Mathematics - III
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Lecture - 7
Bessel Functions and their Properties (Contd.)

Dear viewers, in my last lecture we had discussed the Bessel's equation and it is solutions, which are known as Bessel's function of first kind and Bessel's function of second kind. We had also discussed the transformation of Bessel's equation, where why we obtained a fairly general linear differential equation of second order, whose solutions could be written in terms of Bessel functions. Now, today we shall discuss the some properties of the Bessel functions.

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We begin with an important property of Bessel's functions known as orthogonality of Bessel functions. The orthogonality of Bessel functions is significant, it is used in the problems on by variation of circular membranes.

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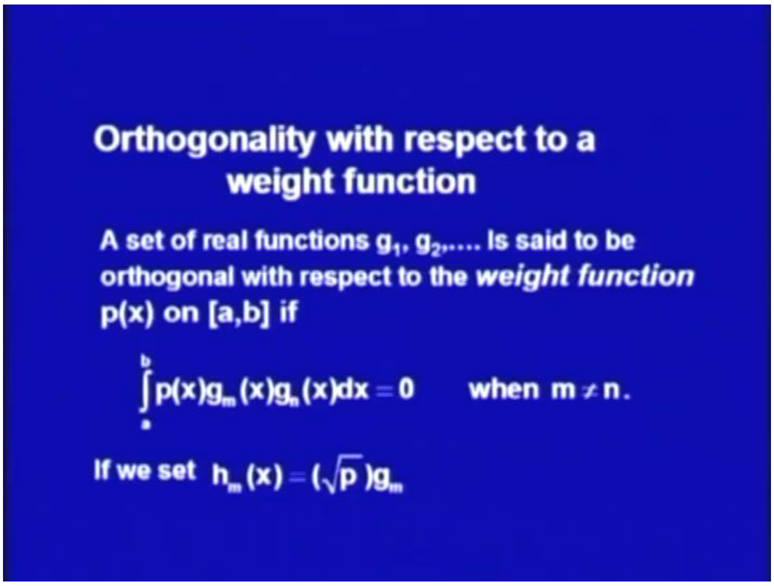

Let us first define the orthogonality with respect to a weight function, a set of real functions g 1, g 2 and so on, is said to be orthogonal with respect to the weight function p x on the interval a, b. If integral a to b p x into g m x into g n x d x is equal to 0, whenever m is not equal to n, if we set h m x equal to square root p x into g m x.

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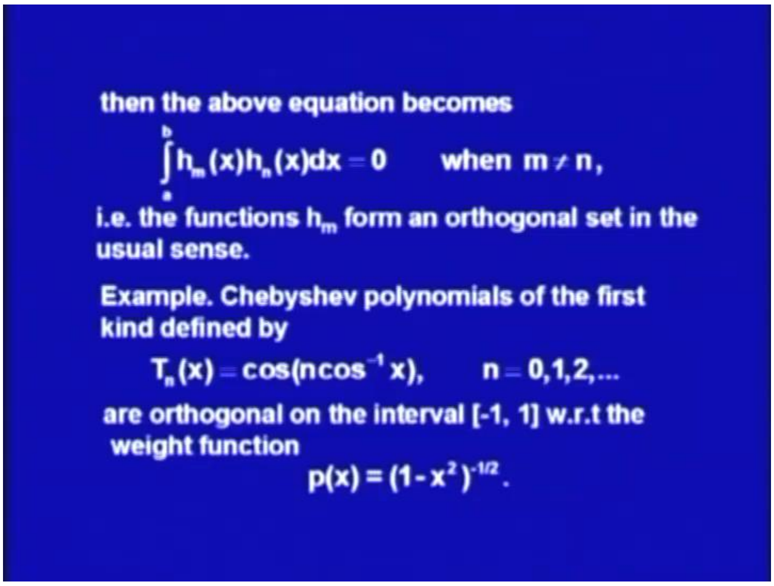

Then, we see that the above equation reduces to integral a to b h m x into h n x d x equal to 0, whenever m is not equal to n which implies that the functions h m x are orthogonal in the usual sense. That is the functions h m x satisfy the orthogonal property integral a to b h m x into h n x d x is equal to 0, so they form an orthogonal set in the usual sense. For example, let us consider Chebyshev polynomials of the first kind, which are defined by T n x equal to cos n cos inverse x, where n is equal to 0, 1, 2, 3 and so on.

Now, from here we can see that T naught x is equal to 1, T 1 x is equal to cos of cos inverse x, so it is x T 2 x is cos 2 cos inverse x, if you put cos inverse x as theta, then you can see that cos 2 theta is 2 cos square theta minus 1. So, T 2 x will be equal to 2 x square minus 1 and similarly, we can see that T 3 x which is cos 3 cos inverse x turns out to be 4 x cube minus 3 x. Because, cos inverse x when we put as theta we have got cos 3 theta and cos 3 theta is 4 cos cube theta minus 3 cos theta.

So, T 3 x turns out to be 4 x cube minus 3 x, so in general T n x is polynomial n x of degree n. And we note that these functions T n x are orthogonal on the interval minus 1 into 1 with respect to the weight function p x equal to 1 minus x square raise to the power minus half, this one can easily show by taking cos inverse x equal to theta. And then, we can show that the integral over minus 1 to 1 T x into T m x into T n x d x is equal to 0 whenever m is not equal to n.

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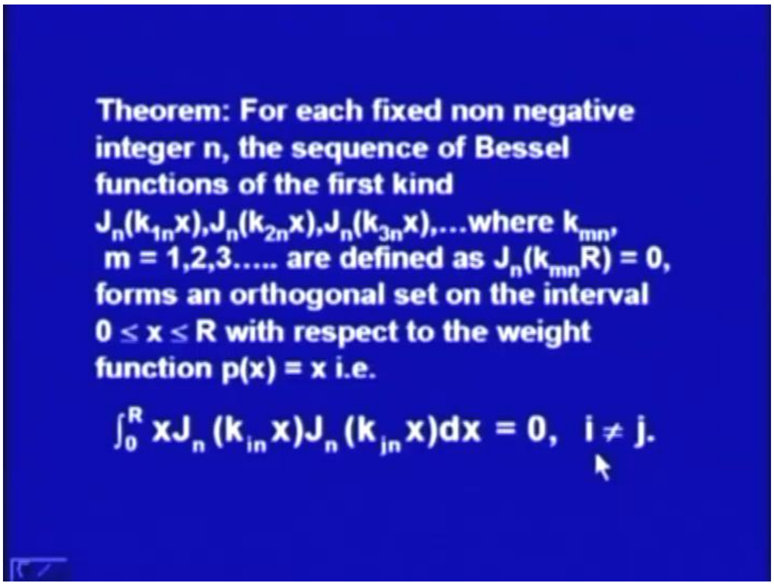

Let us study this theorem on orthogonality Bessel functions, for each fixed a non negative integer n, the sequence of Bessel functions of the first kind J n (k 1 n x), J n (k 2 n x), J n (k 3 n x) and so on. Where k m n, m is equal to 1, 2, 3 are defined as J n (k m n R) equal to 0 that is k m n R are the roots of the Bessel function J n x is equal to 0. Form an orthogonal set on the interval 0 less than or equal to x, less than or equal to R with respect to the weight function p x equal to x, that is we have the result integral 0 to R x J n (k i n x) into J n (k j n x) d x is equal to 0, whenever i is not equal to j.

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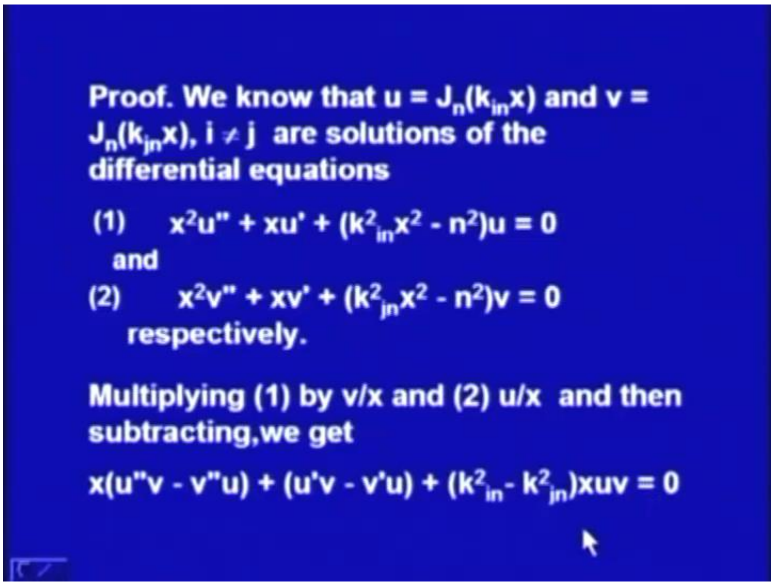

Let us look at the proof of this theorem, we know that u equal to J n (k i n x) and v equal to J n (k j n x), i not equal to j are solutions of the differential equations. X square u double dash plus x u dash plus k i n square x square minus n square into u equal to 0 and x square v double dash plus x v dash plus k j n square x square minus n square into v equal to 0.

We had obtained such a differential equation in the article on transformation of Bessel's equation, when we had made the first substitution, we change the independent variable from T 2 lambda x there. So, from that it comes that u equal to J n (k i n x) is a solution of this, and v equal to J n (k j n x) is the solution of this equation number 2. Now, let us multiply the equation 1 by v over x and equation 2 by u over x and then, subtract we shall get x times u double dash v minus v double dash u plus u dash v minus v dash u plus k i n square minus k j n square x into u into v equal to 0.

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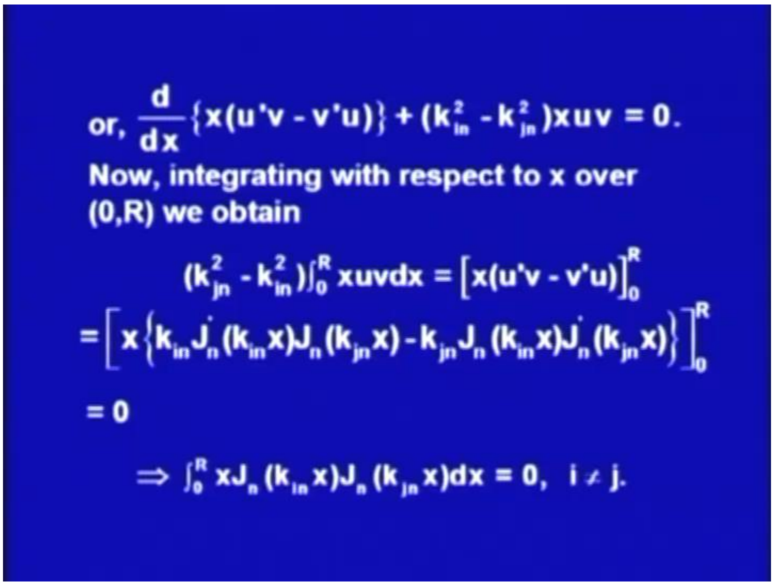

Now, this first two term in the last equation can be combined and we were write them as the differential of x into u dash v minus v dash u and then, the third term is as it is k i n square minus k j n square x into u into v equal to 0. Let us now integrate with respect to x over the interval (0, R), then we shall have k j n square minus k i n square into 0 to R x into u into v d x equal to x into u dash v minus v dash u evaluated 0 R.

