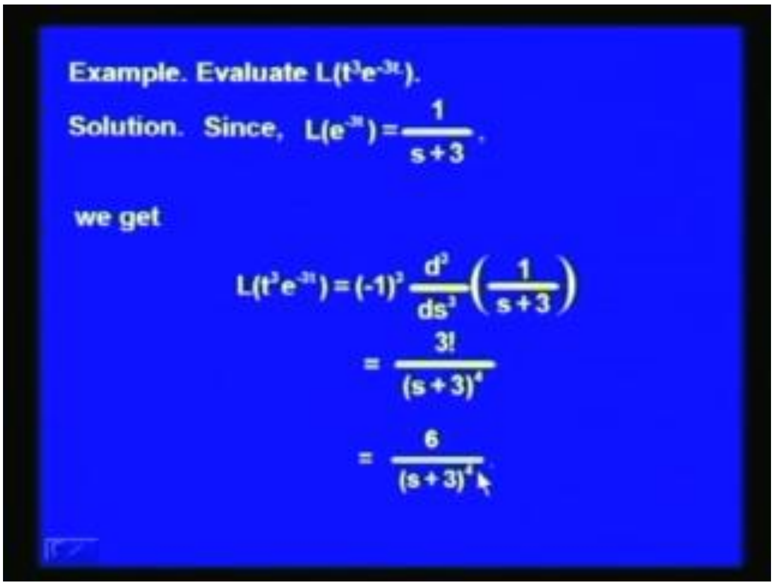

Now, let us evaluate the Laplace transform of t to the power 3 into e to the power minus 3 t. We know that Laplace transform of e to the power minus 3 t is 1 over s plus 3 and therefore, by using the last theorem we get the Laplace transform of t to the power 3 into e to the power minus 3 t as minus 1 to the power 3 into d cube over d s cube 1 over s plus 3 which is equal to 3 factorial over s plus 3 raise to the power 4 and which is equal to 6 over s plus 3 raise to the power 4.

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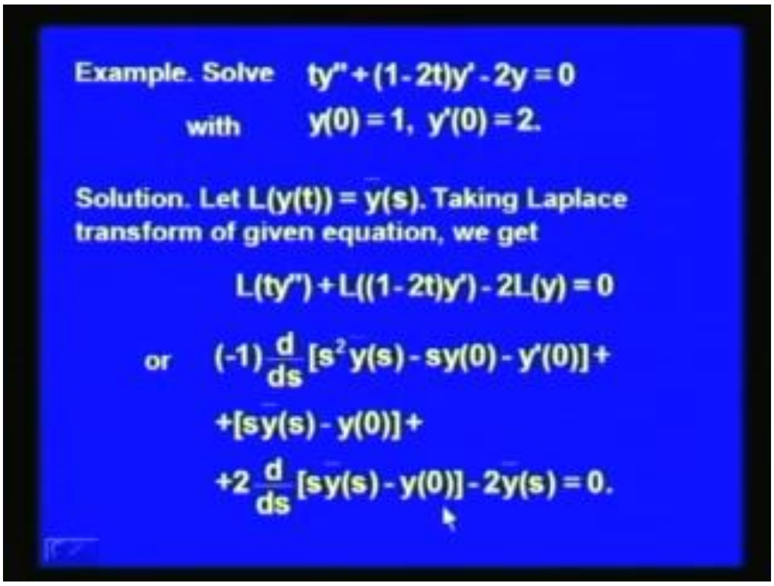

Now, let us solve a differential equation of second order, let us consider the differential equation t y double dash plus 1 minus 2 t into y dash minus 2 y equal to 0 with the given initial condition as y 0 equal to 1 and y dash 0 equal to 2. Let us denote the Laplace transform of y t y y bar s and then when we take the Laplace transform of the given differential equation we will get a Laplace transform t y dash t y double dash plus Laplace transform of 1 minus 2 t into y dash minus 2 times Laplace transform of y equal to Laplace transform of 0 which is equal to 0. This follows using the linearity of the operator l.

Or now Laplace transform of t y double dash is equal to minus 1 into d over d s of Laplace transform of y double dash by the previous theorem. So, we get Laplace transform y double dash as s square into y bar s minus s y 0 minus y dash 0 plus Laplace transform of y dash, we are using the linearity here linearity property the operator l, so firs t if I write the Laplace transform y dash here which is s y bar s minus y 0.

Then, we have minus 2 times Laplace transform of t y dash and Laplace transform of t y dash is minus 1 into d over d s of Laplace transform of y dash which is s y bar s minus y 0. So, Laplace transform of minus 2 t y dash becomes plus 2 times d over d s of s y bar minus y 0 minus 2 times Laplace transform of y is y bar. So, we get minus 2 y bar s equal to 0.

