Mathematics III
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Lecture - 8
Laplace Transformation

Dear viewers, in our talk today we shall be discussing the Laplace Transformation. Laplace transformation is a method for solving ordinary linear differential equations with constant coefficients. Partial differential equations can also be treated by using the Laplace transformation, it has got a tremendous application in the problems on in mechanical and electrical engineering and it is used extensively by the scientist and engineers.

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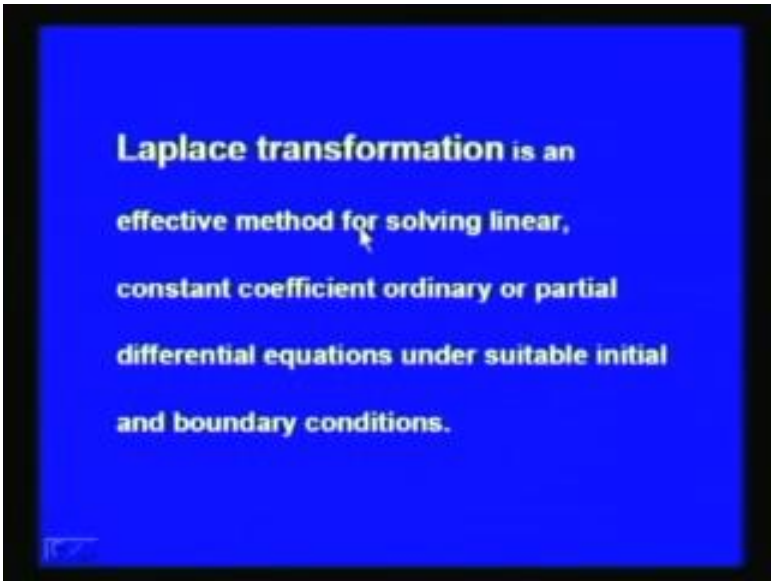

So, Laplace transformation is an effective method for solving linear constant coefficient ordinary or partial differential equations, under the suitable initial and boundary conditions.

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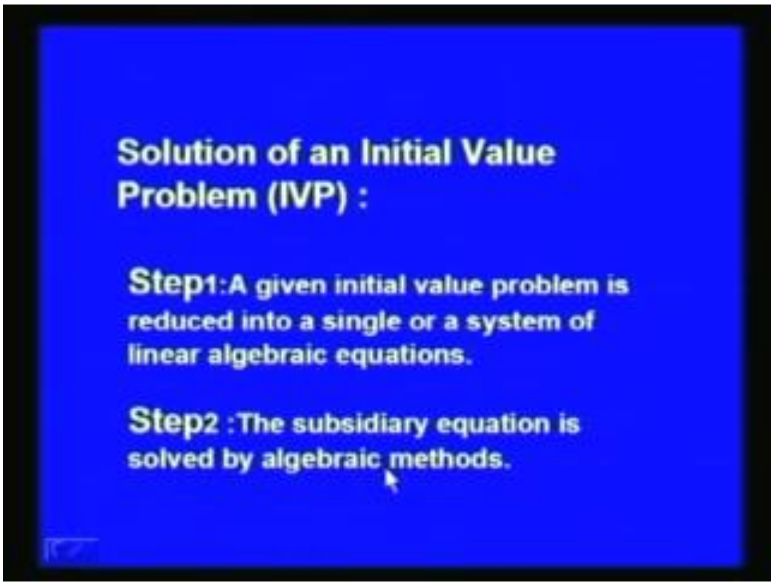

Let us first discuss how do, we find the solution of an initial value problem using the Laplace transformation method. So, the step 1 is first we apply the Laplace transform to be given initial value problem, it will be reduced into a single or a system of linear algebraic equations. If we have a single equation in the initial value problem it will be giving us a single linear algebraic equation and we have a system of differential equations in the initial value problem.

Then, after using the Laplace transform, we will get a system of linear algebraic equations, the algebraic equations are called as the subsidiary equation or subsidiary equations. In the step 2, the subsidiary equation is then solved by purely algebraic methods.

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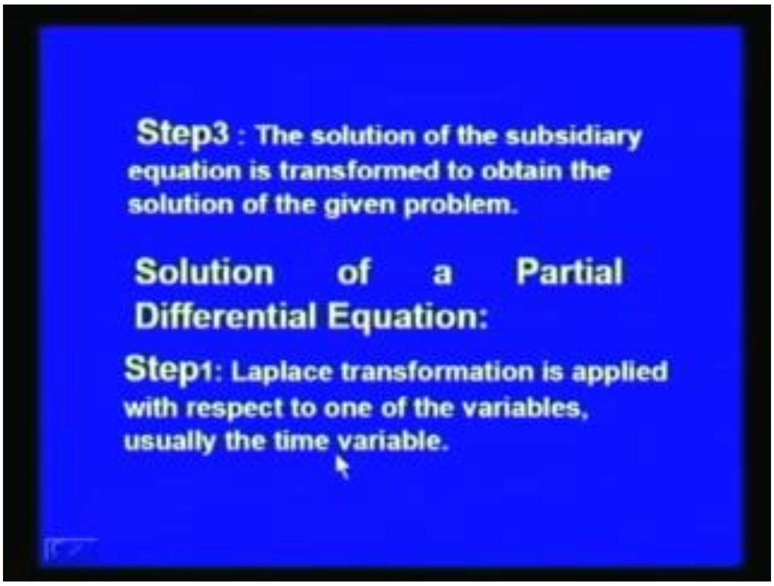

Then, the solution, in the step 3 the solution of the subsidiary equation is transformed to obtain the solution of the given initial value problem, that is we take the inverse Laplace transform of the solution of the subsidiary equation, in order to obtain the solution of the given problem. Let us see how do, we obtain the solution of a partial differential equation. In partial differential equations, first step is we apply the Laplace transformation on the given partial differential equation. It will be applied with respect to one of the two variables usually we take the variable as the time variable.

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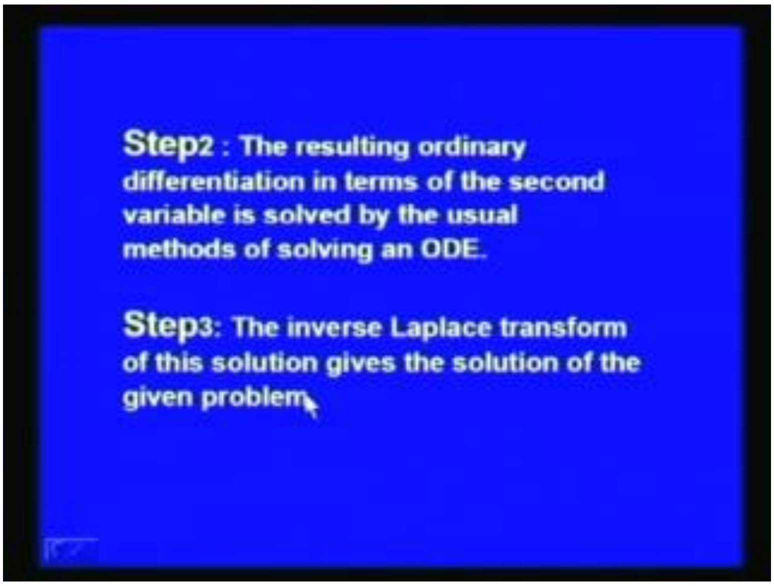

And then the resulting ordinary differentiation in terms of the other variable is solved by using the methods of solving an ordinary differential equation. After getting this solution of the ordinary differential equation so obtained, in terms of the other variable we then take the inverse Laplace transform of the solution, which it gives us the solution of the given boundary value problem.

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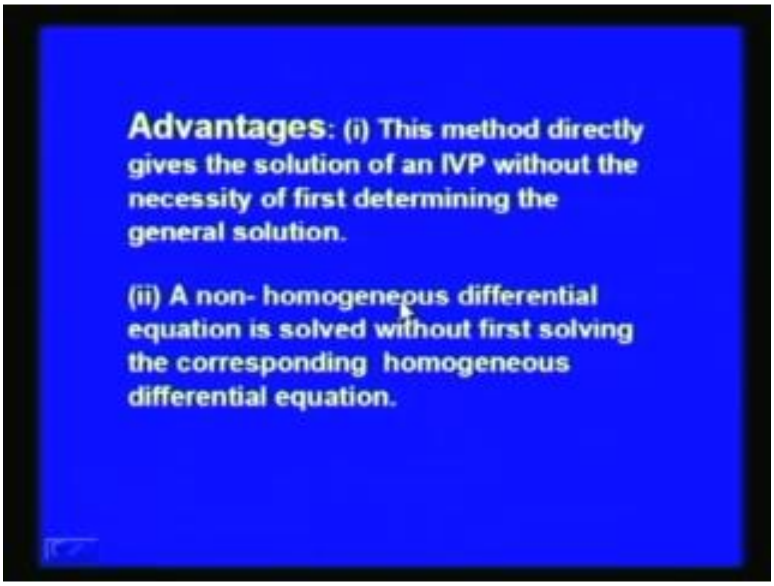

Let us now see, the advantages of the Laplace transformation method, the Laplace transformation method gives us the solution of an initial value problem without the necessity of first determining the general solution. See the when we solve an ordinary differential equation with constant coefficients in the theory of ordinary differential equation, what we do is, we first find the general solution of the given differential equation.

And then use the initial conditions in order to arrive at the solution of the initial value problem. But, in the case of the Laplace transformation method we directly obtain the solution of the initial value problem, we do not have to find the general solution of the problem first. Non homogeneous differential equation can also be solved using the Laplace transformation method. Here, we do not have to solve the corresponding homogeneous differential equation which we do when we solve a differential equation otherwise, a non homogeneous differential equation otherwise.

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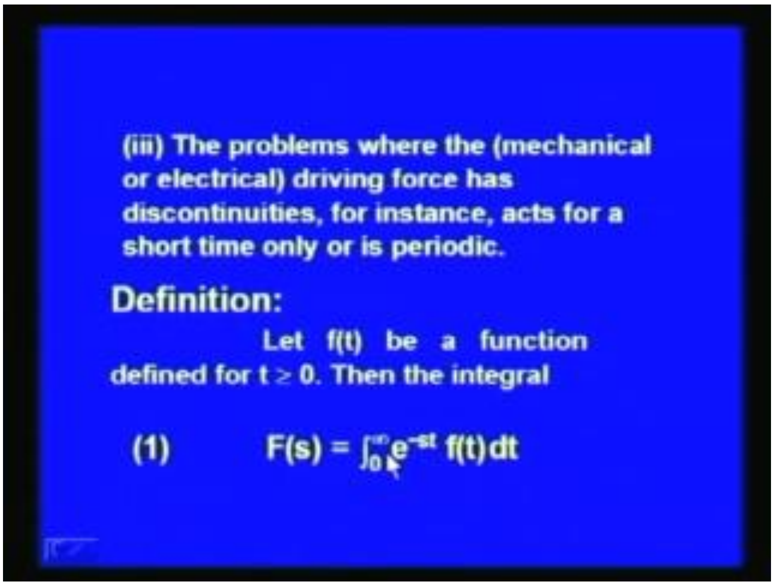

The problems where the driving force has discontinuities say for example, the problems in a simple electric circuit, where when we give the voltage to the circuit from battery or a say it takes for a shorted short time say for the time t, and then it ceases or in the problems where the function is periodic. So, in the problems in mechanical and electrical engineering the Laplace transform is quite useful.