Let us substitute the values of u and v here, the right hand side becomes x into the derivative of u is J n (k i n x), when we differentiate that with respect to x we get k i n into J n dash (k i n x), v was J n (k j n x), then minus v dash is k j n into J n dash (k j n x) and u is J n (k i n x). So, when you put the limits R and 0 here, then because of the effect that J n (k i n x) J n k i n R is equal to 0 and J n k j n R equal to 0.

When we put x equal to R here, this becomes 0 and also this becomes 0 and when you put the lower limit x is equal to 0, again because of x here this whole thing becomes 0, so we have the right hand side equal to 0. And this therefore, implies that if i is not equal to j, we can divide by k j n square minus k i n square in this equation and that will give us integral to 0 to R x into u into v d x equal to 0 replacing the values of u and v, we get integral 0 to R x into J n (k i n x) into J n (k j n x) d x equal to 0.

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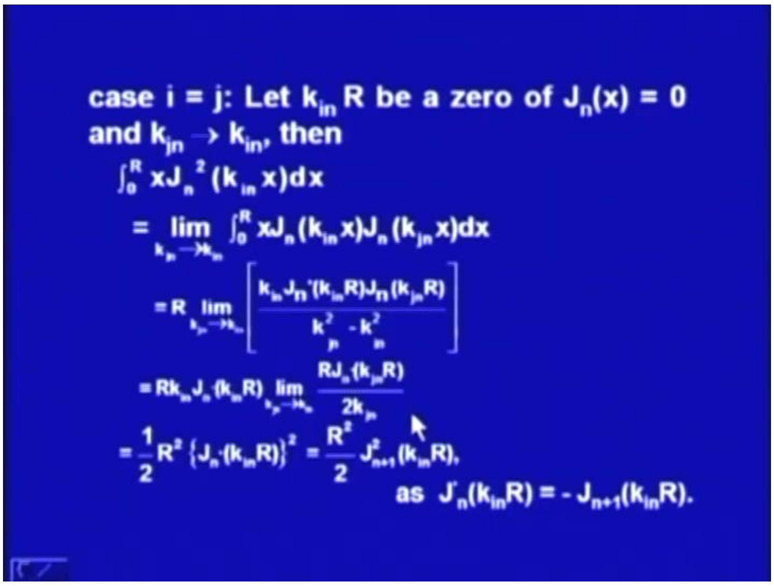

Next let us study the case i is equal to j, so let us assume that k i n R is a 0 of J n x equal to 0 that is we assume that J n k i n R is equal to 0. And let us assume that k j n goes to k i n, then integral 0 to R x J n square (k i n x) d x can be expressed as the limit of integral 0 to R x J n (k i n x) into J n (k j n x) d x as k j n goes to k i n. Now, from our last slide we see that integral 0 to R x J n into (k i n x) d x into x J n (k i n x) into J n (k j n x) d x can be written as k i n into J n dash k i n R into J n k j n R over k j n square minus k i n square, because J n k i n R is equal to 0.

So, this limit of this multiplied by R we have and then, this becomes R times k i n into J and x k i n R into limit k j n goes to k i n R j n dash k j n R over 2 j n, this we get by using the L'Hospital's rule, because this is of the form 0 by 0, then k j n goes to k i n. So, differentiating with respect to k j n we get this expression as this and which is equal to half R J n dash k i n R whole square. And that is further equal to R square by 2 J n plus 1 square k i n R, as we know that J n dash k i n R is equal to minus J n plus 1 k i n R in view of the fact that J n k i n R is equal to 0.

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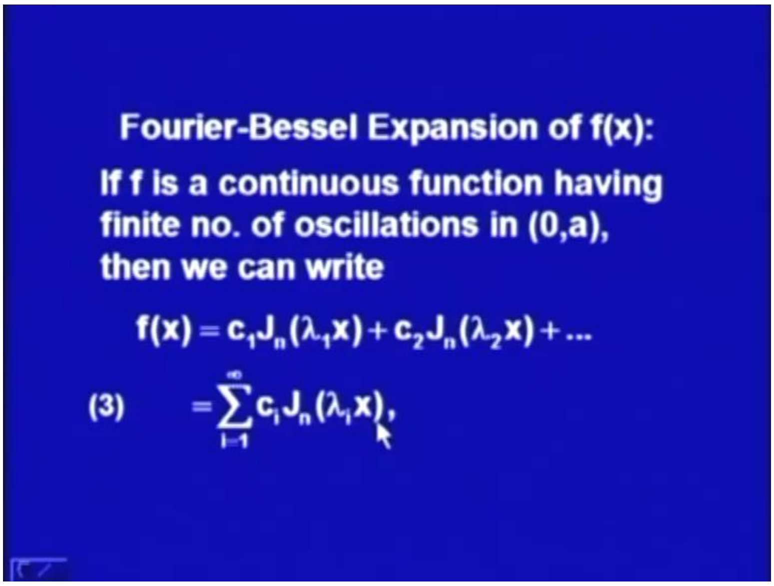

Next we study the Fourier Bessel expansion of f x, if we have a continuous function which has finite number of oscillations in an interval 0, a, then it can be expressed as in terms of Bessel's functions of a given order n. So, let us say f x, we want to write f (x)as an in finite series in terms of Bessel functions of order n, then we assume that f (x)is equal to c 1 into J n lambda 1 x plus c 2 J n lambda 2 x and so on. Or in short we can write the function f (x) as sigma i equal to 1 to infinity c i J n lambda i x.

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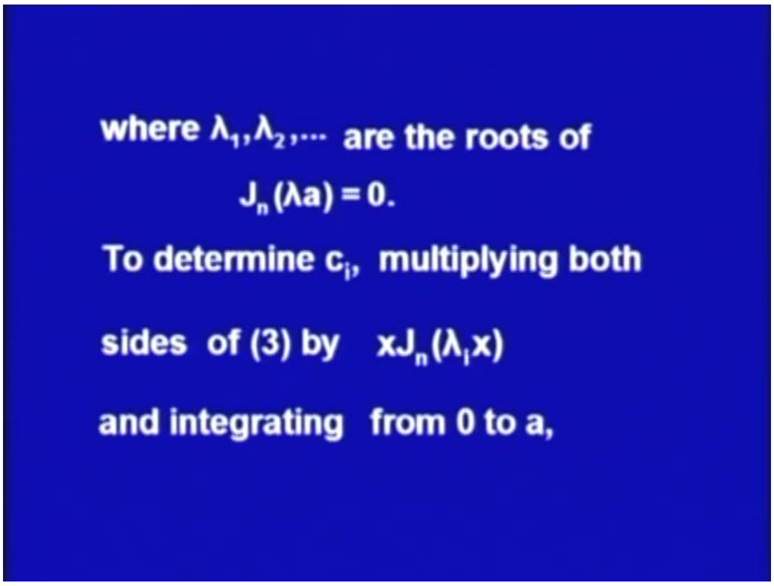

Where lambda 1, lambda 2, lambda 3 and so on, are the roots of the equation J n lambda a equal to 0 to determine the unknown coefficient c i is, let us multiply both sides of the equations 3 by x J n lambda i x and integrate from 0 to a.

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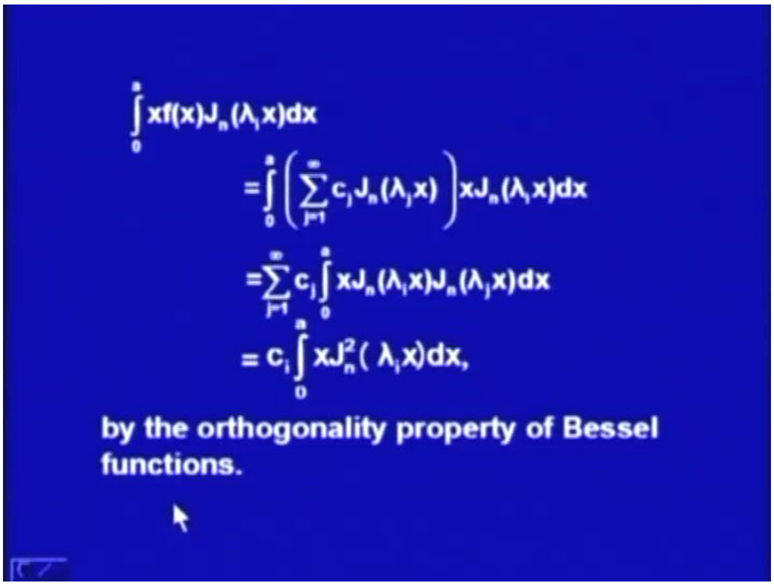

Then, we shall have integral 0 to a x f (x)into j n lambda i x d x equal to integral 0 to a sigma j equal to 1 to infinity c j J n lambda j x into x J n lambda i x d x, which can be also written as equal to sigma j equal to 1 to infinity c j integral 0 to a x J n lambda j x into J n lambda j x d x. Now, making use of the orthogonality of Bessel functions, we ignore that integral 0 to a x J n lambda x into J n lambda j x d x is equal to 0, whenever j is not equal to i. And therefore, this equation further reduces to c i times integral 0 to a x J n square lambda i x d x by the orthogonal property of Bessel functions.

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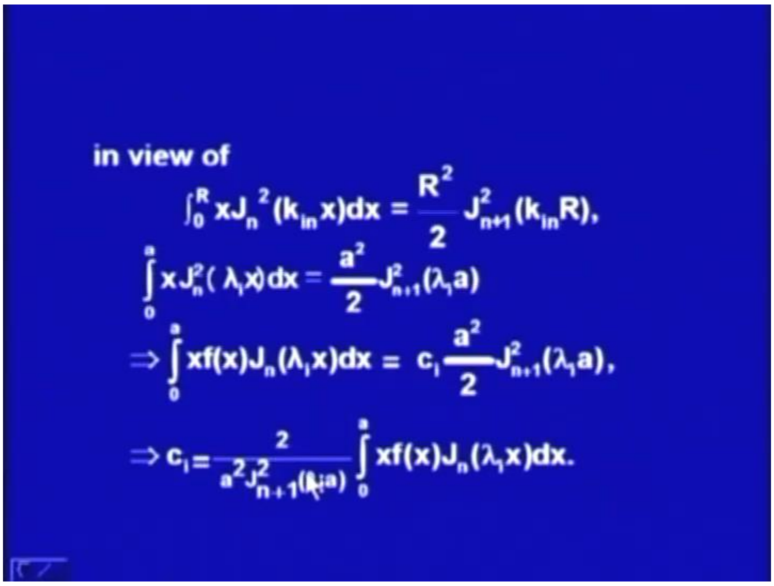

Now, we have seen that when i is equal to J integral 0 to R x J n square k i n x d x is equal to R square by 2 J n plus 1 square k i n R, therefore integral 0 to a x J n square lambda x d x will be equal to a square by 2 J n plus 1 square lambda i a. And hence, integral 0 to a x f (x)J n lambda i x d x is equal to c i times a square by 2 J n plus 1 square lambda i a. From where we get the value of c i as 2 over a square into J n plus 1 square lambda i a multiplied by integral 0 to a x f (x) J n lambda i x d x, and thus we get the values of the unknown coefficients c i is which occur in the Fourier Bessel expansion of the function f x.