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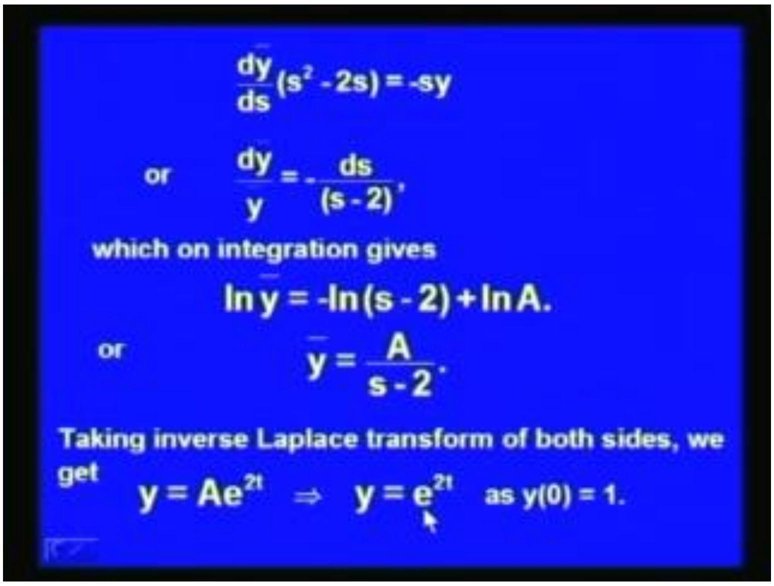

And, on simplification what we get is d y bar over d s multiplied by s square minus 2 s equal to minus s y using the initial conditions. So, we can now separate the variables y bar and s and we get d y bar over y bar equal to minus d s over s. Now, on integration what we get is l n y bar equal to minus l n s minus 2 plus l n A which gives us y bar equal to A over s minus 2. And now, if we take the inverse Laplace transform of this equation we will get inverse Laplace transform of y bar s by t and the right side will be using linearity property a times inverse Laplace transform of 1 over s minus 2 which will be e to the power 2 t.

So, y becomes equal to A times e to the power 2 t, now let us make use of the initial condition that is y 0 is equal to 1. The value of a turns out to be equal to 1 and we get the solution of the given differential equation as y equal to e to the power 2 t.

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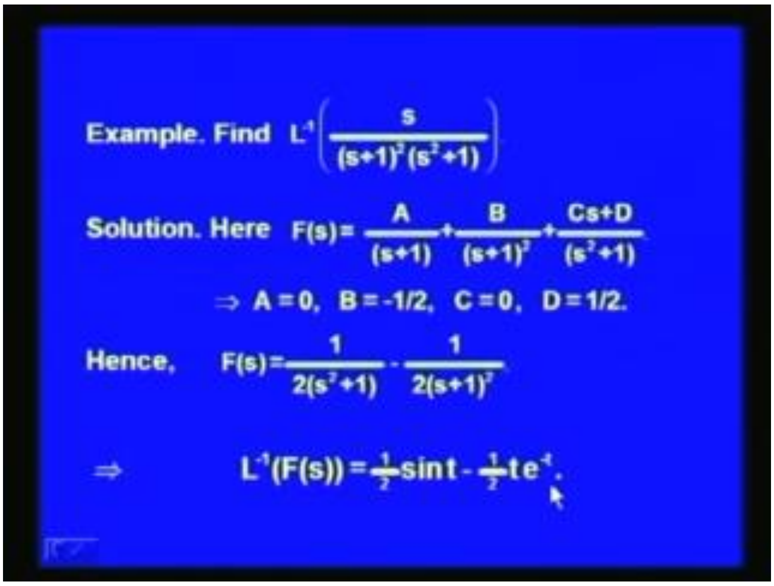

Next, we find the inverse Laplace transform of s over s plus 1 whole square into s square plus 1. So, let us denote by F s s over s plus 1 whole square into s square plus 1, when we break it into when we break F s into its partial fractions F s is we taken as s over s plus 1 whole square into s square plus 1 here. So, when we break F s into its partial fractions we write it as a over s plus 1 plus b over s plus 1 whole square plus c s plus d over s square plus 1.

And, when we take the LCM here, and then use F s equal to s over s plus 1 whole square into s square plus 1 and evaluate the values of the constants a b c d what we get is a is equal to 0 b equal to minus half c equal to 0 and d equal to half. And hence, the value of F s is equal to after making F s into its partial fractions we get F s as 1 over 2 into s square plus 1 minus 1 over 2 into s plus 1 whole square.

So, now let us take inverse Laplace transform of F s and L inverse of F s is equal to then half of L inverse of 1 over s square plus 1 which is sin t minus half of L inverse of 1 over s plus 1 whole square. Now, we know that Laplace inverse Laplace transform of 1 over s plus 1 is e to the power minus t.

So, when we differentiate 1 over s plus 1 with respect to s that is d over d s of 1 over s plus 1 becomes minus 1 over s plus 1 whole square. And that is the inverse Laplace transform of that is equal to t into e to the power minus t inverse Laplace transform of 1 over s plus 1 whole square is t into e to the power minus t by the previous theorem. So, we get the inverse Laplace transform of the desired Function of s s half sin t minus half t into e to the power minus t.