Let us define, what to be mean be a Laplace transform, let us say let ft be a function which is defined for all t greater than or equal to 0; then the integral given by F s equal to integral 0 to infinity e to the power minus s t into f t d t is called the Laplace transform of f t provided the integral in the equation 1 exists.

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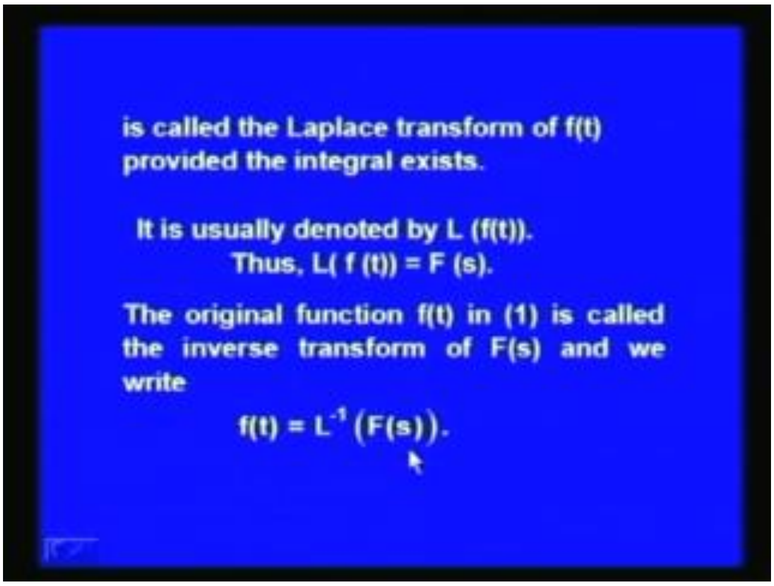

Now, we usually denote this integral by L f t, so we get L of f t equal to F s this L means the Laplace transforms of the function f t. The original function f t in the equation 1 is called the inverse transform of the function F s and we write f t equal to L inverse of F s.

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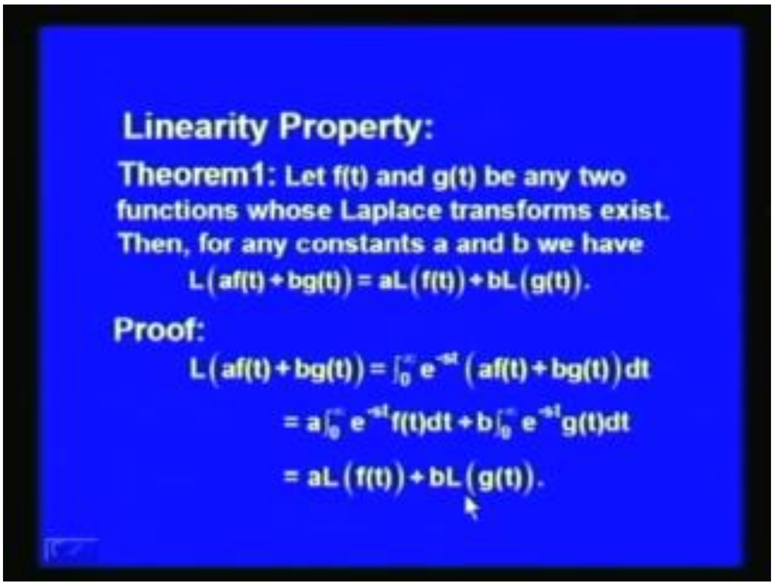

Now, let us look at the linearity property of the Laplace transform, it says that let f t and g t be any two functions whose Laplace transforms exist. Then for any constants a and b could be real or complex constants, we have L of a f t plus b g t equal to a times L f t plus b times L g t.

Let us, look at the proof of this theorem, we can write L of f t plus b g t by definition of the Laplace transform as integral 0 to infinity e to the power minus s t into a f t plus b g t d t; which will be equal to a times integral 0 to infinity e to the power minus s t f t d t plus b times integral 0 to infinity e to the power minus s t into g t d t, and which is equal to a times Laplace transform of f t plus b times Laplace transform of g t.

We have already assumed that the Laplace transform of f t and d t exists, so the integral that are occurring on the right side here integral 0 to infinity e to the power minus s t f t d t exists and integral 0 to infinity e to the power minus s t g t d t exists and thus we have the right hand side as a times L f t plus b times L g.

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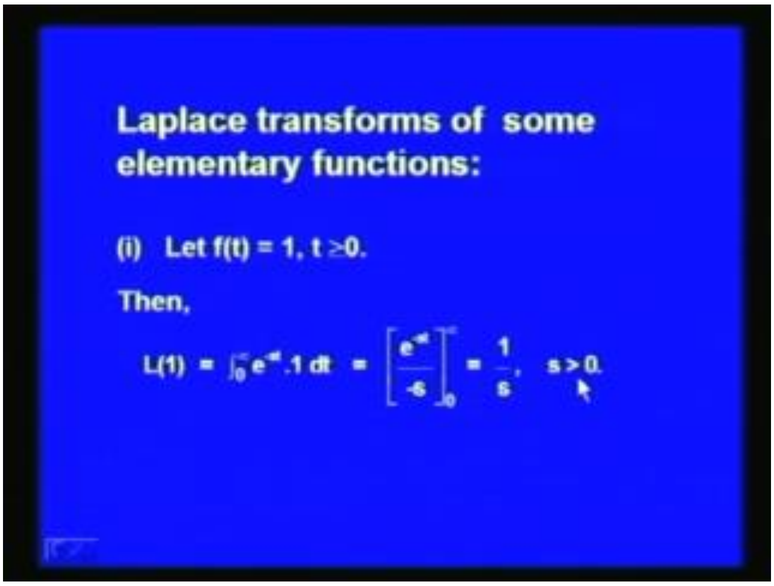

Now, let us discuss the Laplace transforms of some elementary function, let us assume that f t is the constant function 1 for all t greater than greater than or equal to 0. Then Laplace transform of 1 by definition will be equal to integral 0 to infinity e to the power minus s t into 1 d t, the integral of e to the power minus s t is e to the power minus s t over minus s; when t goes to infinity is the upper limit here, when t goes to infinity e to the power minus s t goes to 0, provided we assume that s is greater than 0. So, here the condition on s comes into picture in order to make e to the power minus s t go to 0, we have to assume that s is greater than 0. So, when t goes to infinity this goes to 0 and at the lower limit its value is minus 1 over s and therefore, when we substitute the limits we get the result as 1 over s, so Laplace transform of 1 is 1 over s for all s greater than 0.

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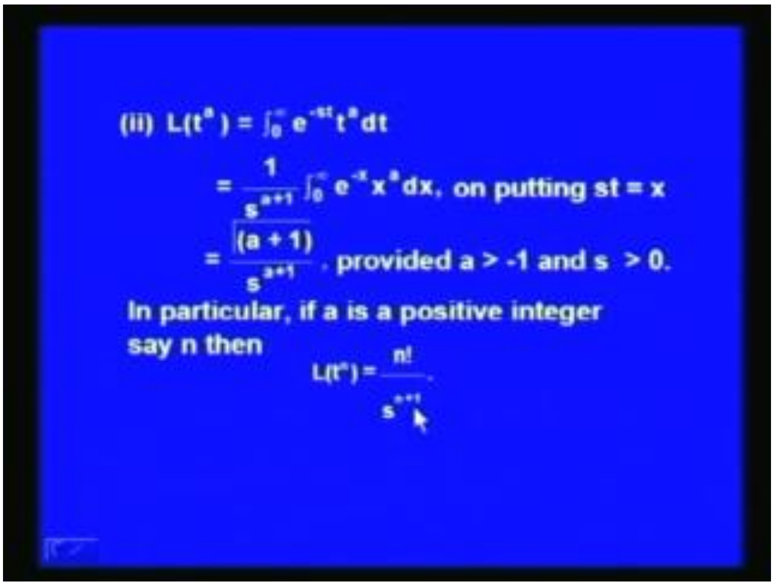

Next let us discuss, the Laplace transform of t to the power a let f t be the t to the power a here, so the Laplace transform of this is integral 0 to infinity e to the power minus s t into t to the power a d t. Let us now put s t equal to x then e to the power minus s t will become e to the power minus x t to the power a will be x to the power a over s to the power a and d t will be d x by s. And therefore, we will have the integral we have we have the right hand side as 1 over s to the power a plus 1, integral 0 to infinity e to the power minus x x to the power a d x where we have assumed that a this s is greater than 0. Because, when t goes to infinity the upper limit when we have change the variable from t to x will become infinity only when we assume s to be greater than 0. So, here the assumption is that s is greater than 0.

Now, this is equal to gamma a plus 1 over s to the power a plus 1 if you recall the definition of gamma function, then 0 to infinity, integral 0 to infinity e to the power minus x x to the power a x to the power a we can write as x to the power a plus 1 minus 1. So, integral 0 to infinity e to the power minus x x to the power a d x becomes gamma of a plus 1 and therefore, we have the right hand side as gamma of a plus 1 over s to the power a plus 1.

And, we know that the gamma function or the integral 0 to infinity e to the power minus x into x to the power a d x is gamma of a plus 1 provided a is greater than minus 1. So, we the condition on a that it must be greater than minus 1 and we have the condition on s that it must be positive this can be said that s is positive we had used here in order to have the limit upper limit of s as infinity.

Now, let us discuss a particular case of this example let us take a to be a positive integer say a equal to n. Then Laplace transform of t to the power n will be gamma n plus 1 over s to the power n plus 1, and below that gamma of n plus 1 is n factorial if n is a positive integer. So, L of t to the power n is n factorial over s to the power n plus 1.

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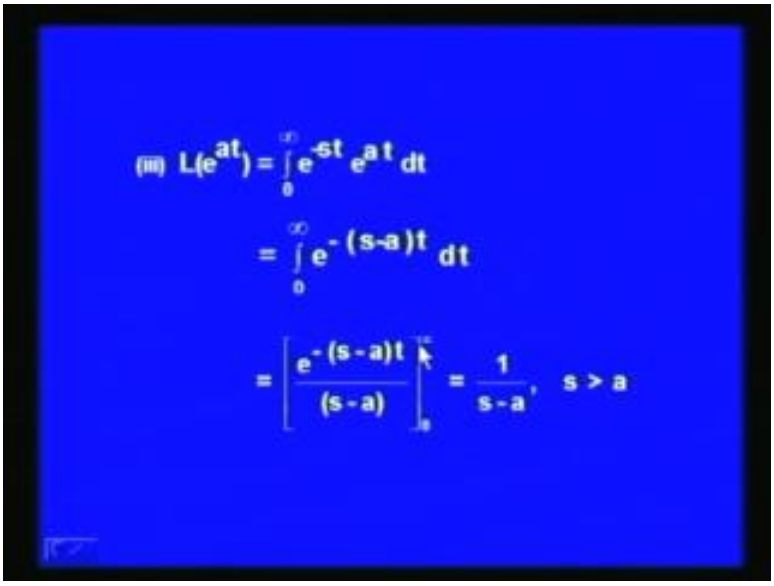

Let us discuss. now the Laplace transform of e to the power a t by definition e to the power a t we may write as integral 0 to infinity e to the power minus s t into e to the power a t d t. And this is equal to integral 0 to infinity e to the power minus s minus a into t d t. When we integrate this we get e to the power minus s minus a into t over s minus a, integral the limits are 0 and infinity. This equal to now when s is greater than a e to the power minus s minus a into t will go to 0 as t goes to infinity. So, this will become 0 when t goes to infinity and at the lower limit the value is minus 1 over s minus a. So, the value of the right hand side is 1 over s minus a provided s is greater than a, the condition on s is required to make this exponential of minus s minus a into t 10 to 0 as t goes to infinity.