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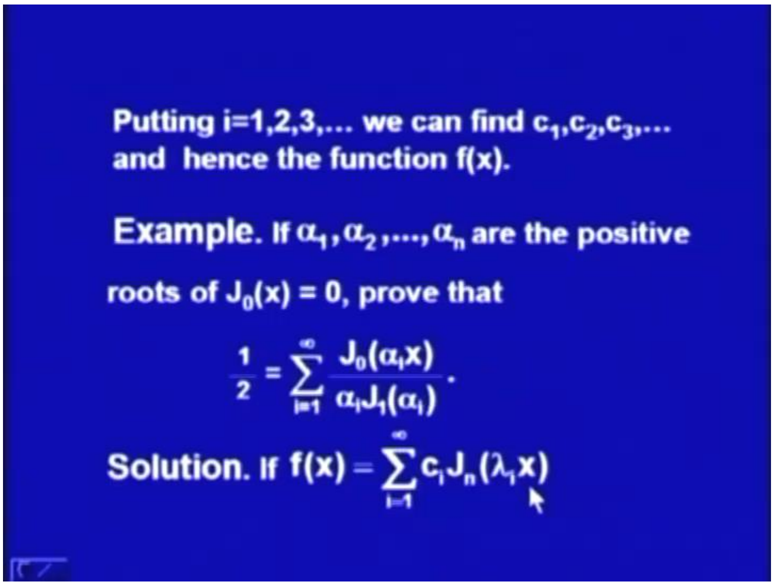

So, putting i equal to 1, 2, 3, we can find the values of c 1, c 2, c 3 and so on, and hence the function f (x) can be expanded in the desired in finite series of Bessel functions. Now, let us study an example based on this article if alpha 1, alpha 2 and so on, alpha n are the positive roots of the equation J naught x equal to 0, J naught x is the Bessel function of order 0, so J naught x is equal to 0.

Let us show that half is equal to sigma i is equal to 1 to infinity J naught alpha x over alpha i J 1 alpha i or we may say that 1 is equal to sigma i equal to 1 to infinity 2 times J naught alpha i x over alpha i J 1 alpha i. So, we wish to expand the function f (x) equal to 1, in terms of n infinite series of Bessel functions of order 0, now we know that if f (x) is equal to sigma i equal to 1 to infinity c i J n lambda i x.

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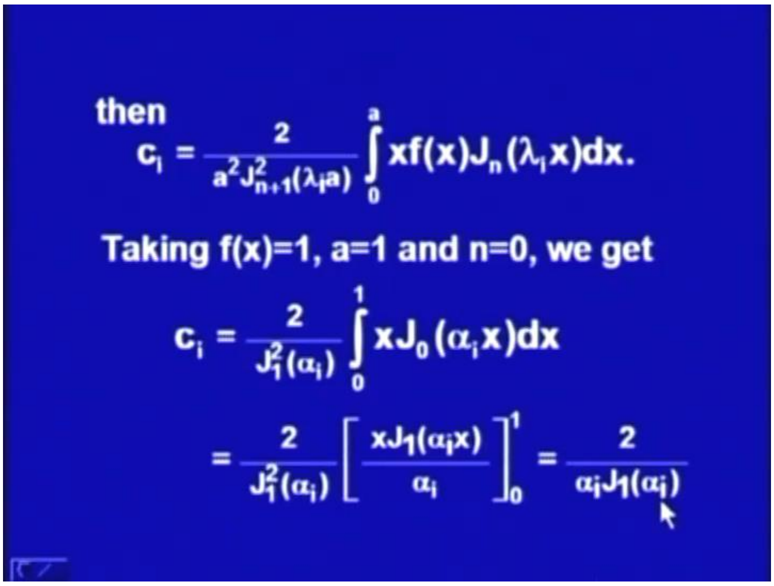

Then, we have seen that c i is are given by 2 over a square J n plus 1 square lambda i a integral 0 to a x f (x) J n lambda i x d x, where lambda 1, lambda 2 and lambda 3 and so on, are the roots of the equation J n lambda a equal to 0. So, in this let us take f (x) equal to 1, a equal to 1 and n equal to 0, because we wish to expand the function f (x) equal to 1, in terms of Bessel functions of order 0.

So, we take f (x) equal to 1 and more over we are given that alpha 1, alpha 2, alpha n are the roots of the equation, J naught x equal to 0, so we take a equal to 1, then we will get c i is equal to 2 over J 1 square alpha i integral 0 to a x J naught alpha i x d x. Now, we know that from the differential formulae for Bessel function, we know that integral of x J naught x is x J 1 x. So, integral of x J naught alpha i x d x over the interval 0 to 1 will be equal to x J 1 alpha i x over alpha i evaluated at over 0, 1. And therefore, c i is will be equal to 2 over J 1 square alpha i into x J 1 alpha i x over alpha i, when you put the limits 0 and 1 here, we get this c i is to be equal to 2 over alpha i J 1 alpha i.

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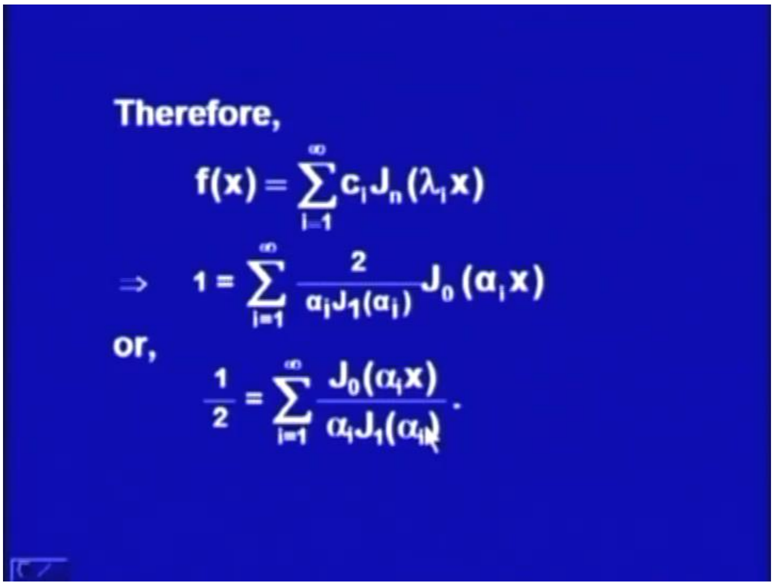

Therefore, f (x) is equal to sigma i equal to 1 to infinity c i J n lambda i x implies that 1 is equal to sigma i equal to 1 to infinity 2 over alpha i into J 1 alpha i into J naught alpha i x. And then, dividing by 2 we see that, half is equal to sigma i equal to 1 to infinity J naught alpha i x over alpha i J 1 i, which gives us the record proof.

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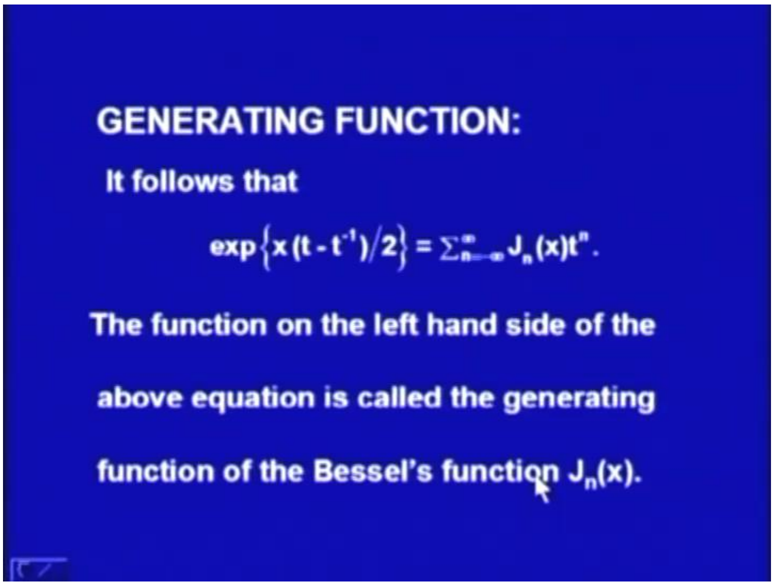

Let us now study the generating function for Bessel functions, we are going to show that exponential of x t minus t to the power of minus 1 over 2 can be written as sigma n equal to minus infinity to plus infinity J n x into t to the power n. So, the Bessel functions of various orders can be derived from the coefficients of the powers of t, and that is why we call this exponential of x t minus t to the power minus 1 over 2 as the generating function for the Bessel's functions. The function on the left hand side of the above equation is called the Bessel generating function of the Bessel's function J n x.

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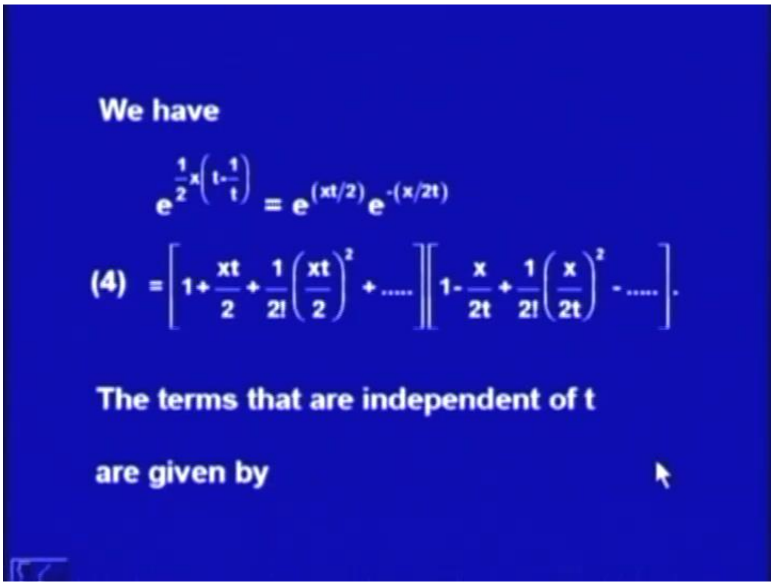

We can write e to the power half x into t minus t to the power minus 1 equal to e to the power x t by 2 into e to the minus x by 2 t. Let us then put the Maclaurin series expansion of e to the power x t by 2 and e to the power minus x by 2 t to have the right hand side as 1 plus x t by 2 plus 1 by 2 factorial x t by 2 whole square and so on. And then, 1 minus x by 2 t plus 1 by 2 factorial x by 2 t whole square and so on, then let us collect the coefficients of by this powers of t.