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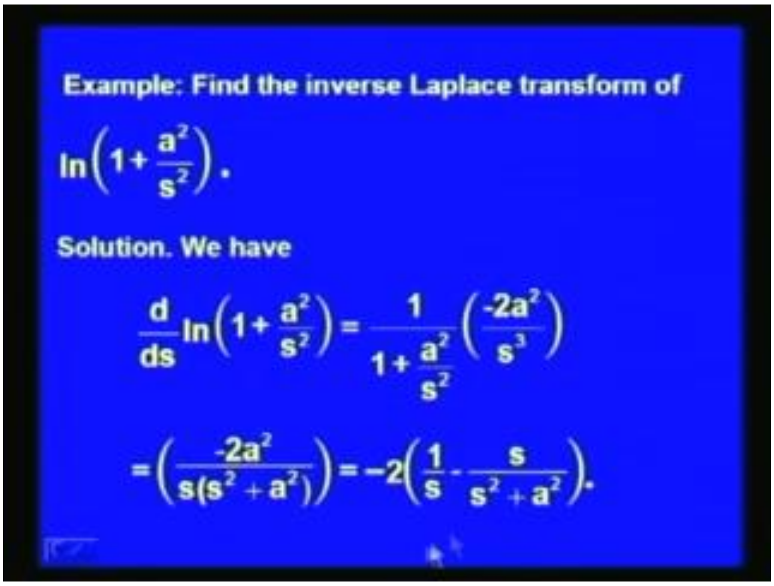

Now, let us find the inverse Laplace transform of the logarithmic function l n 1 plus a square by s square, what we will do is, we will firs t take the derivative of l n of 1 plus a square by s square with respect to s. So, we see that d over d s l n 1 plus a square by s square is 1 plus 1 over 1 plus a square by s square into minus 2 a square by s cube, which on simplification gives us minus 2 a square over s into s square plus a square when we break it into its partial fractions what we get is minus 2 times 1 over s minus s over s square plus a square.

So, if we now take the inverse Laplace transform on both sides of this equation, we will get L inverse of d over d s of l n 1 plus a square by s square equal to minus 2 times L inverse of 1 by s as 1 minus l inverse of s over s square plus a square s cos a t.

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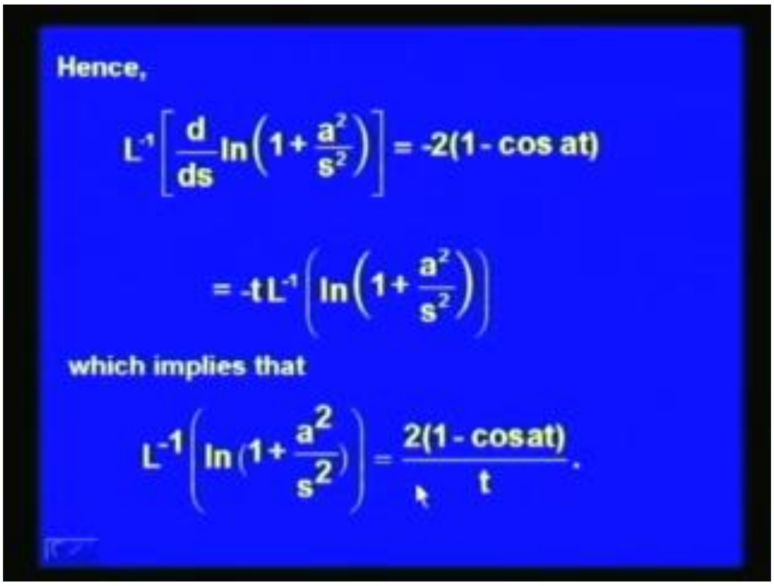

So, we get L inverse of d over d s of l n 1 plus a square by s square as minus 2 times 1 minus cos a t and L inverse d over d s of F s we know l inverse of d over d s of F s is minus t times f t. So, we get l inverse of d over d s of l n 1 plus a square by s square as minus t times L inverse of F s, this is minus t times of f t. So, minus t times L inverse of l n 1 plus a square by s square, so what we get is minus t times L inverse l n 1 plus a square by s square equal to minus 2 times 1 minus 1 minus cos a t. So, this implies that an inverse if l n 1 plus a square by s square is equal to 2 times 1 minus cos a t over t.

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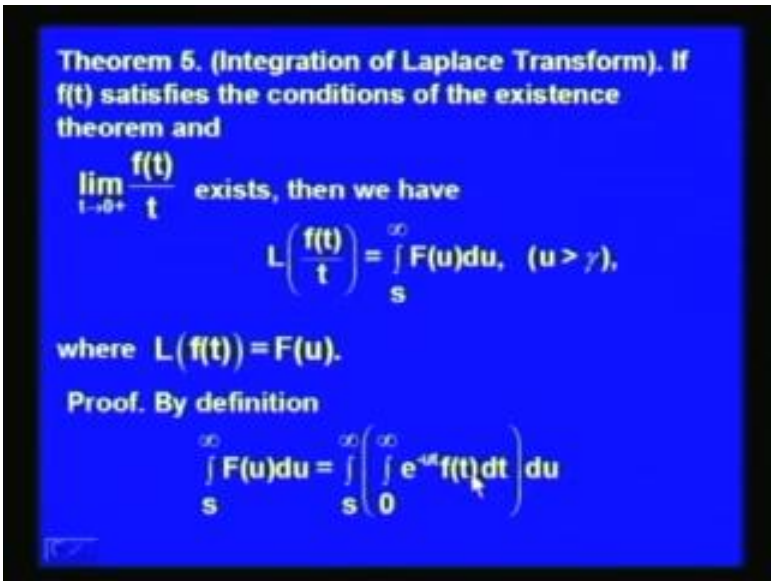