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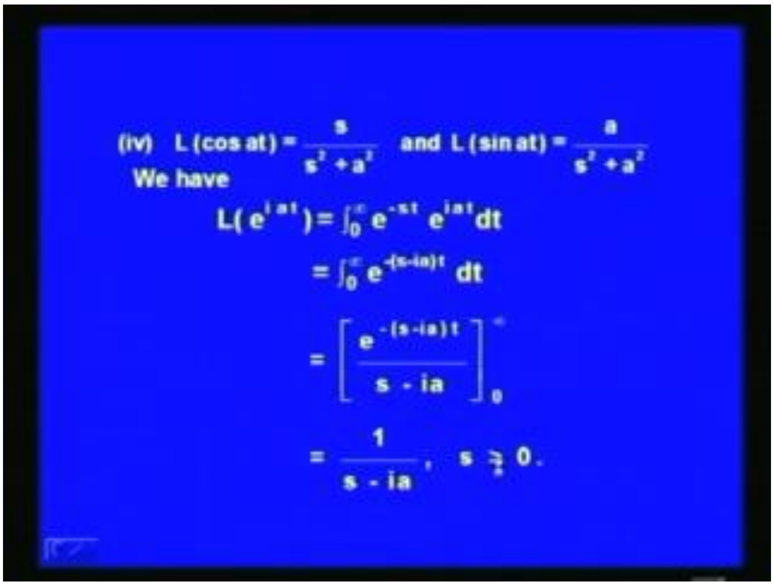

Let us, now discuss the Laplace transform of cos a t and sin a t, we will see that Laplace transform of cos a t is s over s square plus a square and Laplace transform of sin a t is a over s square plus a square. So, in order to find the Laplace transform of cos at and sin a t we consider Laplace transform of e to the power i a t.

We know that by the Euler's formula e to the power i a t is cos a t plus i sin a t, and then we will make use of the linearity property to write the Laplace transformer of e to the power i a t as Laplace transform of cos a t plus i times Laplace transform of sin a t. And then we will find the right hand side, equate the real and imaginary parts on both sides to get the values of Laplace transform of cos a t and sin a t.

So, both Laplace transforms can be found just by considering the Laplace transform of e to the power i a t and using the linearity property of the Laplace transforms. Now, Laplace transform e to the power of i a t can be written as integral 0 to infinity e to the power minus s t into e to the power i a t d t which we can write as integral 0 to infinity e to the power minus s minus a s minus i a into t d t.

When we take, when we find the integral of this we get e to the power minus s minus i a into t over s minus i a the limits are 0, and infinity. Then t goes to infinity e to the power minus s minus i a into t, what is the limit of this, you see we can write e to the power minus s minus i a into t as e to the power minus s t into e to the power i a t.

And then if you take the modulus of e to the power minus s t into e to the power i a t, then what you get is e to the power minus s t, because modulus of e to the power i a t is equal to 1. Now, e to the power i s minus s t, e to the power minus st then goes to 0 s t goes to infinity, so we get the limit of e to the power minus s minus i a into t, s t goes to infinity equal to 0. And therefore, the right hand side is equal to 1 over s minus i a provided s is greater than 0.

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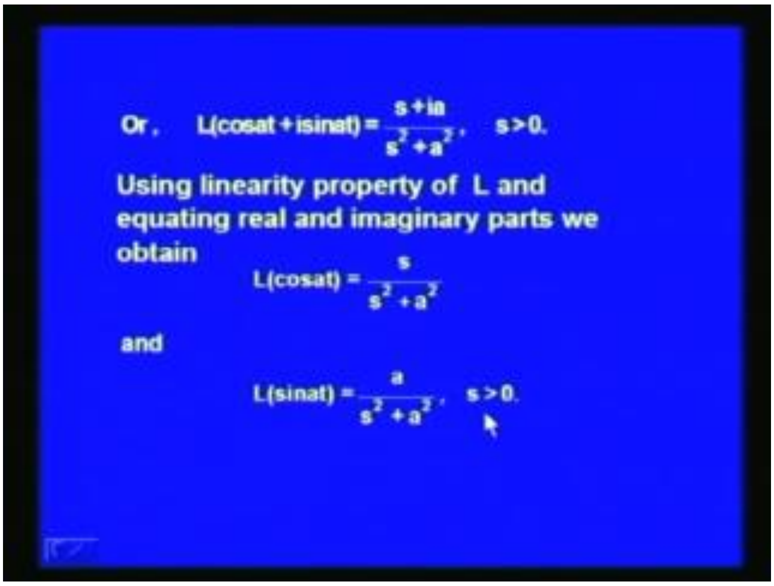

So, Laplace transform of cos a t plus i sin a t is equal to s plus i a over s square plus a square, we have multiplied s plus i a in the numerator and denominator on the right side, in order to bracket into real and imaginary in order to bracket into its real and imaginary part. Now, we will use the linearity property of L, and then equate the real and imaginary parts to be able to get the values of Laplace transform of cos a t and Laplace transform of sin a t. So, when we use the linearity property here, the left hand side becomes Laplace transform of cos a t plus i times Laplace transform of sin a t right hand side is s over s square plus a square plus i times a over s square plus a square. So, when we equate the real and imaginary parts on both sides we get the Laplace transform of cos a t as s over s square plus a square and Laplace transform of sin a t as a over s square plus a square, where the condition on that s is greater than 0.

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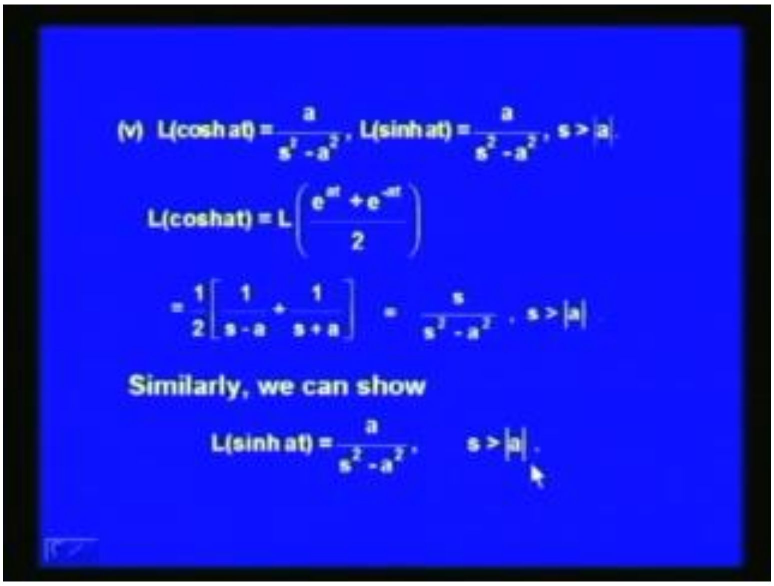

Now, we will find the Laplace transform of hyperbolic functions the cos hyperbolic a t and sin hyperbolic a t, and we shall see that Laplace transform of cos hyperbolic a t is a over s square minus a square, where the Laplace transform of sin hyperbolic a t is a over s square minus a square the condition on s is that it must be greater than the absolute value of a. We know the definition of cos hyperbolic a t it is equal to e to the power a t plus e to the power minus a t by 2. So, Laplace transform of cos hyperbolic a t is Laplace transform of e to the power a t plus e to the power minus a t by 2.

Now, let us use the linearity property of L here, so that we can write it as half of Laplace transform of e to the power a t plus half of Laplace transform of e to the power minus a t. And then we call the Laplace transform of e to the power a t it is 1 over s minus a provided s is greater than a, similarly Laplace transform of e to the power minus a t is 1 over s plus a provided s is greater than minus a.

Thus, the right hand side of Laplace transform of cos hyperbolic a t is equal to half of 1 over s minus a plus 1 over s plus a provided s is greater than a as well as s is greater than minus a, which means that s must be greater than the absolute value of a. Hence, we get the Laplace transform of cos hyperbolic a t as s over s square minus a square whenever s is greater than minus a. Similarly, we can show that Laplace transform of sin hyperbolic a t is equal to a over s square minus a square whenever s is greater than mod of a.

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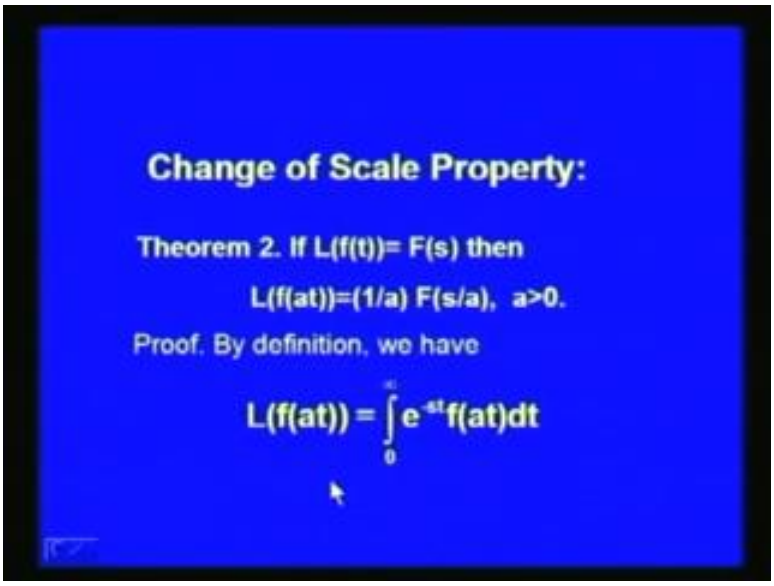

Now, let us discuss the change of scale property, let us assume that Laplace transform of f t is equal to F s. Then we shall show that Laplace transform of f of a t is equal to 1 by a into f of s by a whenever a is greater than 0. Now, by definition of the Laplace transform, we can write Laplace transform of f of a t as integral 0 to infinity e to the power minus s t into f of at d t.

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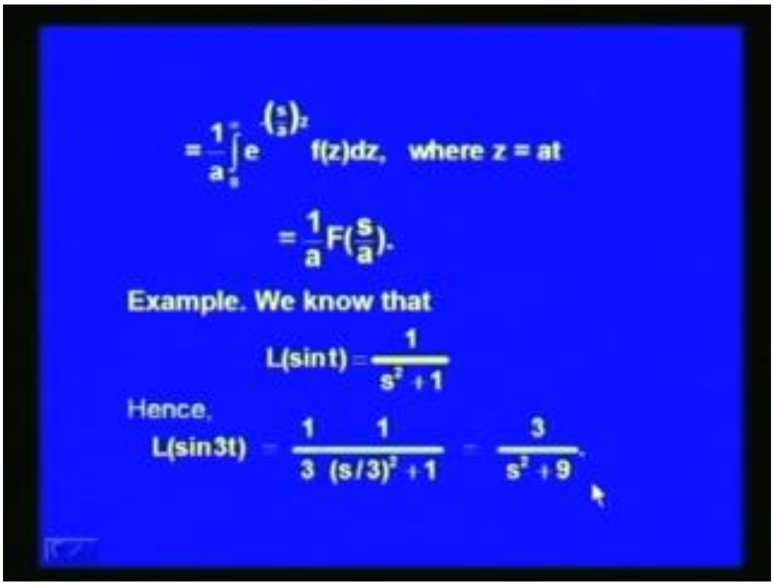

Let us, put z equal to a t now, so then e to the power minus s t will become e to the power minus s over a into z, f of a t will become f of z, d t will become d z by a and the limits will become 0, the lower limit will be 0, because when t is 0, z is 0. The upper limit will be infinity because we have assumed a to be positive, so whenever t goes to infinity z goes to infinity. And thus the Laplace transform of f a t is equal to 1 over a integral over 0 to infinity e to the power minus s over a into z into f z d z.