First we look at the coefficients of t to the power 0, that is the terms that are independent of t, such terms will come when we will multiply 1 by 1 had been multiplied t by 1 by t or t square by 1 by t square. That is the t to the power and coefficient of t power and is multiplied by the coefficient of t to the power minus and from here, we will just gives us the terms that are independent of t.

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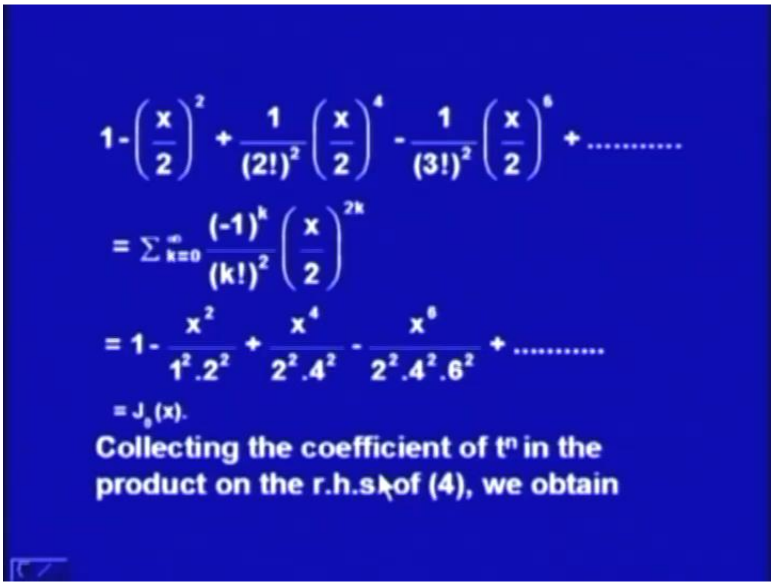

So, we collect those terms that are independent of t and we see that, that are come out to be 1 minus x by 2 whole square plus 1 by 2 factorial whole square into x by 2 raise to the power 4 minus 1 by 3 factorial whole square x by 2 raise to the power 6 and so on, which can be written in the form of sum sigma k equal to 0 to infinity minus 1 to the power k over k factorial square multiplied x by 2 raise to the power 2 k.

And then, we can put at in this convenient form 1 minus x square by 1 square into 2 square plus x to the power 4 over 2 square 4 square minus x to the power 6 over 2 square 4 square 6 square and so on, which we know is nothing but J naught x. So, the coefficient of t to the power 0 is nothing but J naught x, now let us collect the coefficient of t to the power n in the product on the right hand side of 4.

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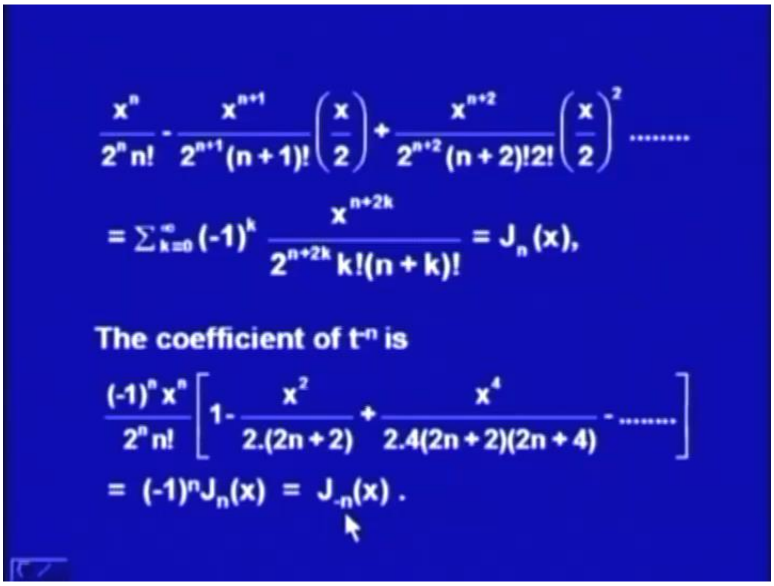

So, that means in the first series, we will multiply the term with that power n by 1, then we will multiply t to the power n plus 1 by t, then t to the power n plus 2 by 1 by t square and so on, and to have the coefficients of t to the power n. And they come out to be x to the power n over 2 to the power n over n factorial minus x to the power n plus 1 over 2 to the power n plus 1 into n plus 1 factorial into x by 2.

Or you can say the second term is x by 2 raise to the power n plus 2 and then, x by 2 raise to the power n plus 2 is into x by 2 whole square is x by 2 raise to the power n plus 4 here divided by n plus 2 factorial into 2 factorial and so on, which can be writ10 as sigma k equal to 0 to infinity minus 1 to the power k, then x to the power n plus 2 k over 2 to the power n plus 2 k into k factorial n plus k factorial and which is equal to J n x.

Next the collect the coefficient of t to the power minus n, so for that we may multiply the coefficient of t to the power minus n by 1, then t to the power minus n minus 1 by t and then t to the power minus n minus 2 by t square and so on. And we see that we get the coefficients of t to the minus n as minus 1 to the power n into x to the power n over 2 to the power n into n factorial multiplied by this infinite series 1 by x square over 2 into 2 n plus 2 plus x to the power 4 over 2 into 4 into 2 n plus 2 into 2 n plus 4 and so on, which is equal to minus 1 to the power n into J n x. And we know that when n is an integer, minus 1 to the power n into J n x is J minus n x.

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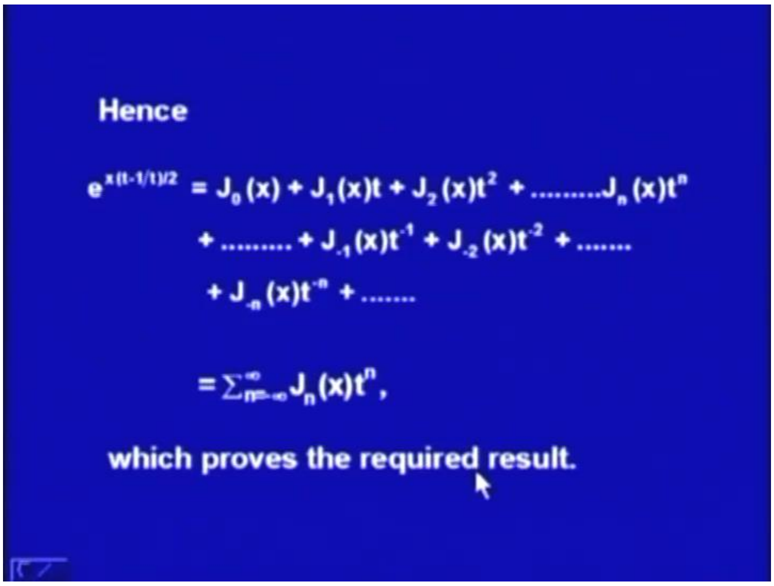

Hence, we can say that e to the power x into t minus 1 over t by 2 is equal to the term that contains that are independent of t is J naught x, the terms which give us the terms in t, t square, t cube, t to the power n and so on, their coefficients are J 1 x, J 2 x, J n x and so on. Because, we saw that the coefficient of t to the power n is J n x and then, the coefficient of t to the power minus n and we have seen is J minus n x, so the coefficients of t to the power minus 1 is J minus 1 x.

The coefficient of t to the power minus 2 is J minus 2 x and so on, the coefficient of t to the power minus n is J minus n x and so on. So, we can write this in the form of the summation sigma n equal to minus infinity to infinity J n x into t to the power n, which gives us the required result.

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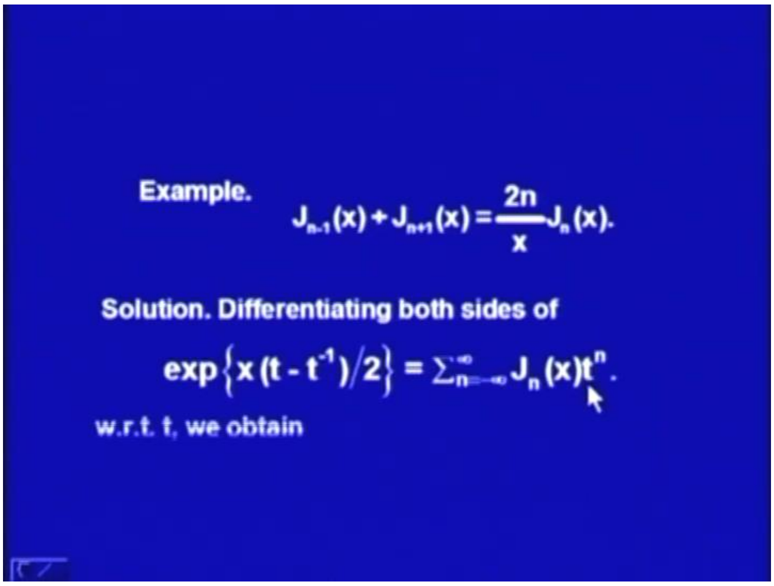

Now, let us study an example based on the generating function, let us establish this formula for Bessel functions J n minus 1 x plus J n plus 1 x is equal to 2 n over x J n x, we had earlier shown this. Now, we are going to prove this, using the article on generating functions, so we know that e to the power x into t minus 1 by t divided by 2 is equal to sigma n equal to minus infinity to infinity J n x into t to the power n.

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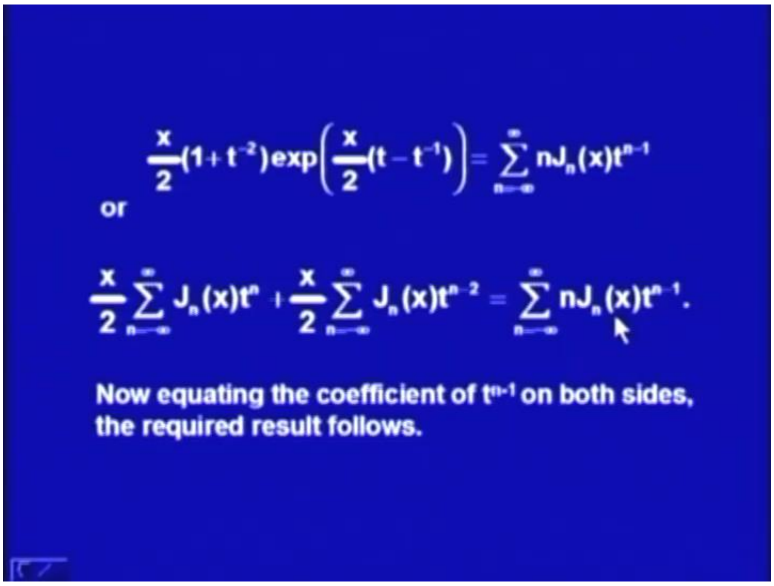

When we differentiate this equation with respect to t what we get is this, x by 2 1 plus t to the power minus 2 into exponential of x by 2 t minus 1 by 2, t is equal to sigma n equal to minus infinity to infinity n into J n x into t to the power n minus 1. Now, we know that the left hand side can be written as x by 2 into e to the power x by 2 t minus t to the power minus 1. And the generating function e to the power x by 2 t minus t to the power minus 1 is equal to sigma n equal to minus infinity to infinity J n x t to the power n.