Now, let us study what happens when we integrate the Laplace transform of a given function we say we see in the theorem that when we integrate the Laplace transform of a given function f t. Let us denote the Laplace transform of f t by F u. Then what we get is that it the function f t is divided by t we get the Laplace transform of f t by t equal to integral s to infinity F u d u. So, this theorem tells us that if f t satisfies the conditions of the existence theorem and more over limit of f t over t as t goes to 0 from the right exist. Then we have Laplace transform of f t over t equal to integral over s to infinity Fu du where u is greater than gamma and l f t is equal to Fu. So, let us see what is the value of integral s to infinity Fu d u integral s to infinity F u d u integral s to infinity F u d u when we put the value of F u here becomes integral s to infinity integral 0 to infinity e to the power minus u t into f t d t d u.

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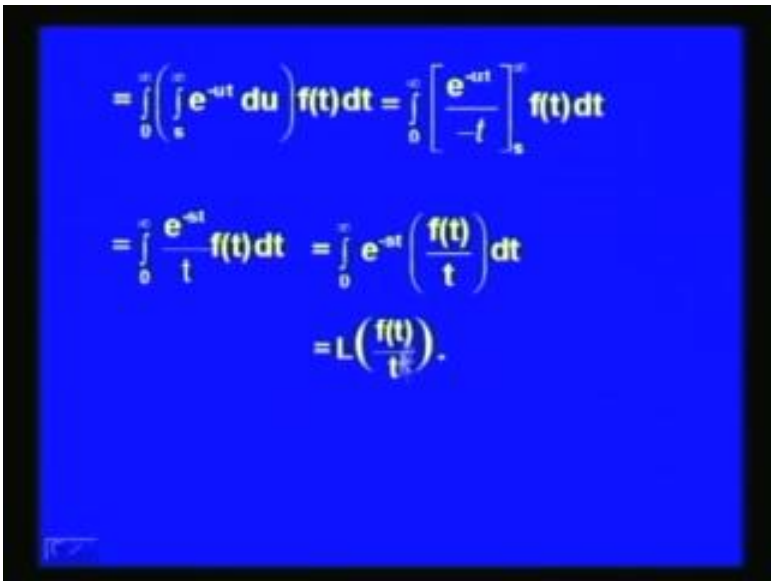

And, which is equal to integral 0 to infinity integral we change the order of integration here. So, integral 0 to infinity integral s to infinity e to the power minus e t u t d u into f t d t which is equal to integral 0 to infinity integral of e to the power minus u t is e to the power minus u t over minus t because they are integrated here with respect to u and the limits are s and infinity into f t d t.

Now, when t goes to infinity, because u is greater than gamma and gamma in the existence theorem it can be taken to be positive. So, e to the power minus u t goes to 0 as t goes to infinity and when we put the lower limit, the value of this expression becomes e to the power minus s t over t. And so we get the right hand side as integral 0 to infinity e to the power minus s t over t into f t d t which can be written in the form of the Laplace transform of f t over t that is integral 0 to infinity e to the power minus s t into f t over t d t. So, we get the right hand side this equal to l of f t over t.

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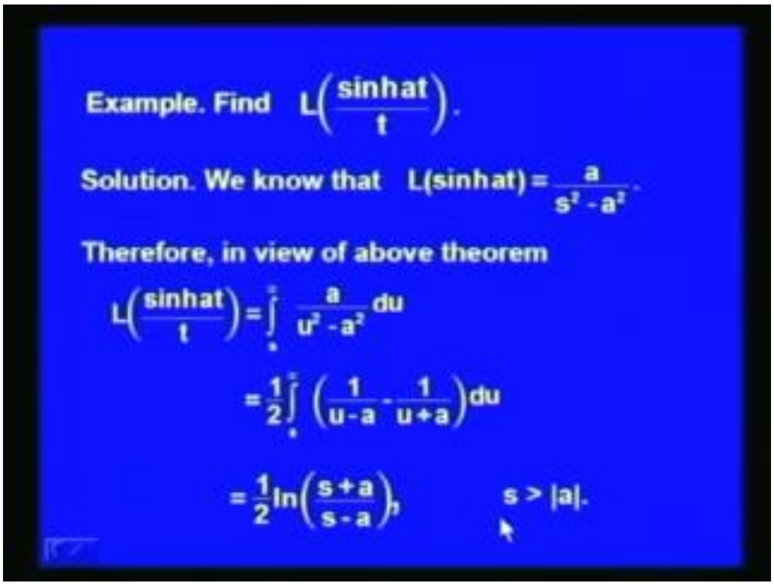

Let us, take an example based on this theorem let us find the Laplace transform of sin hyperbolic a t over t. We know from our previous lecture on Laplace transformation that Laplace transform of sin hyperbolic a t is a over s square minus a square. Therefore in view of the above theorem Laplace transform of sin hyperbolic a t over t will be equal to integral s to infinity Laplace transform of sin hyperbolic a t which is a over u square minus a square. So, we integrate a over u square minus a square with respect to u and substitute the limits as an infinity to get the desired Laplace transform.