And, which is equal to 1 by a into f of s by a if F s is equal to we know that F s is equal to integral 0 to infinity e to the power minus s t into f t d t. So, in place of the variability here we have the variable z, and s in F s is replaced by s by a here, so we get the right hand side here as 1 over a into f of s by a.

Let us, look at an example which is based on this change of scale property, we know that Laplace transform of sin t is equal to 1 over s square plus 1, because Laplace transform we have shown that Laplace transform of sin at is 1 over s square plus a square. So, taking a equal to 1 we get the Laplace transform of sin t as 1 over s square plus 1.

Hence, Laplace transform of sin 3 t, so here we are taking ft is equal to sin t f of a t is equal to sin of 3 t. So, a is equal to 3 here, so a is clearly positive. So, Laplace transform of sin 3 t by change of scale property will be equal to 1 over 3 into 1 over s by 3 whole square plus 1. F s here Fs is this, Fs is 1 over s square plus 1, so we replace F s by f of s by 3; that means, we replace s by s by 3, so we get 1 over 3 into 1 over s by 3 whole square plus 1 which is equal to 3 over s square plus 9. Now, this Laplace transform of sin 3 t we can get directly also by using the formula of Laplace transform of sin a t equal to s over a over s square plus a square in that you put a equal to 3, you get Laplace transform of sin 3 t as 3 over s square plus 9.

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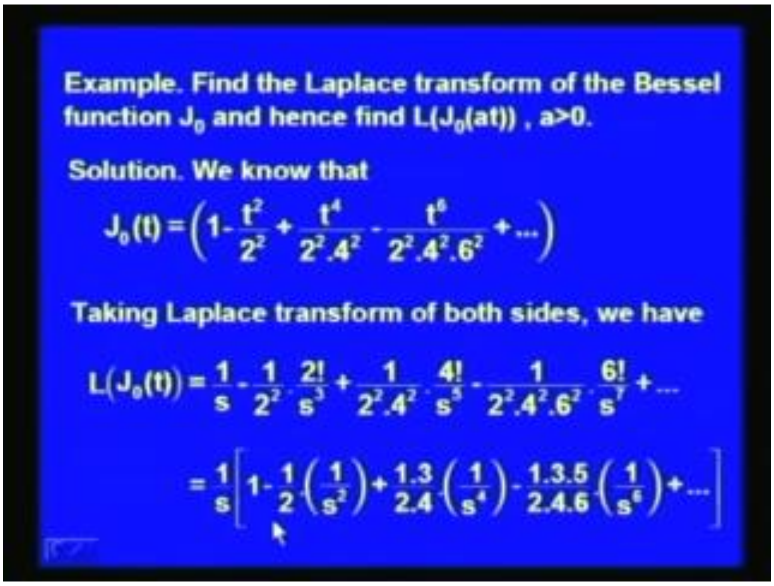

Let us, now find the Laplace transform of the Bessel function of order 0 which we denoted by J naught t. And then find the Laplace transform of J naught a t where a is greater than 0, we shall make use of the change of scale property here to find the Laplace transform of J naught a t once we have the Laplace transform of J naught t.

Now, we know that the Bessel function of order 0 that is J naught t is given by 1 minus t square over 2 square plus t to the power 4 over 2 square into 4 square minus t to the power 6 over 2 square into 4 square into 6 square and so on.

So, when we take the Laplace transform on both sides we get Laplace transform of J t as Laplace transform of 1 which is equal to 1 over s minus Laplace transform of t square over 2 square which is 1 over 2 square into Laplace transform of t to the power n we know is n factorial over s to the power n plus 1. So, taking n equal to 2 there we get the Laplace transform of t square as 2 factorial over s to the power 3 plus Laplace transform of t to the power 4 is 4 factorial over s to the power 5, and then minus 1 over 2 square 4 square 6 square. Laplace transform of 6 similarly is 6 factorial over s to the power 7.

Now, let us take 1 over s common from all the terms, then we get 1 over s times 1 minus 1 by 2 into 1 by s square plus 1 into 3 over 2 into 4 into 1 over s to the power 4 minus 1 into 3 into 5 over 2 into 4 into 6 into 1 by s to the power 6 and so on. So, we have written the right hand side in a proper form, so that we can write it as follows.

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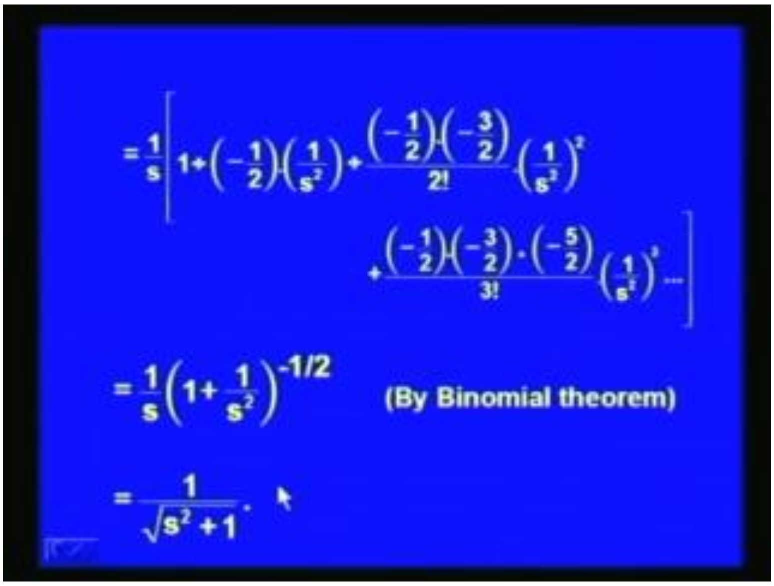

This is further equal to 1 by s times 1 plus minus half into 1 by s square plus minus half into minus 3 by 2 over 2 factorial into 1 by s square over raise to the power 2 plus minus half into minus 3 by 2 into minus 5 by 2 over 3 factorial into 1 by s square raise to the power 3 and so on.

And, this expression inside the brackets we can write as 1 plus 1 by s square raise to the power minus half. So, which follows by binomial theorem we know that 1 plus 1 by s square raise to the power minus half can be written as infinite series given here, inside the bracket. So, this is through right hand side is equal to 1 by s into 1 plus 1 by s square raise to the power minus half provided s is greater than 1. And, this after simplification gives us 1 over under root s square plus 1. So, Laplace transform of the Bessel function of order 0 is equal to s square plus 1 raise to the power minus half.

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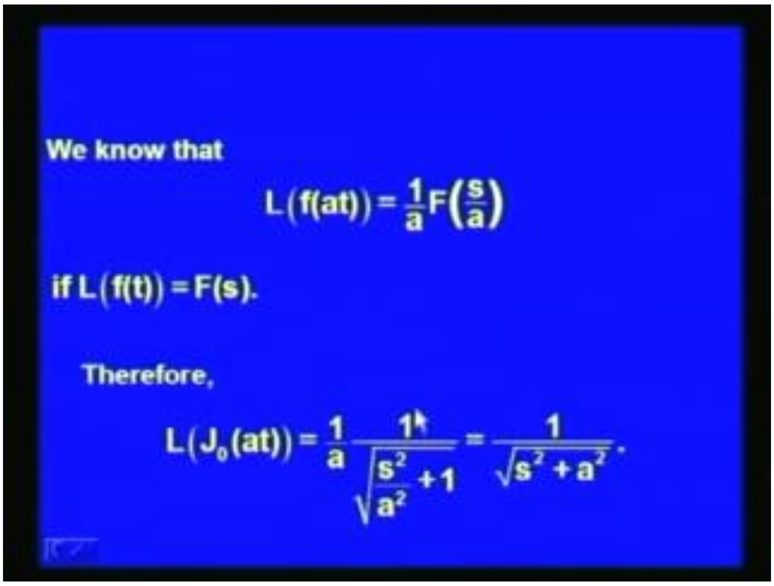

Now, we know that by change of scale property Laplace transform of f of a t is 1 over a F of s by a provided a is greater than 0. So, let us apply this F s here, we let us we call that F s here is L of f t. So, making use of this change of scale property we can write the Laplace transform of J naught a t as 1 over a times f of s by a and F s, we have seen is 1 over square root s square plus 1. So, in that s is replaced by s over a to have the Laplace transform of J naught a t which is equal to 1 over a into 1 by under root s square by a square plus 1 which after simplification gives us 1 over square root s square plus a square.

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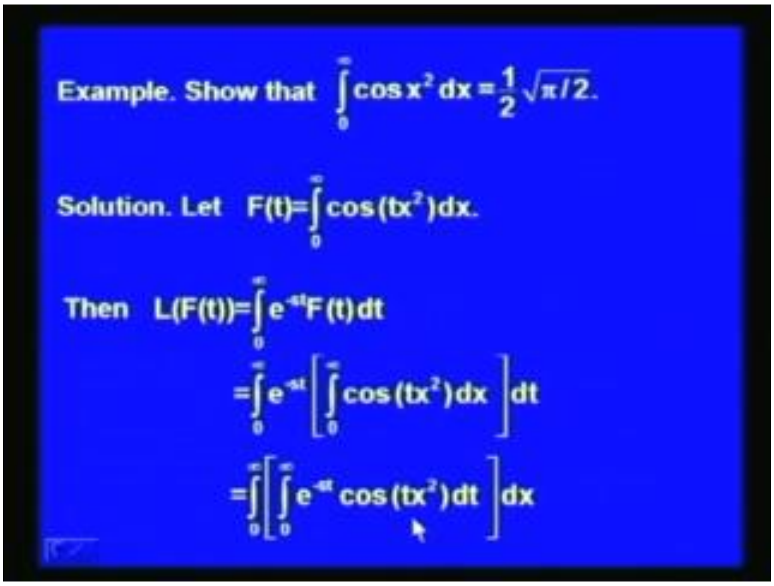

Now, let us take an example where we have to evaluate the integral of 0 to infinity cos x square d x, we know that the integral of cos x square d x over 0 infinity cannot be evaluated by the known methods of integration. So, we will see that this can be obtained using the Laplace transform technique, let us assume that ft is equal to integral 0 to infinity cos of t into x square d x.

Then, Laplace transform of f t which is equal to integral 0 to infinity e to the power minus s t into f t d t by definition will become equal to integral 0 to infinity e to the power minus s t into integral 0 to infinity cos of t into x square d x into d t changing the order of integration here, we have integral 0 to infinity into integral 0 to infinity, then integral 0 to infinity e to the power minus s t cos t x square d t into d x. Now, this inner integral which is integral 0 to infinity e to the power minus s t cos t x square d t this is nothing but Laplace transform of cos t x square.