So, we get x by 2 into sigma n equal to minus infinity to infinity J n x t to the power n and then, plus x by 2 into t to the power minus 2 multiply to sigma n equal to minus infinity to infinity J n x t to the power n gives you x by 2 sigma n equal to minus infinity to infinity J n x t to the power n minus 2. So, left hand side becomes the sum of these two terms and then, it is equal to sigma n equal to minus infinity to infinity and J n x t to the power n minus 1.

So, in this equation what we have done is simply we have put the value of e to the power x by 2 t minus t to the power minus 1 as sigma n equal to minus infinity to infinity J n x t to the power n. Now, we equate the coefficient of t to the power n minus 1 on both sides, coefficient n minus 1 here will be x by 2 into J n minus 1, the coefficient of t to the power n minus 1 here will be x by 2 J n plus 1 and the coefficient of t to the power n minus 1 will be n J n x and that will give us the required result.

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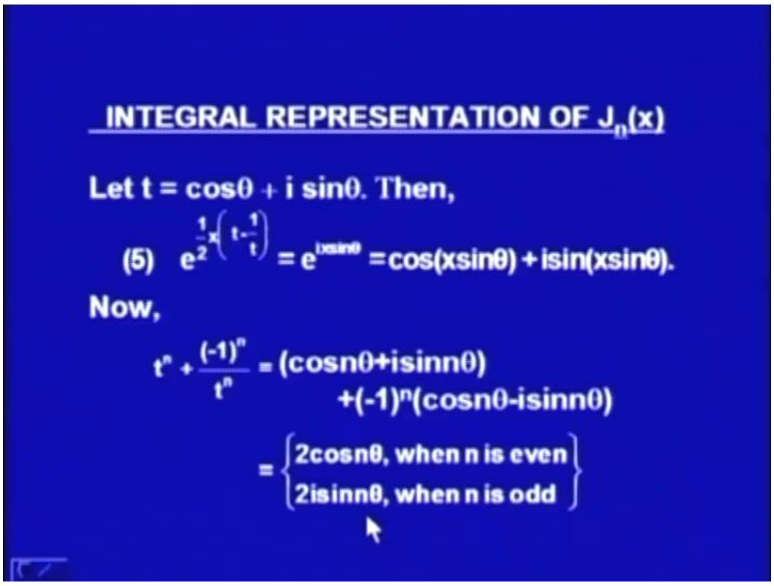

Now, let us study the integral representation of Bessel functions of first kind that is J n x, we are going to show that the Bessel function of first kind of order n which is an in finite series can be expressed as an integral, which we call as Bessel integral, if n is an integer. So, let us put t equal to cos theta plus i sin theta, let t be equal to cos theta plus i sin theta, then the generating function for the Bessel functions e to the power half x t minus 1 by t will be equal to e to the power i x sin theta.

Because, t is cos theta plus i sin theta, so 1 by t will be cos theta minus i sin theta, so t minus 1 by t will be 2 i sin theta and therefore, e to the power half x into 2 i sin theta will give us e to the power i x sin theta. And by the Euler's formula we know that e to the power i x sin theta is cos x sin theta plus i sin x sin theta. Now, t to the power n plus minus 1 to the power n over t to the power n will be equal to t to the power n by De Moivre's theorem will be cos theta plus i sin theta to the power n.

So, n is an integer, therefore t to the power n will be cos n theta plus i sin n theta, then minus 1 to the power n, 1 over t to the power n will be t to the power minus n, so cos theta plus i sin theta to the power minus n will be cos n theta minus i sin n theta. If n is an even integer, then the right hand side, therefore is 2 cos n theta and when n is an odd integer the right hand side becomes 2 i sin n theta. So, the value of t to the power n plus minus 1 to the power n over t to the power n is 2 cos n theta, when n is an even integer and it is 2 i sin n theta when n is an odd integer.

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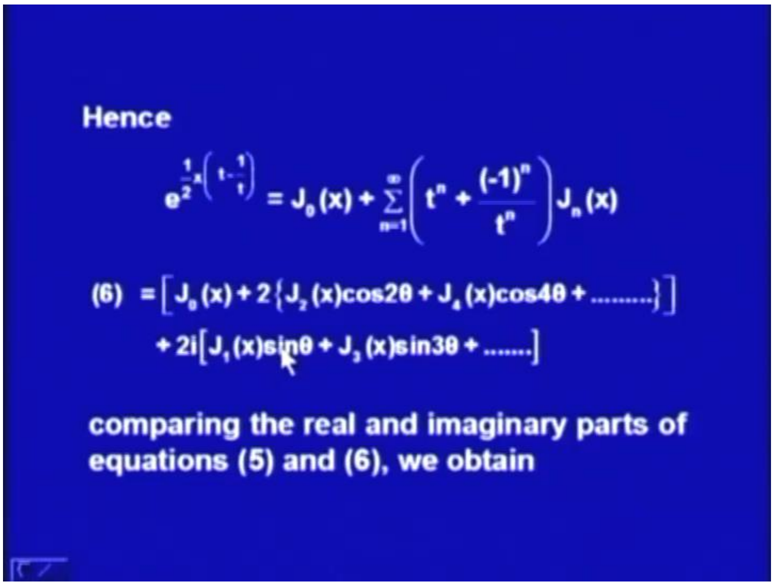

Hence, the generating function e to the power half x t minus 1 over t, which is equal to J naught x plus sigma n equal to 1 to infinity t to the power n J n x plus sigma n equal to 1 to infinity minus 1 to the power n over t to the power n J n x will be equal to J naught x plus 2 times J 2 x cos 2 theta J 4 x cos 4 theta and so on. Because, when n is even here, t to the power n plus minus 1 to the power n over t to the power n becomes 2 cos n theta, so 2 cos n theta into J n x when n is an even integer.

And when n is an odd integer, the value of t to the power n plus minus 1 to the power n over t to the power n is 2 i sin n theta, so for odd integral values of n, this sum will give us the terms 2 i J 1 x sin theta plus J 3 x sin 3 theta and so on. Now, let us compare the real and imaginary parts of equations 5 and 6, in the equation 5 we have seen that e to the power half x t minus 1 by t is cos x sin theta plus i sin x sin theta, so cos x sin theta plus i sin x sin theta is equal to this.

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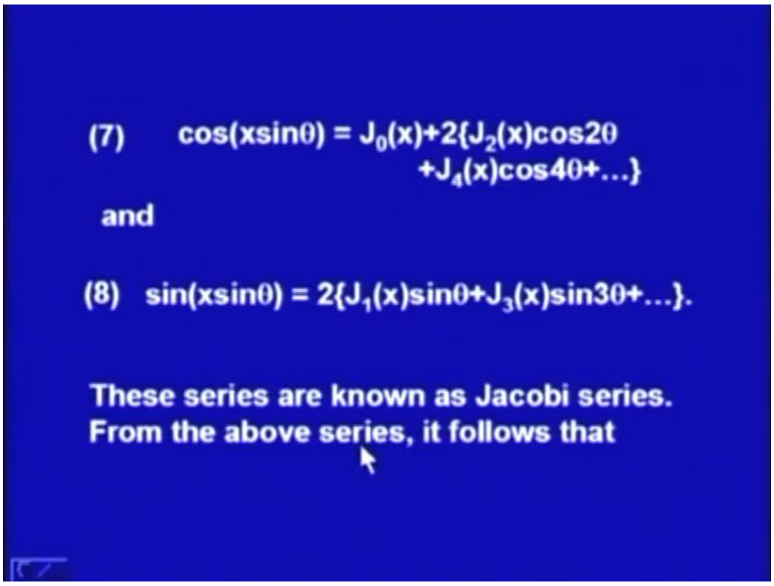

So, when the equate real and imaginary parts on the two sides, we get cos x sin theta equal to J naught x plus 2 times J 2 x cos 2 theta plus J 4 x cos 4 theta and so on. And sin x sin theta equal to 2 times J 1 x times sin theta plus J 3 x sin 3 theta and so on, these series are known as the Jacobi series.

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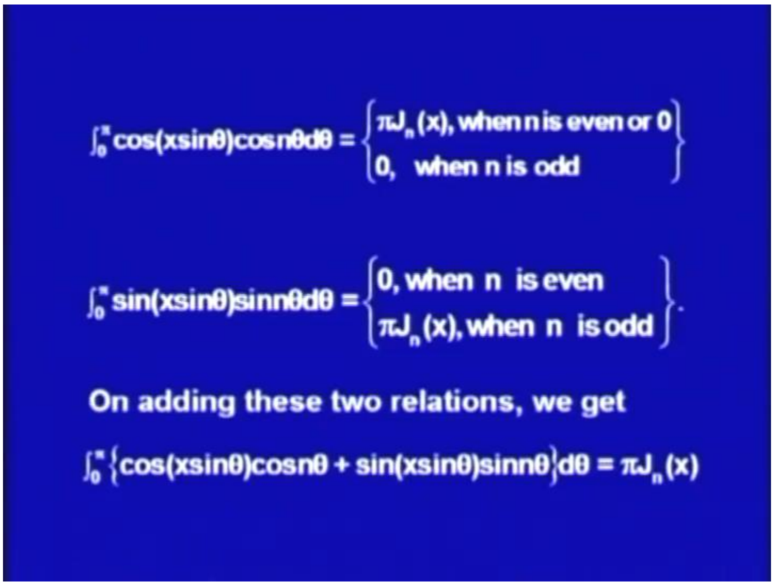

Now, from these series it follows that, if you multiply the Jacobi series for cos x sin theta by cos n theta and integrate over 0 to pi, then we know that the cosine functions cos n theta are orthogonal over the interval 0 to pi that is whenever m is not equal to, that is integral 0 to pi cos n theta into cos n theta d theta is 0 whenever m is not equal to n. And in the case when m is equal to n, we know that 0 to pi cos square n theta d theta is pi by 2, so making use of that 0 to pi cos x sin theta into cos n theta d theta is equal to pi into J n x when n is an even integer or n is equal to 0.

And it will be 0 when n is an odd integer, because when n is an odd integer cos n theta into cos n theta integral of that will be 0, because m will be never equal to n, on the right hand side of cos x sin theta we have only even integral multiples of theta in cosine terms. And then, in the Jacobi series for sin x sin theta, we multiply sin x sin theta by sin n theta and similarly, we see that the value of integral 0 to pi sin x sin theta sin n theta d theta is equal to 0 when n is even.