So, when we integrate in order to integrate this we break it into partial fractions and the partial fractions are half of 1 by u minus a minus 1 by u plus a, when we integrate this with respect to u, what we will get half times l n u minus a minus l n of u plus a. And the limits of integration are s n infinity, when we would like to use the upper limit that is u goes to infinity l n u minus a minus l n u plus a will tend to infinity varies infinities. So, we break we write it in the form of l n of u minus a over u plus a. And dividing by u then it can be writ10 as l n of 1 minus a by u over 1 plus a by u, so when u will tend to infinity it will tend to l n 1 which will go to 0.

So, this expression after integration as u goes to infinity will tend to 0 and when we put the lower limit the value of this integral will come out to be half of l n s plus a over s minus a whenever s is greater than mod of a, the condition that s is greater than mod of a comes from here. Because l n s plus a exists when s is greater than minus and l n s minus a exists when s is greater than a, so s must be greater than mod of a.

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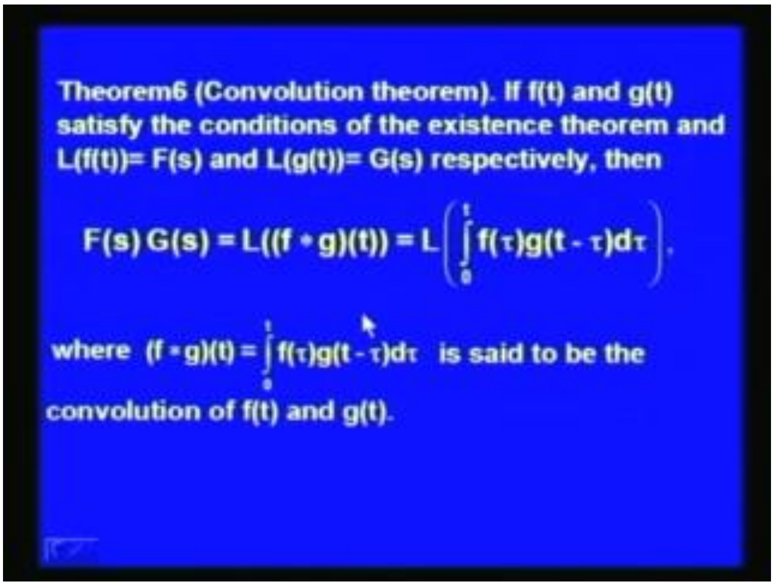

Now, we study a very important property of the Laplace transforms which is known as the convolution theorem. If we know the Laplace transform of two functions f t and g t, we can find the Laplace transform, then wants to know what is means the Function whose when Laplace transform gives us the product of the Laplace transform of f t and g t. It will turn out that that function if we call that function as f t is nothing but the convolution of f t and g t and the convolution of f t and g t which we will denote f s tar g t is defined as integral 0 to t f tau g t minus tau d tau. This theorem is extensively use in the study on integral equations and many other problems of engineering, so it has been lots of applications.

So, let us assume that l of f t is F s l of g t is g s and f t and g t satisfy the conditions of the existence theorem. Then we see that F s into g s, the product of Laplace transform of f t and g t is nothing but the Laplace transform of the convolution of f and g that is f star g t and f star g t is by definition integral 0 t f tau g t minus tau d tau. This f star g satisfy the evolution of f t and g t.

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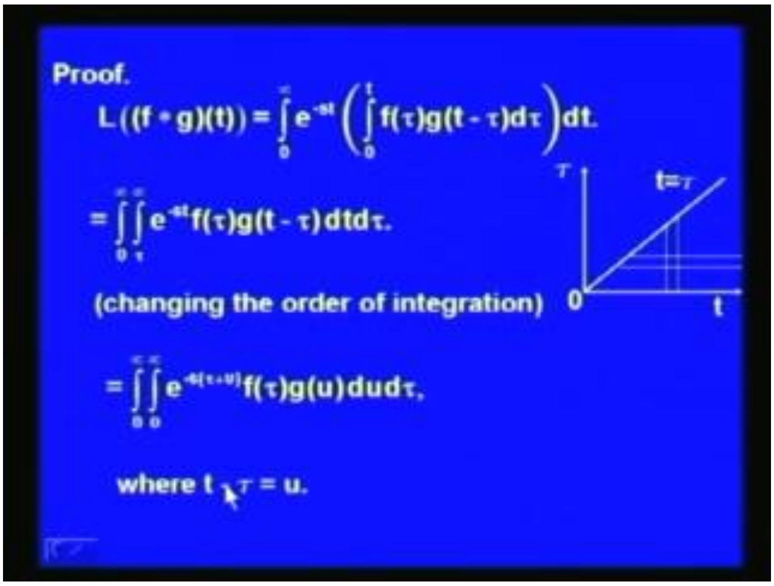

Let us, look at the proof of this theorem L of f star g and t will be equal to integral 0 to infinity e to the power minus s t into f star g and t and f star g and t by definition is integral 0 to t f tau g t minus tau d tau and then d t. So, now we want to change the order of integration here, the inner integral with respect to tau and the outer integral is with respect to t. So, let us look at t tau plane based our real this horizontal axis is t axis the vertical axis is tau axis and this is the line t equal to tau.