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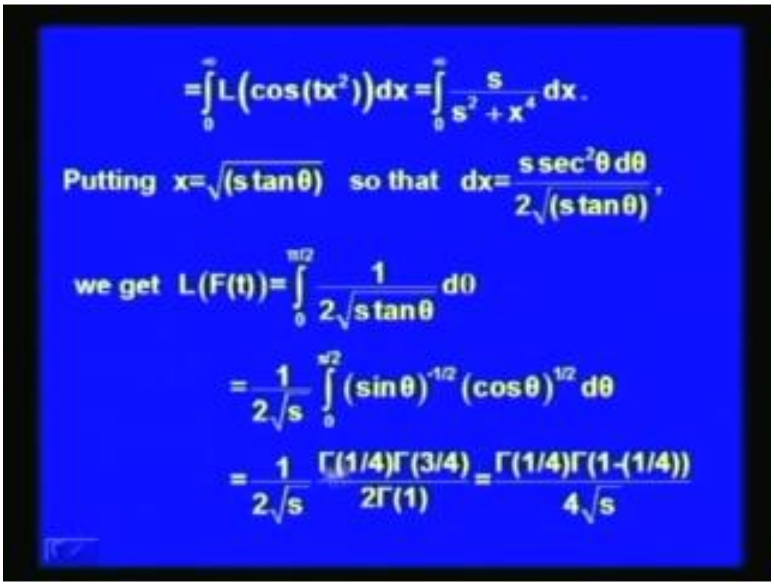

So, we get the right hand side as integral 0 to infinity cos Laplace transform of cos t x square into d x and Laplace transform of cos a t we know, it is s over s square plus a square. So here, a is equal to x square, and therefore Laplace transform of cos of t x square is s over s square plus x to the power 4. So, right hand side becomes integral 0 to infinity s over s square plus x to the power 4 d x.

Now, let us put x equal to square root of s tan theta here, so then we shall have d x equal to 1 over 2 root s tan theta into s into sec square theta d theta. And therefore, we will get the Laplace transform of f t as integral 0 to pi by 2, now here we see that when x is 0 theta is 0.

And then x goes to infinity, theta goes to pi by 2, so the limit lower limit of theta is 0 while the upper limit of theta is pi by 2, and thus we get the Laplace transform of f t as 1 over integral 0 to pi by 2, 1 over 2 root s tan theta d theta, which is equal to 1 over 2 root s into integral 0 to pi by 2, sin theta raise to the power minus half into cos theta raise to the power half d theta.

And, the value of this integral we can obtain by using the gamma function 1 over 2 root s is as it is here, and then we this we know that integral 0 to pi by 2, sin theta raise to the power m into cos theta raise to the power n d theta is gamma m plus 1 by 2 into gamma n plus 1 by 2 divided by 2 times gamma m plus n plus 2 by 2.

So, making use of that formula we have here m equal to minus half n equal to half, so we get gamma m plus 1 by 2 that is gamma 1 by 4. Then gamma n plus 1 by 2 that is gamma 3 by 4 over 2 times gamma of m plus n plus 2 by 2, m is minus half n is half, so m plus n plus 2 by 2 gives you gamma 1. And therefore, we have the right hand side equal to gamma 1 by 4, gamma 3 by 4 which is gamma half 1 minus 1 by 4 divided by 4 into square root s into gamma half 1 which is equal to 1.

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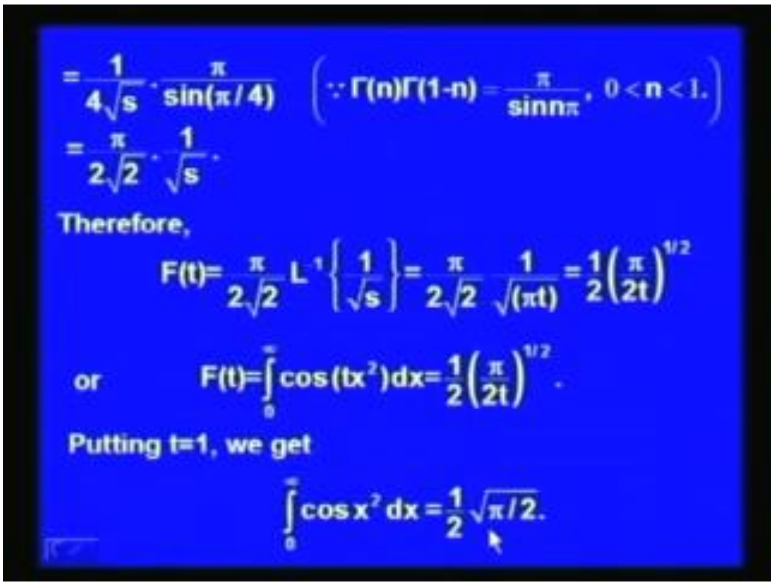

And then the right hand side becomes equal to 1 over 2 root 4 root s into pi over sin pi by 4, because we know that gamma n into gamma 1 minus n is equal to pi over sin n pi whenever 0 is less than n less than 1. So, this is equal to pi over 2 root as 2 root 2 into 1 over root s. And therefore, f t is equal to pi over 2 root 2 into inverse Laplace transform of 1 over root s.

Now, we know that Laplace transform of t to the power a is equal to gamma a plus 1 over s to the power a plus 1 whenever a is greater than minus 1. So, taking a equal to minus half there, we get the Laplace for transform of t to the power minus half s root pi over root s. And therefore, Laplace transform of 1 over root s is 1 over root pi t, so we get the function f t s pi by 2 root 2 into 1 over root pi t.

And, which is equal to 1 over 2 into pi by 2 t raise to the power half, and thus f t becomes equal to integral 0 to infinity cos t x square d x equal to 1 by 2 into pi by 2 t raise to the power half. Now, let us take t equal to 1 to have the value of the desired integral, we get the desired integral s integral 0 to infinity cos x square d x equal to 1 by 2 pi by 2 raise to the power half, that is square root of pi by 2.

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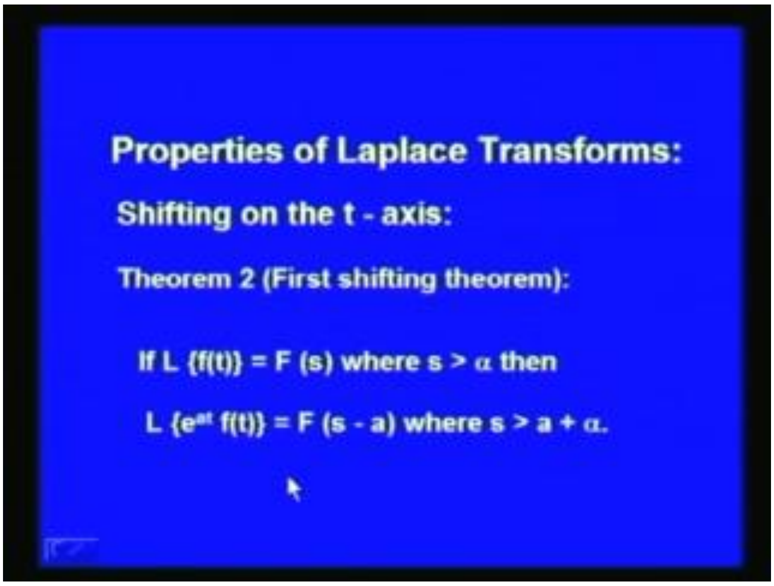

Now, we are going to discuss an important property of the Laplace transform, we call it first shifting theorem. If L f t equal to F s where s is greater than alpha then L of e to the power a t f t is equal to F of s minus a, where s is greater than a plus alpha.

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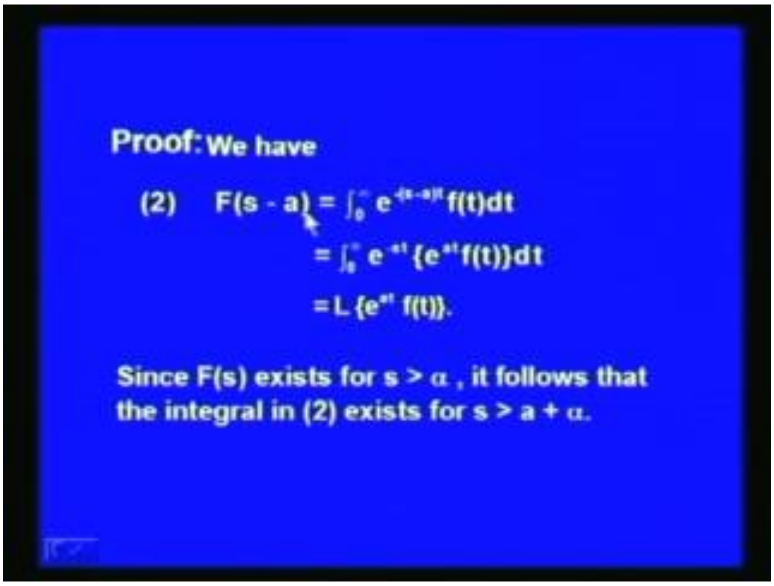

This theorem tells us that s is replaced by s minus a in the Laplace transform of the function f t, then ft is multiplied by e to the power a t. So, when we shift from s to s minus a we get that f t is multiplied by e to the power a t. Now, let us proof this theorem by definition the of the Laplace transform we can write F of s minus a equal to integral 0 to infinity e to the power minus s minus a into t f t d t replacing s by s minus a in the definition of the Laplace transform of the function f t.

And, this e to the power minus s minus a into t, we can write as e to the power minus a t into e to the power a t e to the power a t we can combine with the function f t. So, that we get the right hand side as integral 0 to infinity e to the power minus s t into e to the power a t into f t d t, and we can then write it as the Laplace transform of e to the power a t into f t. Since, F s exists for s greater than alpha by our assumption, therefore the integral here the integral here exits for s minus a greater than alpha, that is s greater s than a plus alpha. And therefore, we may say that L e to a t into ft is equal to Fs minus a provided s is greater than a plus alpha.

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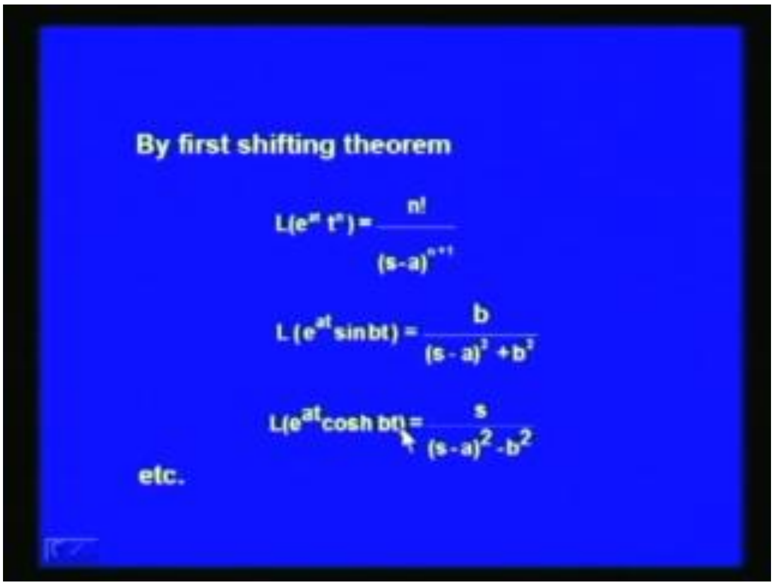

Now, by the first shifting theorem, we know that Laplace transform of t to the power n is n factorial over s to the power n plus 1. So, if you multiply t to the power n by e to the power a t by the first shifting theorem we can say that Laplace transform of e to the a t into t to the power n will be equal to n factorial over s minus a to the power n plus 1 replacing s by s minus a, here we are assuming that n is a n is a positive integer.