Because, in the sin x sin theta expansion of that we have sin of odd multiples of theta on the right side, so when we multiply by sin n theta where n is even, then integral sin m theta into sin n theta d theta will always be 0, because m will be not equal to n. And then, integral of sin square n theta d theta, when n is odd will give us pi into J n x, so now let us add these two equations, when we add these two equations, we find that integral 0 to pi cos x sin theta into cos n theta plus sin x sin theta into sin n theta d theta is equal to pi into J n x.

Because, if n is 0 then this is 0, obviously this is 0 and when n is 0, 0 to pi cos x sin theta into cos n theta d theta is equal to pi into J n x. And when n is an even integer, then integral 0 to pi sin x sin theta sin n theta is 0, we have seen here an integral 0 to pi cos x sin theta into cos n theta d theta is pi into J n x, so again we get the right side. And when n is an odd integer here, then integral 0 to pi sin x sin theta sin n theta is pi into J n x, while the other integral 0 to pi cos x sin theta into cos n theta d theta is 0. So, when we add these two equations for all integral values of n, 0, 1, 2, 3, n equal to 0, 1, 2, 3 and so on, we see that integral 0 to pi cos x sin theta cos n theta plus sin x sin theta sin n theta d theta is pi into J n x.

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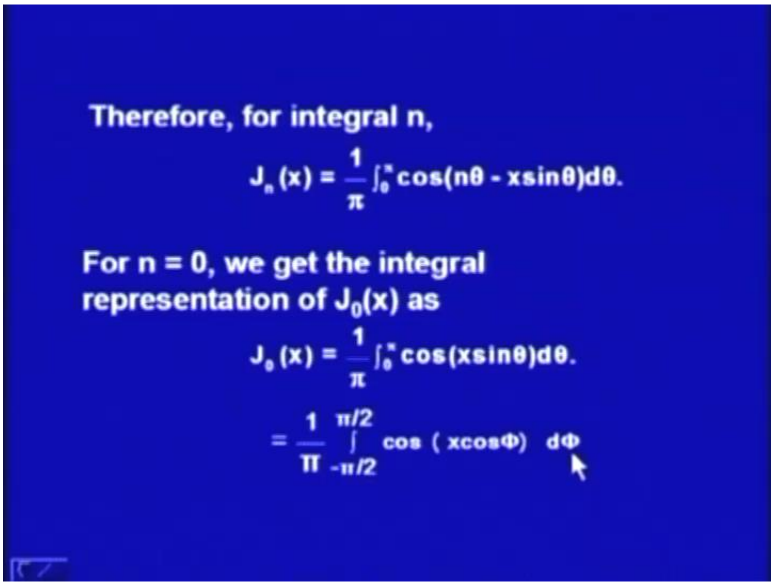

And which can be put in an alternate form as J n x equal to 1 over pi integral 0 to pi cos n theta minus x sin theta d theta, because we know that cos of a minus b is cos a cos b plus sin a sin b. For n equal to 0, from here we can see for n equal to 0 we get that J naught x is equal to 1 over pi 0 to pi cos x sin theta d theta, now using the property of definite integrals we can see that, if you put here theta equal to pi by 2 minus pi,

If you make this substitution theta is equal to pi by 2 minus pi, then this integral representation of J naught x changes into 1 over pi integral over minus pi by 2 to pi by 2 cos x cos phi d phi, here we have put theta equal to pi by 2 minus pi. Now, since cos x cos phi is an even function of phi may be use of the property of definite integrals, we can write the integral over minus pi by 2 to pi by 2 cos x cos phi d phi as 2 times integral 0 to pi by 2 cos x cos phi d phi.

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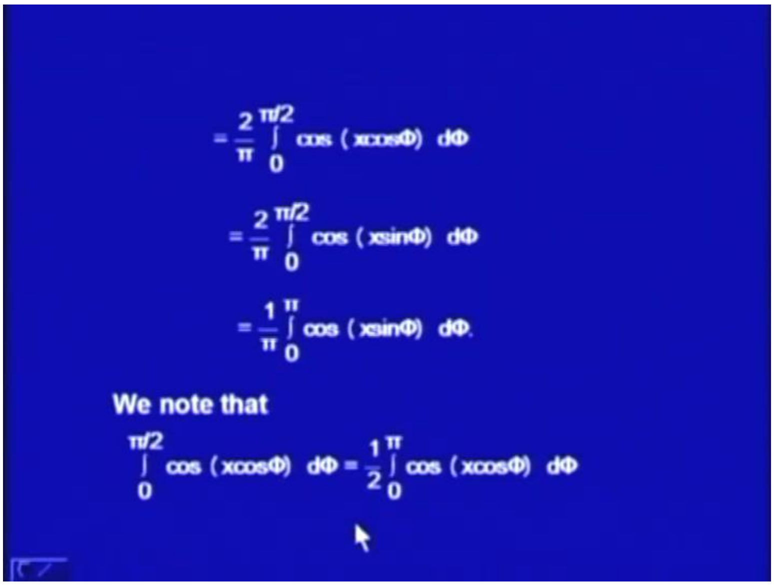

And so we will get 2 over pi 0 to pi by 2 cos x cos phi d phi, which is equal to 2 over pi integral 0 to pi by 2 cos x sin phi d phi, because we have the property of the definite integrals that integral 0 to a, f (x) d x is equal to integral 0 to a f a minus x d x; so when you replace phi by pi by 2 minus phi here, you get cos x sin phi. Now, if you look at integral 0 to pi cos x sin phi d phi, then we see that 0 to pi cos x sin phi d phi is equal to 2 times 0 to pi by 2 cos x sin phi d phi, again by using the property of definite integrals.

Because, we have the property of definite integrals they says that integral 0 to 2 a f (x) d x is equal to 2 times 0 to a f (x) d x provided f 2 a minus x is equal to f x, so making use of that property these two are equal. And therefore, we have J naught x also equal to 1 over pi integral 0 to pi cos x sin phi d phi, now we also note that integral 0 to pi by 2 cos x cos phi d phi is equal to half of integral 0 to pi cos x cos phi d phi.

Because, again when you replace phi by pi minus phi here, you get cos of minus x cos phi which cos of minus theta is cos theta, so you get cos of x cos phi. And therefore, again making use of the property of definite and definite integrals 0 to pi cos x cos phi d phi is nothing but it is equal to 2 times 0 to pi by 2 cos x cos phi d phi.

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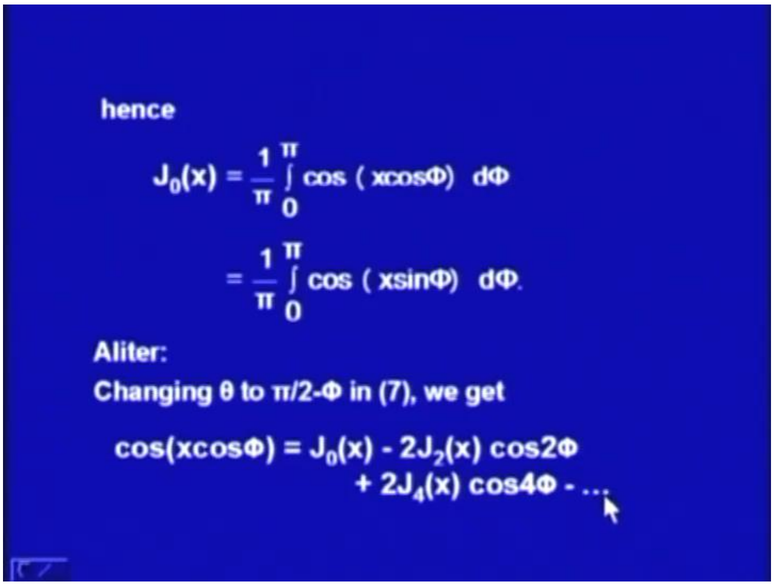

So, the integral representation of J naught x is has another form J naught x can be also written as 1 over pi 0 to pi cos x cos phi d phi, so we have two integral representations, when the limits are 0 and pi of the Bessel functions of other 0. One is J naught x equal to 1 over pi 0 to pi cos x cos phi d phi, another one is J naught x equal to 1 over pi integral 0 to pi cos x sin phi d phi.

Now, we can arrive at this integral representation of J naught x in an another way from Jacobi series, also we can get this integral representation of J naught x directly. If in the Jacobi series equation number 7 you change the theta to pi by 2 minus phi, if you put theta equal to pi by 2 minus phi there in the Jacobi series given in equation 6. Then, we get that cos x cos phi, sin phi becomes cos phi there and then, cos 2 theta, cos 4 theta becomes cos 2 phi minus cos 2 phi and then, plus cos 4 phi. So, right hand side of that equation becomes J naught x minus 2 J 2 x cos 2 phi plus 2 J 4 x cos 4 phi and so on.

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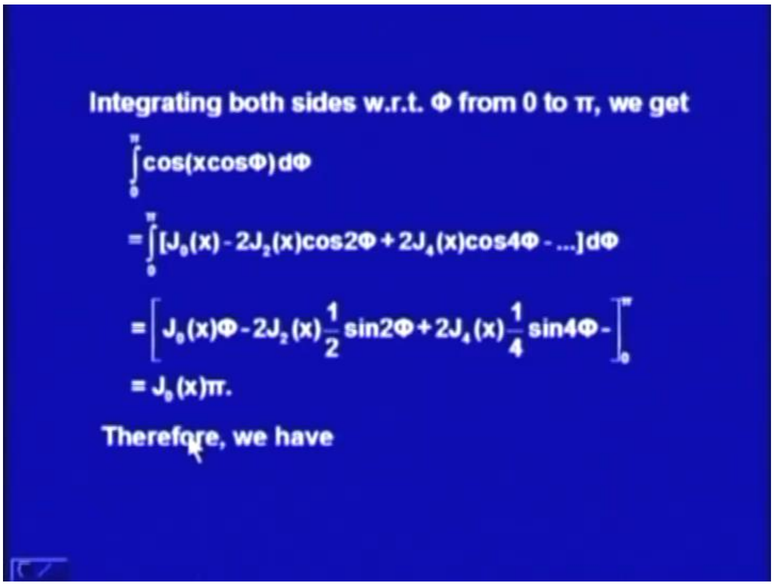

And when we integrate this equation with respect to phi from 0 to pi, what we get is 0 to pi cos x cos phi d phi is equal to integral 0 to pi J naught x minus 2 J 2 x cos 2 phi plus 2 J 4 x cos 4 phi and so on, d phi. Now, let us integrate it we get integral of J naught x d phi with respect to phi is J naught x into phi, then 2 J 2 x by sin integral of cos 2 phi sin 2 phi by 2 and then, 2 J 4 x integral of cos 4 phi is sin 4 phi by 4.

So, now let us put the limits 0 in to pi, we will get the value of this expression as J naught x into pi, because sin n phi is always equal to 0. So, we get sin 2 pi, cos sin 4 phi all are 0's and at phi equal to 0 again this whole expression is 0, so the value of this expression is J naught x into pi.