So, here we are integrating first with respect to tau and then with respect to t tau varies from 0 to t so; that means, we are taking a and the region over which we are integrating is this 1. So, when we are integrating first with respect to tau and tau varies from 0 to t, it means that we are taking a vertical s trip over this region. So, because for this vertical s trip tau is from 0 to tau equal to t and t varies from 0 to infinity in this part of the plane.

So, when we want to change the limits of integration the order of integration, then now we will have to take a horizontal strip here. So, this we take the horizontal strip arbitrary horizontal strip in this area and then the limits of integration for t first we shall write the limits of integration for t. So, t varies from tau to infinity and then we write the limits of integration for tau, so tau varies from 0 to infinity. So, we change the order of integration here, first we have the limits of integration for t varies from tau to infinity and then we have the limits of integration for tau varies from 0 to infinity.

Now, let us put t minus tau equal to u here when we put t minus tau equal to u here, the limits of integration that is t varies from tau to infinity change to t varies from 0 to when u varies from 0 to infinity and we get the integrate as e to the power minus s tau plus u into f tau g u d u d tau. The limits of integration for u are 0 infinity for tau they are again 0 infinity. And here, we can separate the variables tau and u we can write e to the power minus s tau plus u as e to the power minus s tau into e to the power minus s u. So, let us group the terms which depend on tau and which and the terms which depend on u and separate the two integrals we can we have the following.

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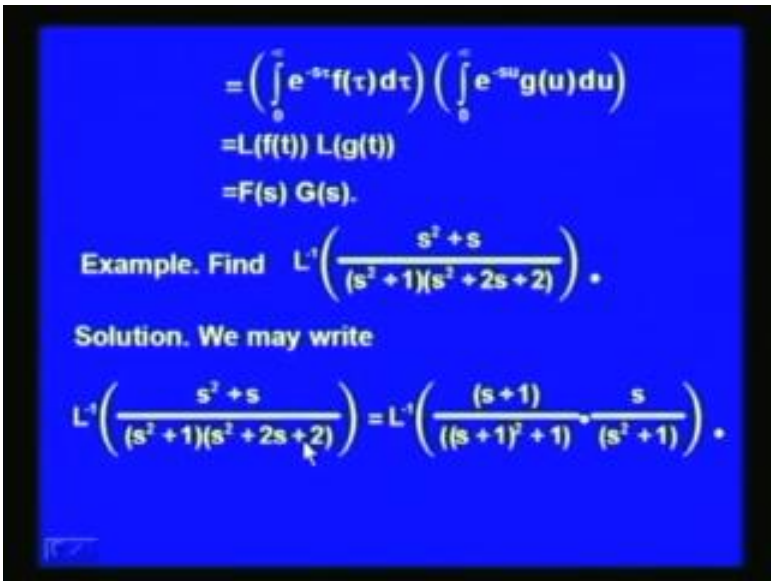

We have we the last double integral may be equal to is then equal to the product of these simple integrals, integral 0 to infinity e to the power minus s tau f tau d tau and then integral 0 to infinity e to the power minus s u g u d u. This integral is the Laplace transform of f t and this integral is Laplace transform of g t and which is equal to F s into g s. Let us, apply this convolution theorem to find the inverse Laplace transform of the Function s square plus over s square plus 1 into s square plus 2 s plus 2. The inverse Laplace transform of this may be this Function of s may be regarded as product of two functions s over s square plus 1 s over s square plus 1 and then another 1 as s plus 1 over s square plus 2 s plus s.

So, we can write the inverse Laplace transform of the given function s equal to L inverse of s plus 1 over s plus 1 whole square plus 1 into s over s square plus 1. So, 1 we will take as F s and other Function of s will take as g s and then we will we will be writing L inverse of F s into g s equal to convolution of f and g by the convolution theorem.

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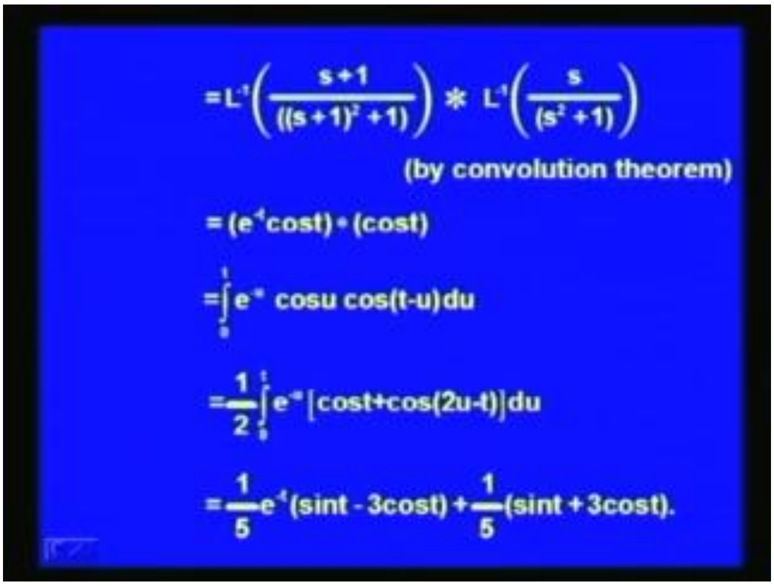

So, we have convolution of f and convolution of g this is L inverse of s plus 1 over s plus 1 whole square plus 1 star L inverse of s over s square plus 1 this is star denotes the convolution. So, this is f t and this is g t by convolution theorem. So, now, inverse Laplace transform of s plus 1 over s plus 1 whole square plus 1 using the first shifting theorem with which we have studied in the previous lecture on Laplace transformation gives us e to the power minus t into cos t. We know that Laplace transform of s over s square plus 1 is cos t.