Now, similarly we know that Laplace transform of sin b t is b over s square plus b square. So, when we multiply sin b t by e to a t by first shifting theorem, Laplace transform of e to the power a t into sin b t will be b is equal to b over s minus a whole square plus b square. And, similarly Laplace transform of e to the power a t into cos hyperbolic b t is equal to s over s minus a whole square minus b square, because we know that Laplace transform of cos hyperbolic b t is s over s square minus b square. So, by using first shifting theorem, the Laplace transform of e to the power a t cos hyperbolic b t will be obtained by replacing s by s minus a in the Laplace transform of cos hyperbolic.

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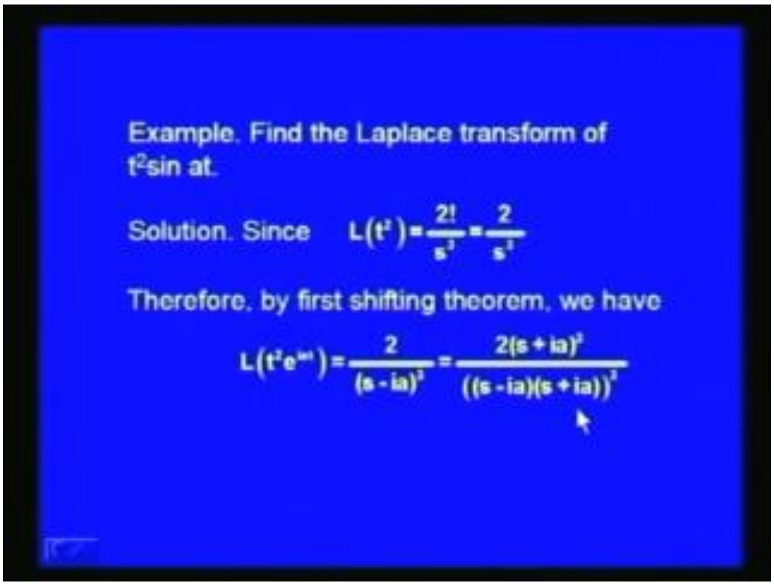

Now, let us find the Laplace transform of t square into sin a t we know that the Laplace transform of t square is 2 factorial over s cube that is it is equal to 2 over s cube. And therefore, by the first shifting theorem we can write the Laplace transform of t square into e to the power i a t as equal to 2 over s minus i a whole cube replacing s by s minus i a and which is equal to 2 into s plus i a whole cube over s minus i a into s plus i a whole cube.

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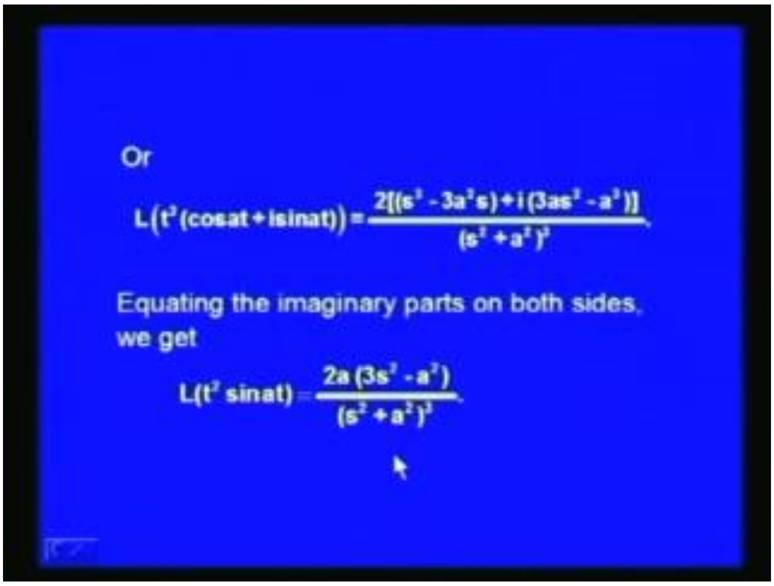

Now, we can write it as L of t square e to the power i t i a t is cos i a t plus i sin a t and then right hand side is 2 times s cube minus 3 a square s plus i times 3 a s square minus a cube divided by s square plus a square raise to the power 3. So, equating the imaginary parts both sides we get Laplace transform of t square into sin a t as equal to 2 into a into 3 s square minus a square over s square plus a square raise to the power 3.

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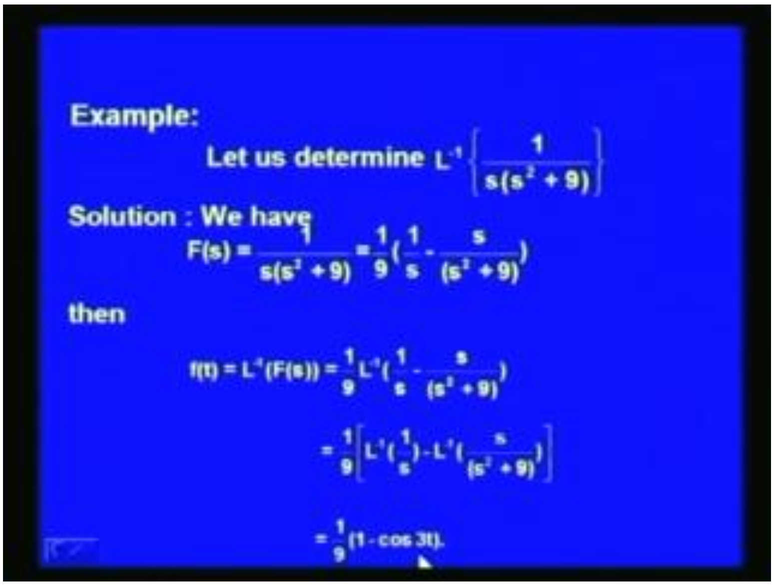

Let us now determine the inverse Laplace transform of 1 over s into s square plus 9. Let us denote by lower s into s square plus 9 by Fs. So, Fs is equal to 1 over s into s square plus 9 breaking 1 over s into s square plus 9 into its partial fractions we get 1 over 9 times 1 over s minus s over s square plus 9. And then f t is equal to inverse Laplace transform of F s, so L inverse of F s it is equal to 1 by 9 using linearity property we have 1 by 9 L inverse of 1 over s minus s over s square plus 9 which is equal to 1 by 9 into L inverse of 1 by s minus L inverse of s over s square plus 9. And, we know that inverse Laplace transform of 1 over s is 1 inverse Laplace transform of s over s square plus 9 is cos 3 t. So, inverse Laplace transform of F s which is ft is equal to 1 over 9 times 1 minus cos 3 t.

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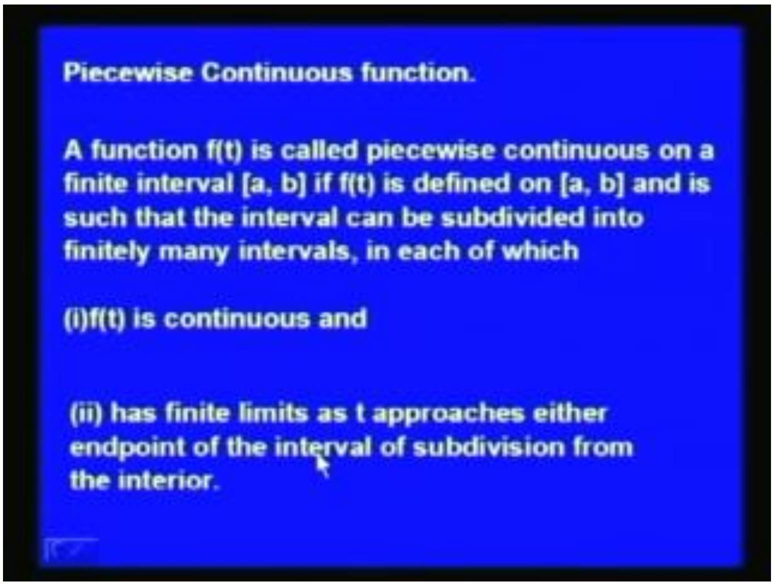

Now, let us we are going to discuss the existence theorem for the Laplace transform, so before that we define what do we mean by a piecewise continuous function. A function f t defined over an interval a to b is said to be piecewise continuous. If it is if the interval a b can be subdivided into finitely many sub intervals in each of which the function f t is continuous and has finite limits as t approaches either end point of the interval of subdivision from the interior that is the function f t has only ordinary discontinuities inside the interval a b.

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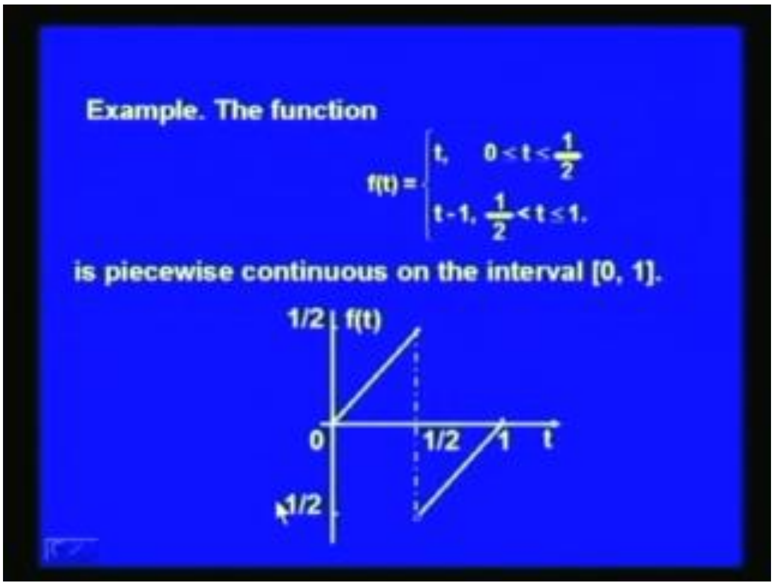

Let us, look at the function f t equal to t where t takes values in the interval 0 half 0 less than or equal to t less than or equal to half. And ft is defined as t minus 1 when half is less than t less than or equal to 1. So, you can see that this function is piecewise continuous, because the left hand limit of this function here is half, while the right hand limit is minus half. The function is defined over the whole interval 0, 1 and the interval function the interval 0 when has been divided into 2 sub intervals 0 half and half 1 in each of which the function f t is continuous.

And, as t approaches to the point of discontinuity that is half, the left hand limit is half while the right hand limit is minus half. So, the function f t has an ordinary discontinuity at the point half, by the function f is said to be have said to have an ordinary discontinuity at a point in the interval a b if it has finite on either side. So, of the point of discontinuity, so here the function f has ordinary discontinuity at the point half.

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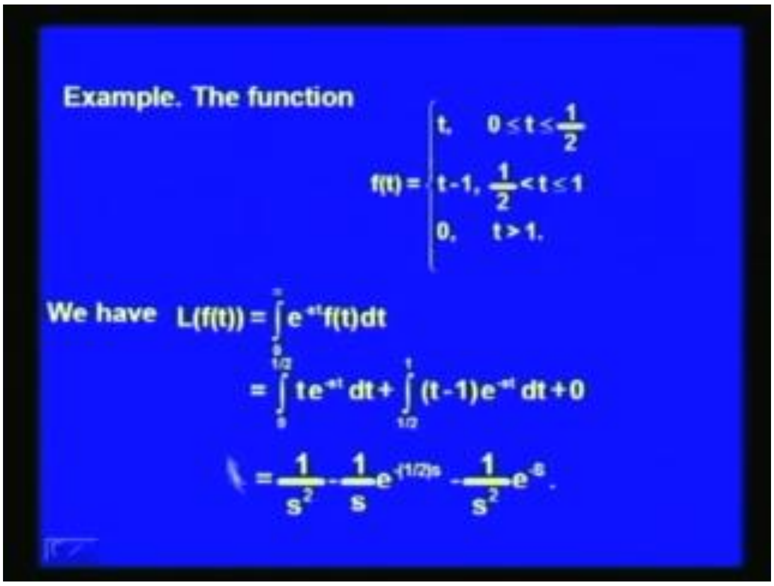

And, we note here that if a function is continuous it is almost piecewise continuous, but the converse is not true. Now, let us take an example of the function f t defined as ft equal to t when 0 less than or equal to t less than or equal to half t minus 1 when half is less than t less than or equal to 1 and we define it equal to 0 or all t greater than 1. So, then let us see what is the Laplace transform of this by definition Laplace transform f t is integral 0 to infinity e to the power minus s t into f t d t, which is which is equal to integral 0 to half t e to the power minus s t d t plus half to integral over half to 1 t minus

1 e to the power minus s t d t plus 0. The third integral which is integral over half 1 to infinity, because 0 because the function f is defined as 0 there.