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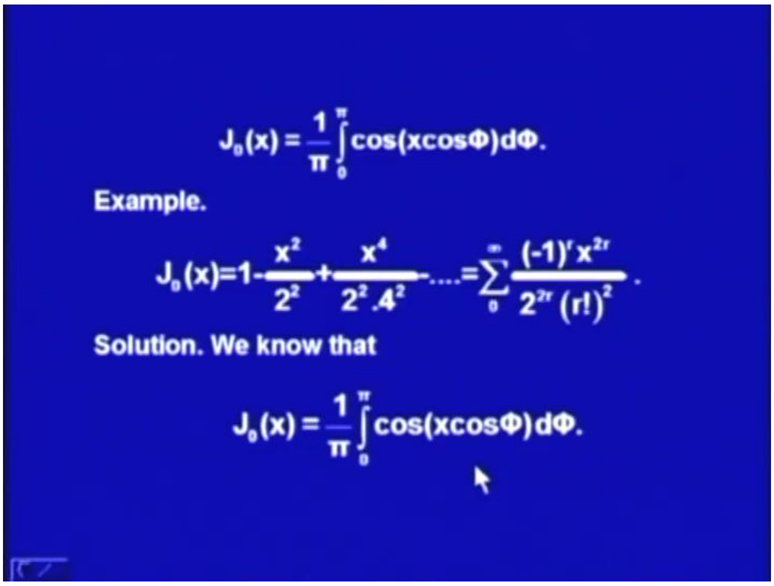

And therefore, we have J naught x equal to 1 over pi integral 0 to pi cos x cos phi d phi, so the value of J naught x can be evaluated in an independent manner directly from the Jacobi series equation number 7, rather than obtaining yet as a particular case of J n x for n equal to 0. Now, let us take an example which is based on the integral representation of Bessel functions, let us show that reading the integral representation of J naught x, the value of J naught x is equal to 1 minus x square by 2 square plus x 4 by 2 square into 4 square and so on.

That is sigma r equal to 0 to infinity minus 1 to the power r x to the power 2 r over 2 to the power 2 r into r factorial square, this expansion infinite series expansion of J naught x we shall derive using the integral representation of J naught x; we know that J naught x is equal to 1 over pi 0 to pi cos x cos phi d phi.

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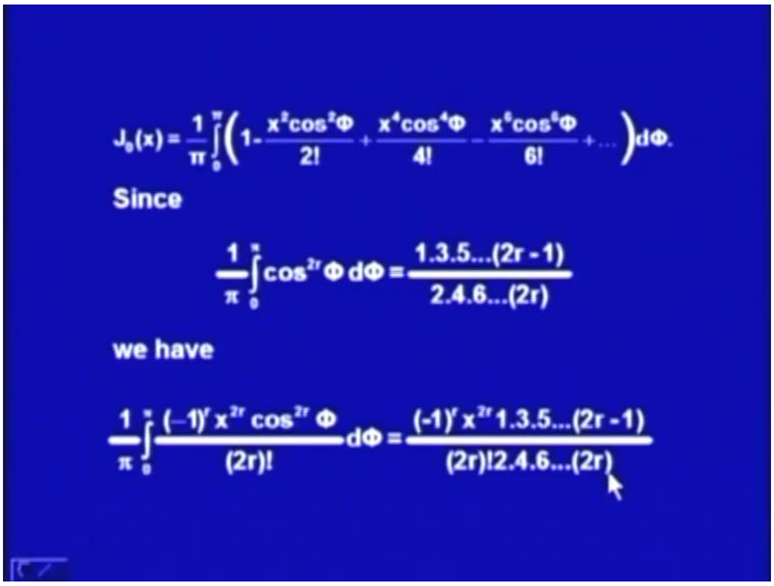

Now, let us put the Maclaurin series expansion of cos x cos phi, then we shall get J naught x equal to 1 over pi integral 0 to pi 1 minus x square cos square phi over 2 factorial plus x to the power 4 cos 4 phi over 4 factorial minus x to the power 6 cos 6 phi over 6 factorial and so on, d phi. Now, since 1 over pi integral 0 to pi cos 2 r phi d phi is equal 1, 3, 5 and so on, 2 r minus 1 over 2 into 4 into 6 and so on, 2 r.

This result can be derived by using the reduction formula for cos 2 r phi or it can also be obtained by the gamma function, because when you replace phi by pi minus phi, because of the even power of cos phi the sin will not change that is cos 2 r phi will remain cos 2 r phi, when we replace phi by pi minus phi. So, the limits will change from 0 to pi by 2 and we will get 2 over pi 0 to pi by 2 cos 2 r phi d phi, from functions we know that integral 0 to pi by 2.

Sin theta raise to the power m cos theta raise to the power n d theta is equal to gamma m plus 1 by 2 into gamma n plus 1 by 2 over 2 times gamma m plus n plus 2 by 2. So, making use of that, we can see that the value of this integral is 1 into 3 into 5 and so on, 2 r minus 1 over 2 into 4 into 6 and so on, 2 r, so then we shall have 1 over pi integral 0 to pi minus 1 to the power r into x to the power 2 r cos 2 r phi over 2 r factorial d phi.

Now, there is nothing but the general term of this expansion, we have taken the general term from this expansion and evaluated the integral of that with respect to phi over the interval 0 to pi and multiply by 1 over pi. So, we see that making use of this result the value of this integral comes out to be minus 1 to the power r x to the power 2 r 1, 3, 5 and so on, 2 r minus 1 over 2 r factorial 2 into 4 into 6 and so on, up to 2 r.

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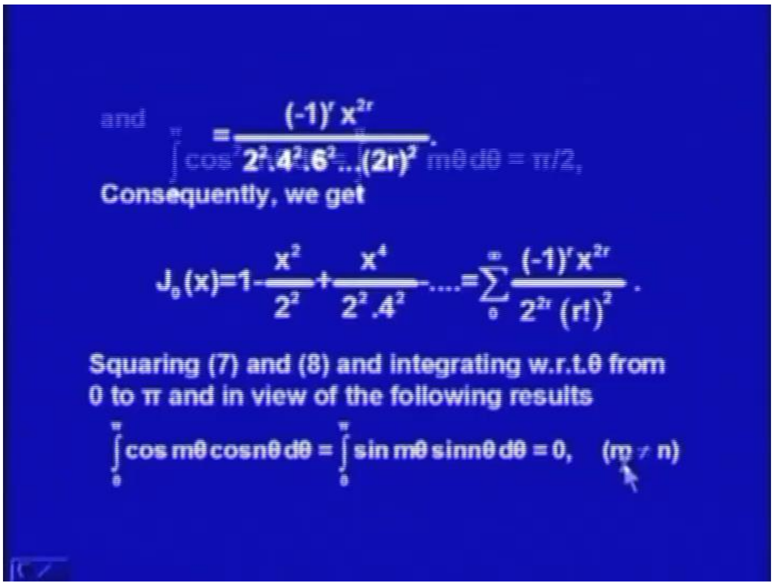

And which is equal to minus 1 to the power r x to the power 2 r over 2 square into 4 square into 6 square and so on, 2 r whole square. And consequently the value of J naught x is equal to sigma r equal to 0 to infinity minus 1 to the power r x to the power 2 r over 2 to the power 2 r into r factorial square, r in the expanded form; it has 1 minus x square by 2 square plus x to the power 4 over 2 square into 4 square and so on.

Now, let us square the equation number 7 and 8 and integrate with respect to theta over the interval 0 to pi, then in view of the following results. Integral 0 to pi cos m theta into cos n theta d theta and 0 to pi sin m theta sin n theta d theta being 0, whenever m is not equal to n.

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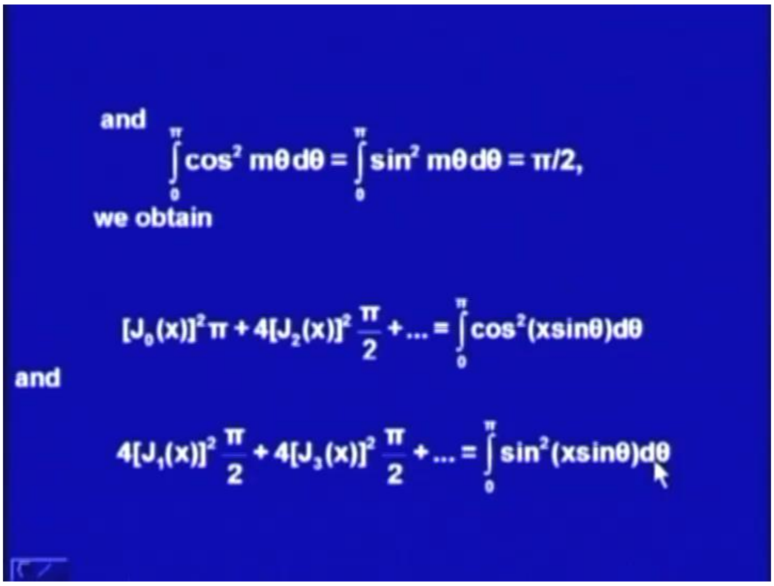

And integral 0 to pi cos square m theta d theta and 0 to pi sin square m theta d theta being equal to pi by 2, we obtain J naught x whole square into pi plus 4 J 2 x whole square into pi by 2 plus and so on, equal to integral 0 to pi cos square x sin theta d theta. And 4 times J 1 x whole square into pi by 2 plus J 3 x whole square into pi by 2 and so on, equal to integral 0 to pi sin square x sin theta d theta.

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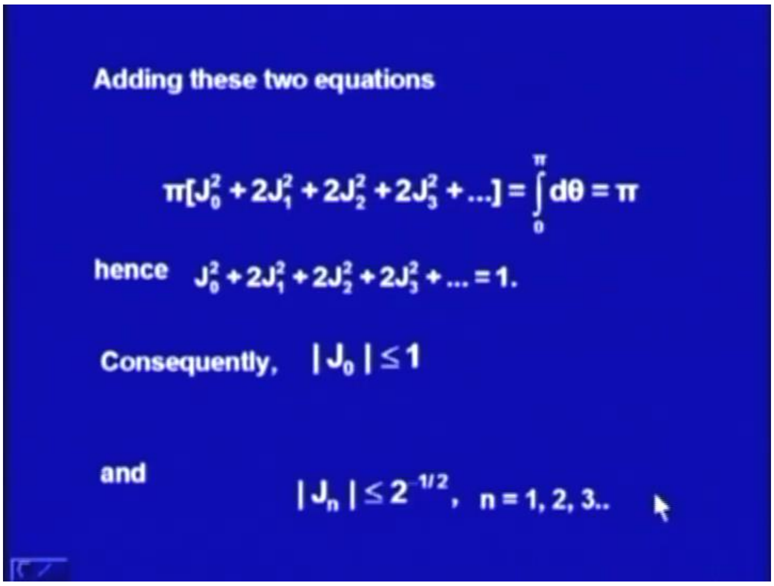

On adding these two equations we get pi into J naught square plus 2 times J 1 square plus 2 times J 2 square plus 2 times J 3 square and so on, equal to 0 to pi integral 0 to pi d theta. Because, cos square x sin theta plus sin square x sin theta is 1, so the right hand side becomes integral 0 to pi d theta which is equal to pi, and hence J naught square plus 2 J 1 square plus 2 J 2 square plus 2 J 3 square and so on, is equal to 1.