And then you multiply e to the power minus t to cos t, then its Laplace transform gives s plus 1 over s plus 1 whole square plus 1. So, inverse Laplace transform of this function of s is e to the power minus t cos t and inverse Laplace transform of s over s square plus 1 we know it is cos t. So, this is equal to integral 0 to infinity e to the power minus u cos u into cos t minus u d u.

Now here, we have two functions one is f and another one is g and we write it as integral 0 to t F u g t minus u d. So, which function is to be chosen as F u and which is to be chosen as g u depends on how conveniently we can evaluate this integral, because f star g is always equal to g star u convolution of f and g same as convolution of g and f. So, we are free to choose any function out of these two as f and the other one as g. And this, Further equal to half of integral 0 to t e to the power minus u cos t plus cos 2 u minus t d u which is equal to after integration we get 1 by 5 e to the power minus t sin t minus 3 cos t plus 1 by 5 sin t plus 3 cos t.

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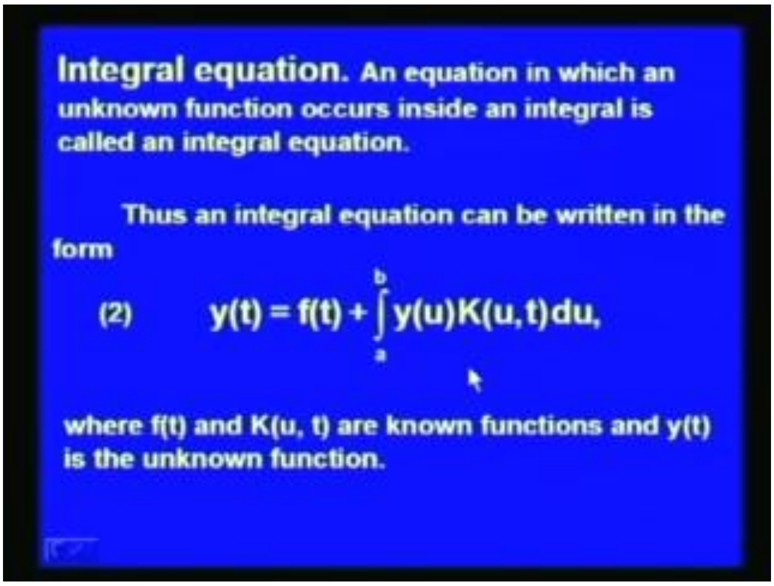

Now, we shall be applying this convolution theorem to find a solution of an integral equation. Let us, see what is an integral equation an integral equation is an equation in which an unknown function occurs inside an integral, thus an integral equation may be written as y t equal to f t plus integral a to b y u into k u t u, where f t and k u t are known functions and y t is the unknown Function k u t is called the kernel function.

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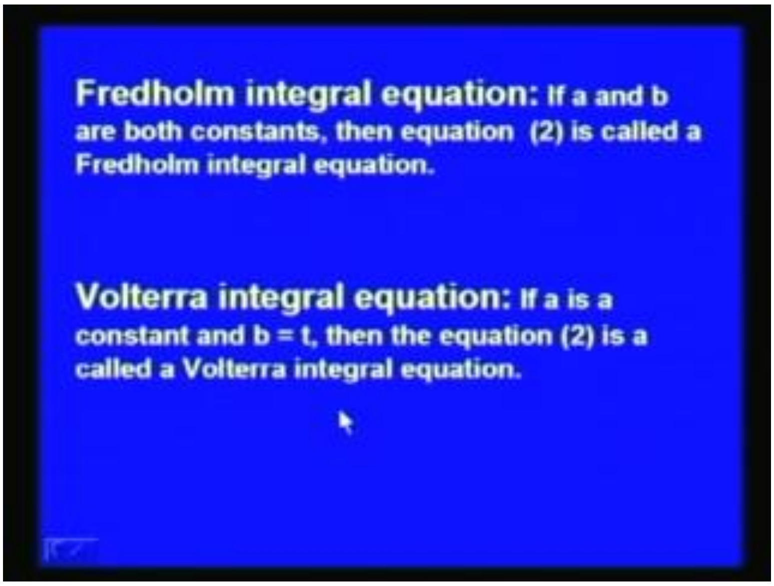

Now, this if a and b are both constants then the integral equation 2 is called a Fredholm integral equation and if a is a constant b and b equals the variable t then the equation 2 is known as a Volterra integral equation.