Now, let us see value of the values of these two integrals, the integral first integral and the second integral here, when we evaluate their values and put the limits it turns out that Laplace transform of f t is equal to 1 over s square minus 1 over s into e to the power minus half into s and then minus 1 by s square e to the power minus s.

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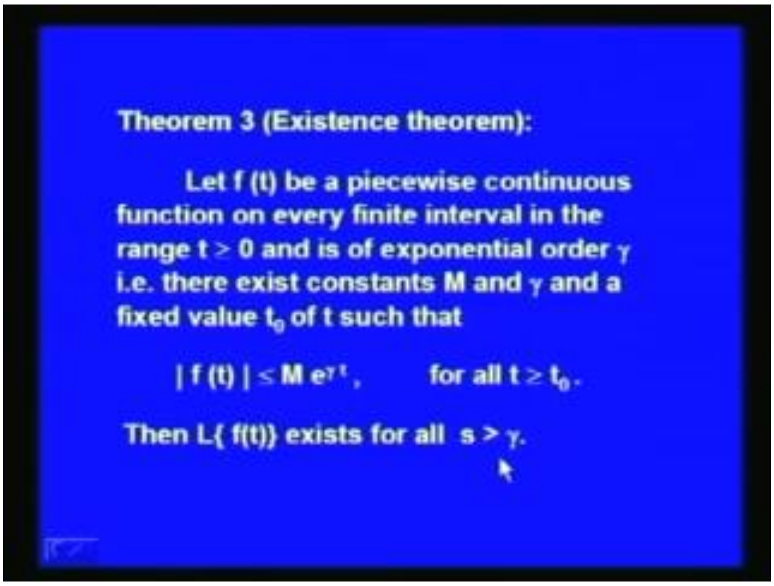

Now, we discuss the existence theorem for the Laplace transform, that is we are going to see the conditions under which the Laplace transform of a given function f exists. So, let us assume that f t is a piecewise continuous function on every finite interval in the range t greater than or equal to 0 and is of exponential order gamma, that is we assume that there exist constants M and gamma and a fixed value t naught of t such that mod of f t is less than or equal to M times e to the power gamma t for all t greater than or equal to 0.

In simple terms, we can say that f function f is said to be of exponential order gamma if limit of e to the power minus gamma t e to the power minus gamma t into f t exists and is finite as t goes to infinity. So, then Laplace transform of the function f t exists for all s greater than gamma.

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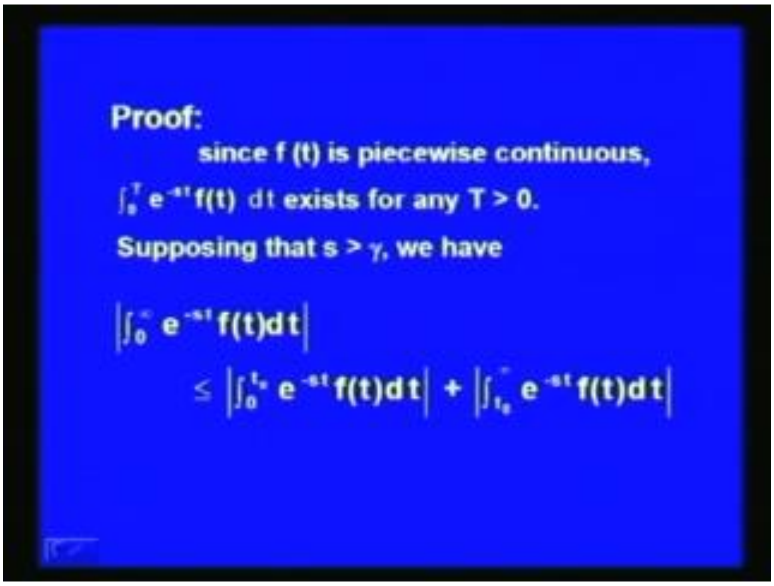

Let us, look at the proof of this theorem we have assume that f t is the piecewise continuous function, therefore integral over 0 to t e to the power minus s t f t d t exists for any T greater than 0. Now, supposing that s is greater than gamma, we can write the modulus of integral 0 to infinity e to the power minus s t f t d t less than or equal to modulus of integral 0 to t naught e to the power minus s t into f t d t plus modulus of integral t naught to infinity e to the power minus s t into f t d t. We bracket into 2 parts integral over t naught 0 to t naught.

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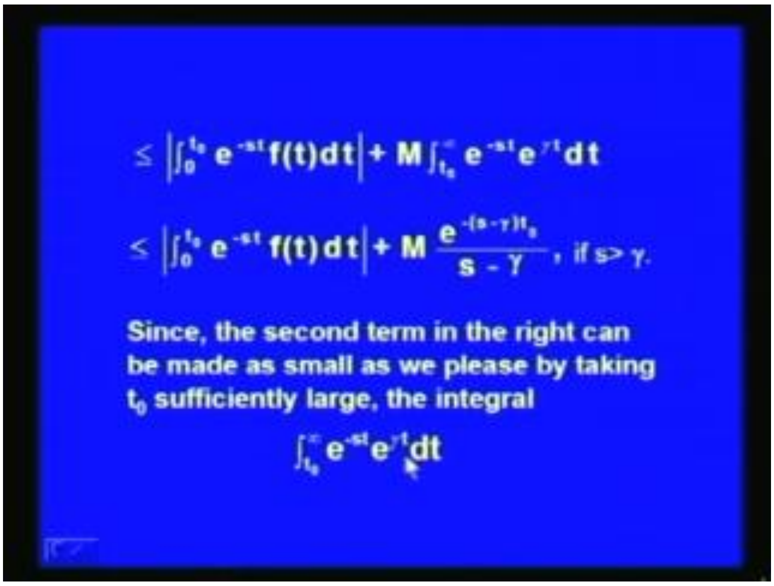

And then t naught to infinity, t naught is the value after which all values of t for all values of t mod of ft less than or equal to m times e to the power gamma t, which is less than or equal to modulus of integral 0 to t naught, e to the power minus s t into f t d t plus M times integral t naught to infinity e to the power minus s t into e to the power gamma t d t and which is further less than or equal to modulus of integral 0 to t naught e to the power minus s t into f t d t, plus M times e to the power minus s minus gamma into t naught over s minus gamma. The integral of e to the power minus s t into e to the power gamma t is e to the power minus s minus gamma into t over s minus gamma if we assume s to be greater than gamma then s t goes to infinity.

The value of e to the power minus s minus gamma into t is 0, so at the lower limit its value is e to the minus s minus gamma into t naught. And therefore, the value of the integral is integral t naught to infinity e to the power minus s t into e to the power gamma t d t is e to the power minus s minus gamma into t naught over s minus gamma wherever s is greater than gamma.

And, the second term here that is e to the power minus s minus gamma into t naught over s minus gamma can be made as small as we please by taking t naught to be sufficiently large. And therefore, the integral t naught to infinity e to the power minus st into e to the power gamma t d t exists.

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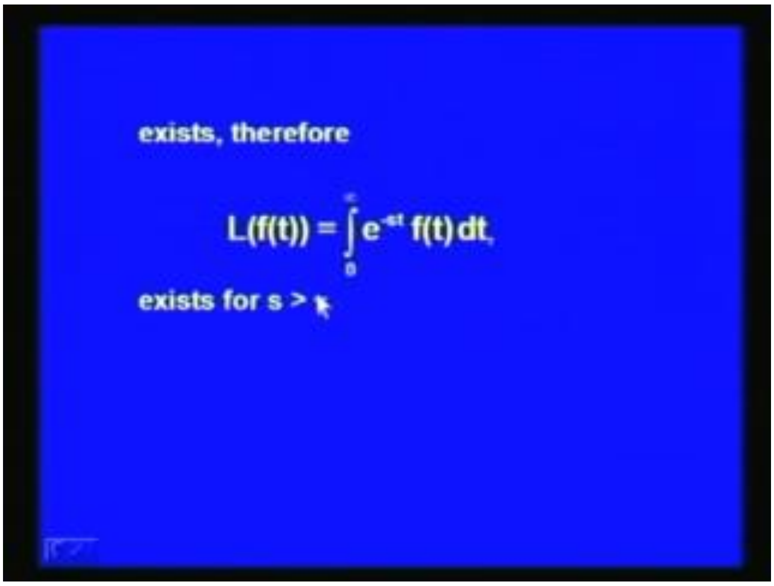

And, thus Laplace transform of f t which is to infinity 0 to infinity e to the power minus s t f t d t exists for all values of s greater than gamma.

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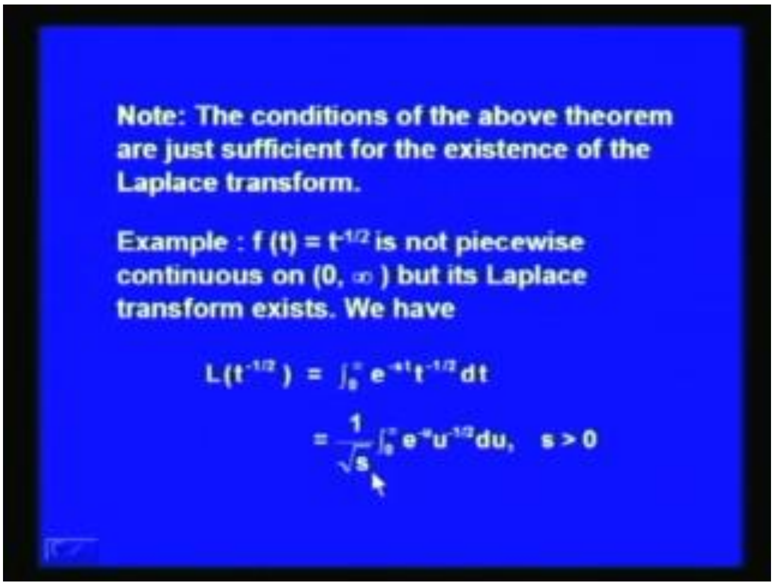

Now, we note here that the conditions of the above theorem are just sufficient for the existence of the Laplace transform.

For example, let us consider the function f t equal to t to the power minus half we can see that this function is not piecewise continuous over the interval 0 to infinity, because s t goes to 0 plus that is s t goes to 0 from the right t to the power minus half goes to plus infinity. And therefore, it is not a piecewise continuous function.

But, we see that its Laplace transform exists, its Laplace transform is square root pi over s which follows from Laplace transform of t to the power a by taking a equal to minus half there. So, this we have independently shown here also the Laplace transform of t to the power minus half is 1 over root s integral 0 to infinity e to the power minus u to the power minus half d u whenever s is greater than 0.