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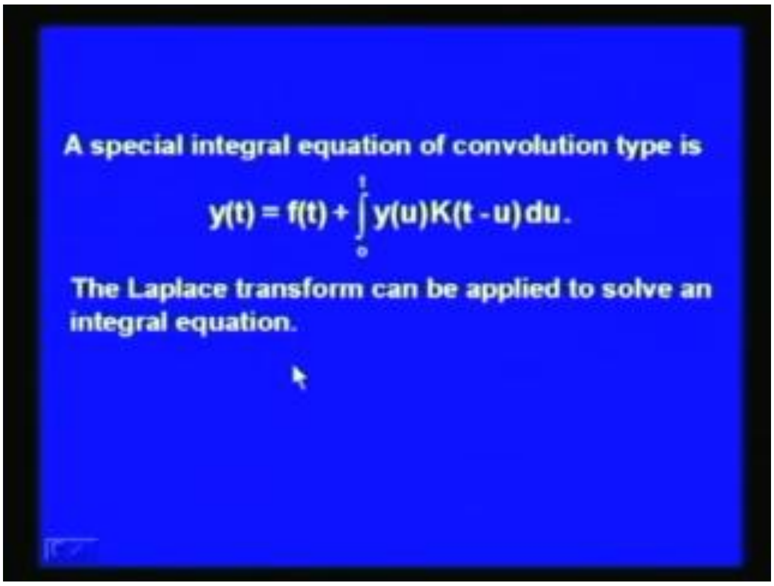

Now, a special integral equation of convolution type is when Volterra integral equation where you take a equal to 0. So, y t is equal to f t plus integral 0 to t y u into k t minus u d u here we take k t u as k t minus u and a as 0. So, it is a special kind of Volterra integral equation we are going to find the Laplace transform we are going to find a solution of an equation of this type.

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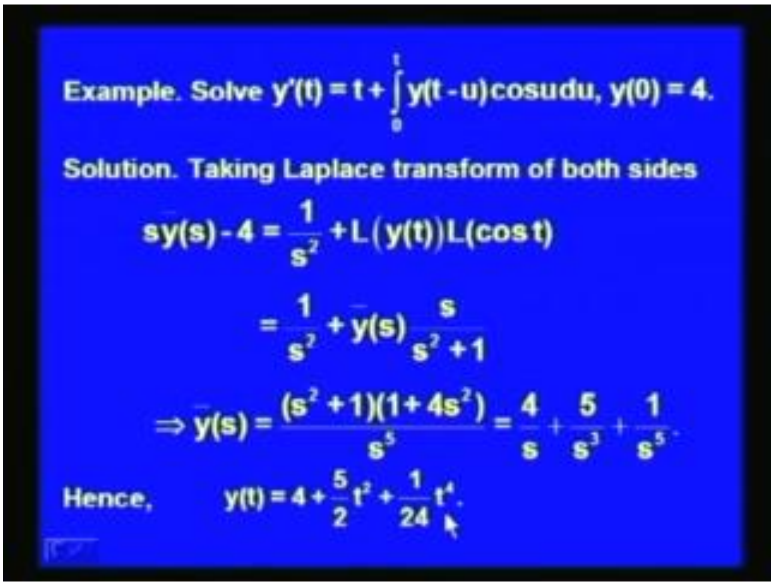

So, let us solve the differential integral equation y dash t equal to t plus integral 0 to t y t minus u into cos u d u. So here, f t is equal to t a is equal to 0 and instead of y here, we have y dash t, y t is the unknown function of t, which we and we are given the initial condition that t equal to 0 y takes the value 4.

So, let us take the Laplace transform of on both sides of this integral equation l of y dash t is equal to s y bar s minus y 0, y 0 is equal to 4 Laplace transform of t is 1 over s square and then the Laplace transform of 0 to t y t minus u cos u d u if we take fu equal to cos u and g u as y t minus u, then we know by convolution theorem that Laplace transform of 0 to t y t minus u into cos u d u is equal to Laplace transform of y t into Laplace transform of cos t.

So, the right hand side of this equation becomes 1 by s square Laplace transform of y t is y bar s and Laplace transform of cos t is s over s square plus 1. Now, we solve this equation for the value of y bar s it turns out that y bar s is equal to s square plus 1 into 1 plus 4 s square over s to the power 5. When we break into its partial fraction, we get the partial fraction as 4 by s plus 5 by s cube plus 1 by s to the power 5. Now, we take the inverse Laplace transform of this equation, inverse Laplace transform of y bar s is y t. And then the right hand side when we take the inverse Laplace transform we get 4 plus 5 into t square by 2, and then t to the power 4 by 4 factorial. So, we get y t equal to 4 plus 5 by 2 t square plus 1 by 24 into t to the power 4.

Now, in our lecture today, we have consider some of the properties of the Laplace transforms like Laplace transform of derivatives and integrals and then the differentiation and integration of Laplace transforms and the convolution theorem. In my next lecture we will be doing some more properties of the Laplace transforms like the second shifting theorem which refers to the shifting on the s axis. If you recall the first shifting theorem refers to the shifting on the t axis the second shifting theorem refers to the shifting of the s axis.

And then in the theorem on shifting on the s axis that is second shifting theorem we shall be making use of a special function known as unit step function which is also called as Heaviside's function. And we will be discussing Dirac delta function which is also known as an impulse function.

So, unit step function and Dirac delta functions have very special importance and the in the study of problems that occur in mechanical and electrical engineering. So, we will be looking some of the applications of those functions in the problems of electrical and mechanical engineering. And we will also be considering Laplace transform of periodic function. So, all these we will be covering in our next lecture.

Thank you.