Here, we have taken s t equal to u, so the limits of integration for u are remain become 0 to infinity if we assume s to be greater than 0. And which is equal to using the definition of gamma function, gamma half over root s that is root pi over root s whenever s is greater than 0.

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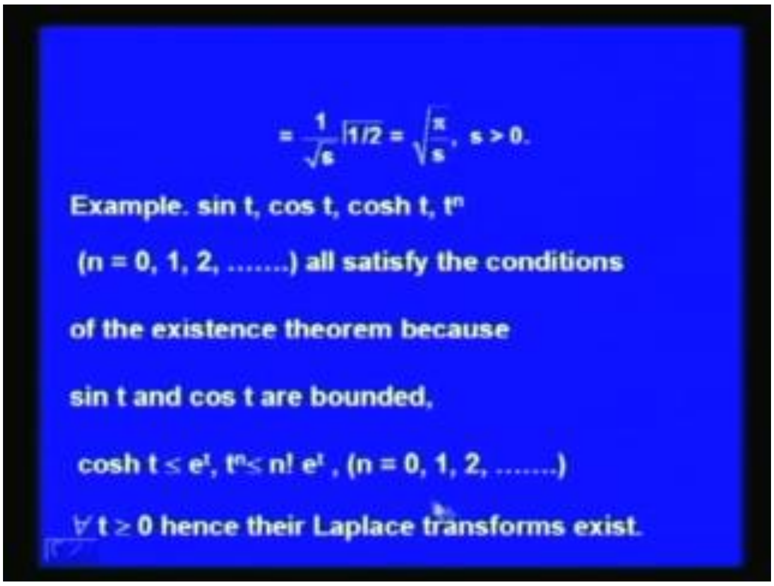

Now, let us look at some of the functions which satisfy the conditions of the existence theorem say let us see take the function f t equal to sin t f t equal to cos t, we know that f t equal to sin t and cos t are both bounded functions modulus of sin t and modulus of cos t is less than or equal to 1 for all values of t. So, they are both bounded function for all t greater than or equal to 0 in particular. And, cos hyperbolic t is e to the power t plus e to the power minus t by 2, so cos hyperbolic t is less than or equal to e to the power t for all t greater than or equal to 0 and t to the power n. Then n takes the non negative integral values 0 1 2 and so 1 is less than or equal to n factorial into e to the power t for all t greater than or equal to 0.

So, sin t and cos t being both bounded functions that is mod of sin t and mod of cos t are both less than or equal to 1 means they are both of exponential order. I mean you can write in mod of sin t less than or equal to 1 which is further less than or equal to e to the power t, and mod of cos t less than or equal to 1 can be written less than or equal to e to the power t. So, they are both of exponential order.

And, similarly cos hyperbolic t is of exponential order, because it is less than or equal to e to the power t, t to the power n for a given n is less than or equal to n factorial e to the power t, so it is also of exponential order. And they are all continuous functions for all t greater than or equal to 0, so in particular they are all piecewise continuous functions.

And therefore, they satisfy all the conditions of the existence theorem, and the hence they are Laplace transforms exists.

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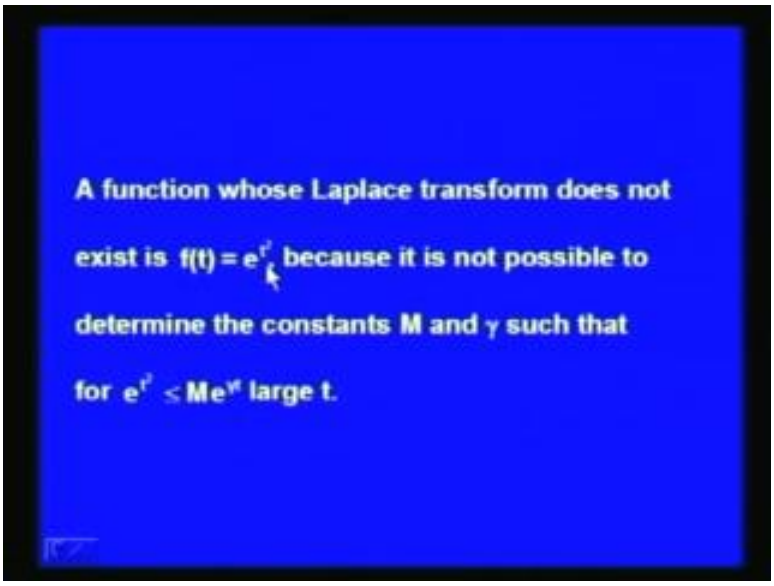

Let us, take now example of a function whose Laplace transform does not exists, when such function is ft equal to e to the power t square, f t equal to e to the power t square is a continuous function for all t greater than or equal to 0 being in the exponential function of t square. So, it is piecewise continuous, but it is not of exponential order because for any value of a greater than 0 e to the power minus a t into e to the power t square s t goes to infinity is always finite is always infinite, it does not exist. So, therefore, we can say that it is not possible to determine the constants M and gamma such that e to the power t square is less than or equal to M times e to the power gamma t for large t and there it is therefore, its Laplace transform does not exists.

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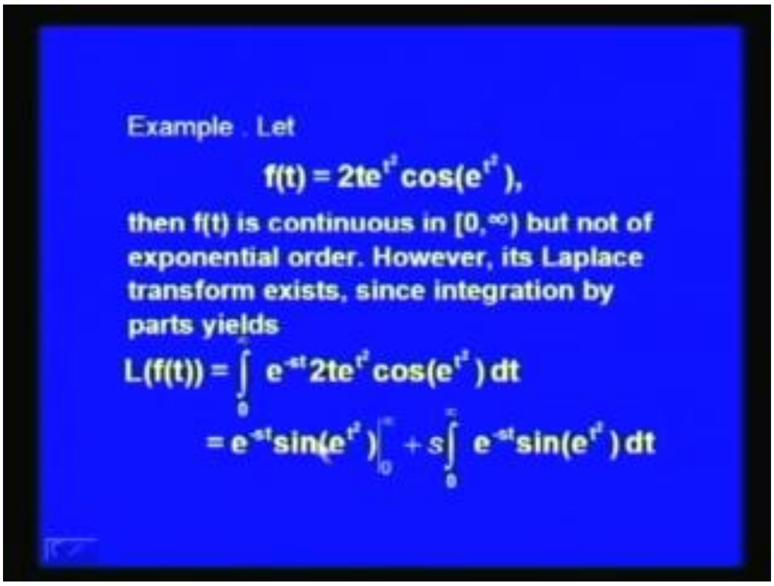

Now, let us take an example of a function which is not of exponential order, but still its Laplace transform exists f t equal to 2 into t into e to the power t square cos e to the power t square, you can see that it is not of exponential order. Now, we can see that t e to the t square cos of e to the power t square are all continuous functions, so their produce is a continuous function in 0 infinity. And therefore, it is a piecewise continuous function in 0 infinity, but it is not of exponential order as we can see. Because e to the power minus a t into f t for any a greater than 0 as t goes to infinity will not goes to a finite limit. So; however, its Laplace transform still exists.

Since, integration by parts yields Laplace transform of f t is integral 0 to infinity e to the power minus s t into 2 t into e to the power t square cos of e to the power t square d t. When we integrate by parts here 2 t e to the power t square into cos of e to the power t square is nothing but the differential for sin of e to the power t square.

So, let us take e to the power minus first function and 2 t e to the power et square cos e to the power t square as second function. We have by integration by parts using integration like parts we have first function that is e to the power minus s t integral of second function that is sin of e to the power t square evaluated at 0 infinity and then derivative e to the power minus s t into e to the power minus s t into minus s. So, we get plus s times integral 0 to infinity e to the power minus s t into sin e to the power t square d t.

Now, when t goes to infinity modulus of e to the power minus s t into sin e to the power t square is less than or equal to e to the power minus s e to the power minus s t. Because, mod of sin e to the power t square is less than or equal to 1, so if we assume that s is greater than 0, then e to the power minus s t, s t goes to infinity will goes to 0. And for therefore, the limit of e to the power minus s t into sin e to the power t square is 0, s t approaches infinity. And at the lower limit t equal to 0 its value is 1, this is 1 and sin e to the power 0 is e sin e to the power 0 is 1. So, with the lower limit its value is sin 1.

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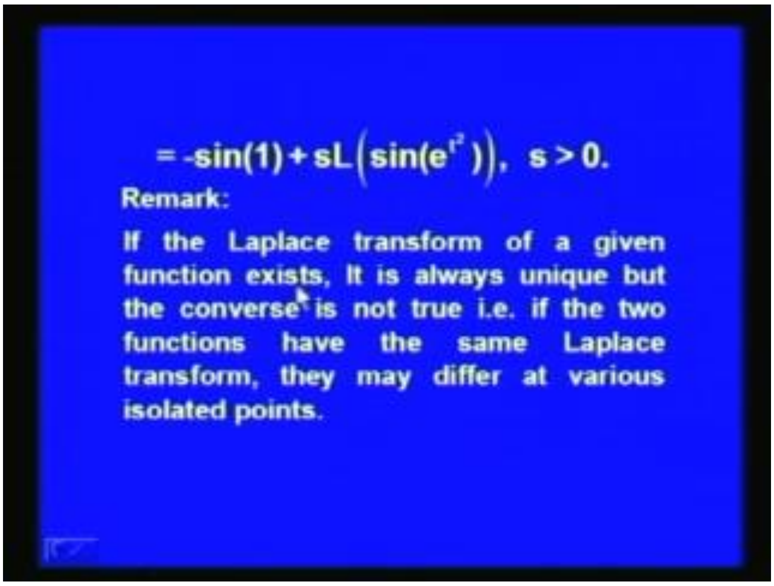

So, we will get the right hand side as minus sin 1 plus s times Laplace transform of sin e to the power t square, whenever s is greater than 0. And, sin of e to the power t square being a bounded function mod of sin e to the power t square is less than or equal to 1. So, it is of exponential order and it is a continuous function for all t greater than or equal to 0, so it is piecewise continuous. For all t greater than or equal to 0, and therefore it satisfies all the conditions of the existence theorem, so its Laplace transform exists.

And therefore, the Laplace transform of the given function f t exists for all s greater than 0. Now, we may remark here that if the Laplace transform of a given function exist, it is always unique, but the converse is not true, if the two functions have the same Laplace transform they may differ at various isolated points.

This is our lecture on Laplace transformation today we have seen some of the properties of the Laplace transform like the first shifting theorem which refers to the shifting on the x axis, and then we discussed the change of a scale property. Then we have discussed the existence theorem which gives us the sufficient conditions under which the Laplace transform of a given function exist. The conditions are that the function must be piecewise continuous for all t greater than or equal to 0, and then it should be of exponential order.

In our next lecture, we shall be discussing some other properties of the Laplace transform that is when you differentiate the function f, then if you take the Laplace transform what happens will see that the transform of the function gets multiplied by s. And then we shall see when we take the Laplace transform of the integral of a function it is then what we get is the Laplace transform of the function gets divided by s.

And then we shall also find see that when we take the derivative of the Laplace transform of the function. Then the function gets multiplied by minus t and similarly when we take the integral of the Laplace transform will get that the function f t gets divided by t. So, we will be seeing all those properties of the Laplace transform and then we shall see how to apply those properties of the Laplace transforms to the various problems. The problems that we shall take up will be how to find the solution of an ordinary differential equation with constant coefficients and some more problems.

Thank you.