Now, from here we can easily see that we are adding positive quantity 2 J 1 square, 2 J 2 square, 2 J 3 square are positive quantities which are added to J naught square. And then, we get this as equal to 1, so J naught square is less than or equal to 1, which implies that mod of J naught is less than or equal to 1 and 2 times j n square where n is equal 1, 2, 3 and so on is less than or equal to 1.

So, mod of J n or you can say J n square is less than or equal to half, so mod of J n is less than or equal to 2 to the power minus half r 1 by root 2, this whole square n equal to 1, 2, 3 and so on. So, these are the properties of Bessel functions of order 0 and Bessel functions of order n, where n is equal to 1, 2, 3 and so on.

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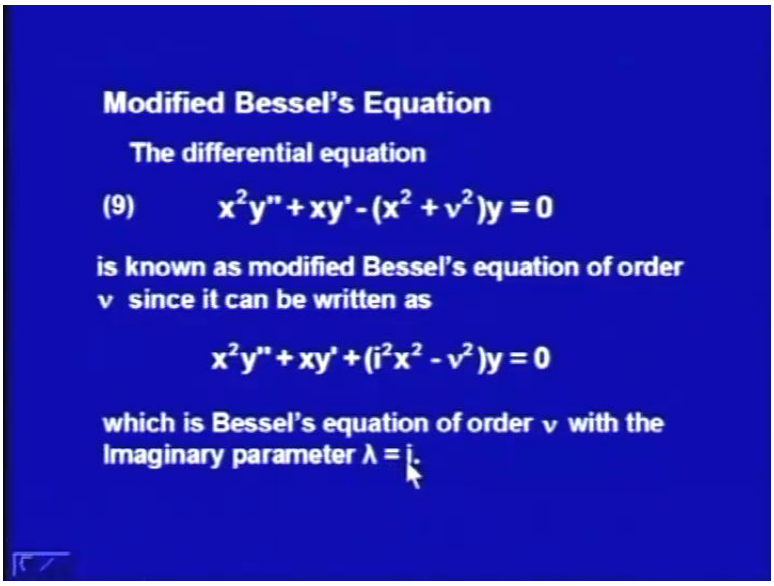

Now, we shall study a modification of Bessel's equation, the differential equation x square y double dash plus x y dash minus x square plus nu square into y equal to 0 is called as the modified Bessel's equation of order nu. Because, we were write it as x square y double dash plus x y dash plus i square into x square minus nu square into y equal to 0, where i is the ((Refer Time: 45:30)) square root minus 1 and so on.

And we know that this is Bessel's equation of order nu with the imaginary parameter lambda equal to, the parameter lambda here is the imaginary lambda equal to i.

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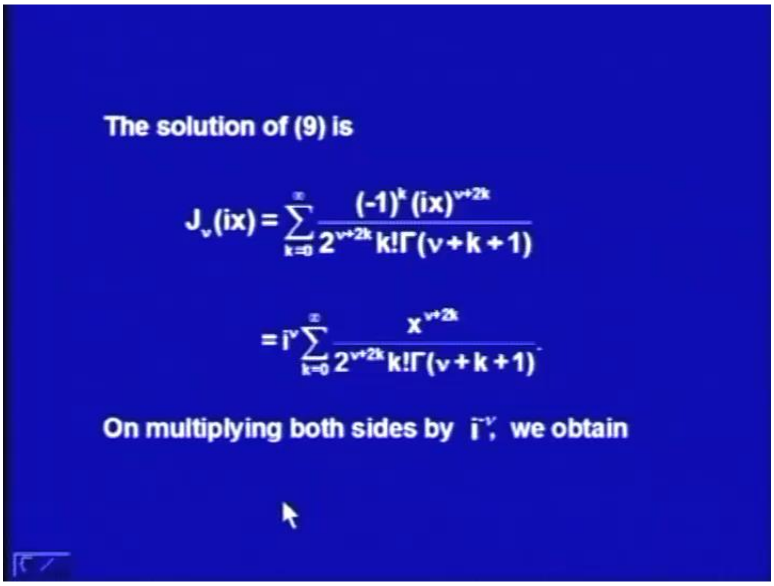

So, the solution of 9 is J nu i x equal to sigma k equal to 0 to infinity minus 1 to the power k i x to the power nu plus 2 k over 2 to the power nu plus 2 k into k factorial gamma nu plus k plus 1; which below also write as i to the power nu into sigma k equal to 0 to infinity x to the power nu plus 2 k over 2 to the power nu plus 2 k into k factorial gamma nu plus k plus 1.

Because, i to the power 2 k is i square raise to the power k, which is minus 1 to the power k and minus 1 to the power k here also we have, so minus 1 to the power 2 k will give us plus 1. Now, let us multiply both sides of this equation by i to the power minus nu.

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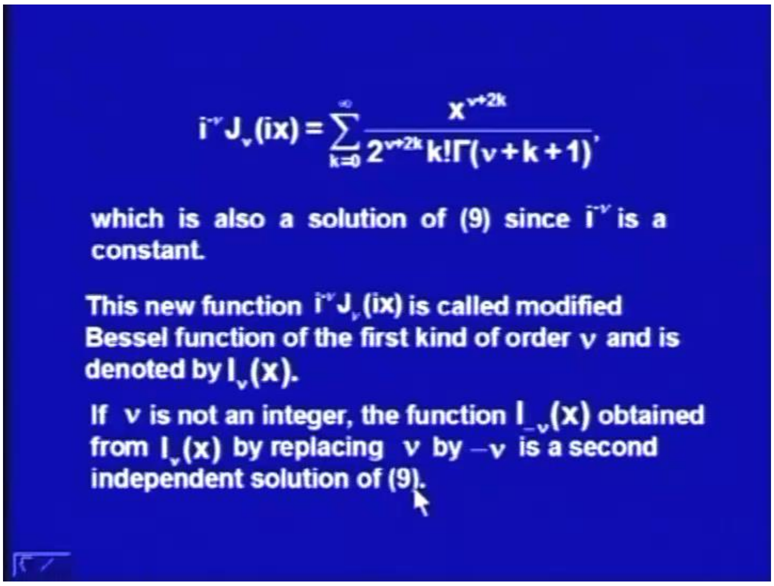

Then, we will obtain i to the power minus nu into J nu i x equal to sigma k equal to 0 to infinity x to the power nu plus 2 k over 2 to the power nu plus 2 k into k factorial gamma nu plus k plus 1, which is also a solution of 9. Since i to the power minus nu is a constant, then i to the power minus nu into J nu i x is called modified Bessel function of first kind of order nu and we denoted by i nu x. Now, if nu is not an integer, the function i minus nu x, which is obtained from i nu x by replacing nu by minus nu is then a second independent solution of the equation 9.

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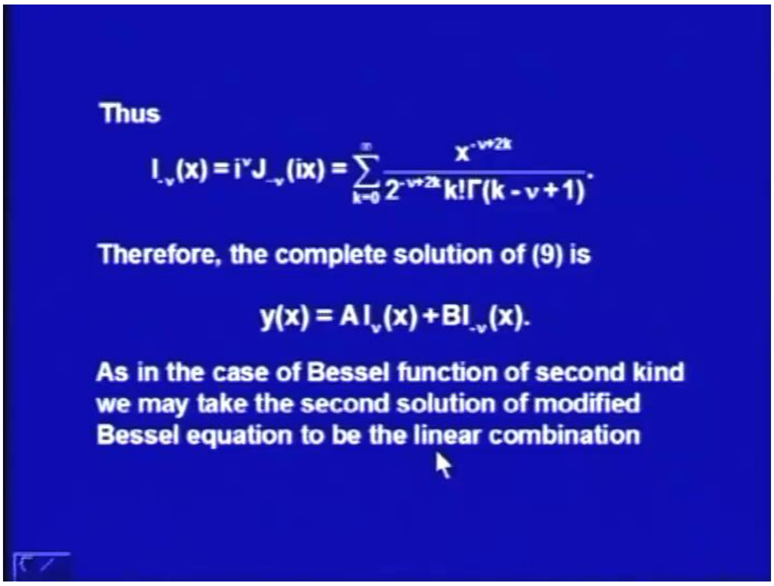

And thus i minus nu x is equal to i to the power nu into j minus nu i x, which is equal to sigma k equal to 0 to infinity x to the power minus nu plus 2 k over 2 to the power minus nu plus 2 k into k factorial gamma k minus nu plus 1. And thus the complete solution of equation 9 we may write as y x equal to a times i nu x plus b times i minus nu x.

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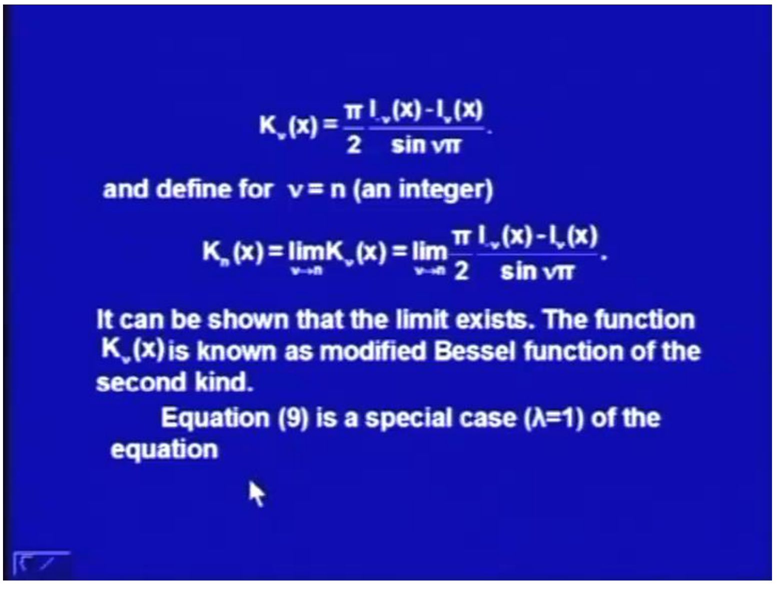

Now, as in the case of Bessel function of second kind, we may take the second solution of modified Bessel equation to be the linear combination of i nu x and i minus nu x when nu is not an integer, that is we can take K nu x equal to pi by 2 into i minus nu x minus i nu x over sin nu pi. And for nu equal to n an integer we define K n x equal to limit of K nu x as nu goes to n equal to limit nu goes to n pi by 2 into i minus nu x minus i nu x over sin nu pi.

Now, we can show that this limit exists, so when nu is not only integer K nu x being the linear combination of i nu x and i minus nu x is also a solution of the equation 9. And when nu is the integer, K n x being the limit of K nu x and since the limit exists is also a solution of the equation 9. And therefore, we call the function K nu x as the modified Bessel function of the second kind, now equation 9 is a special case of the following equation.

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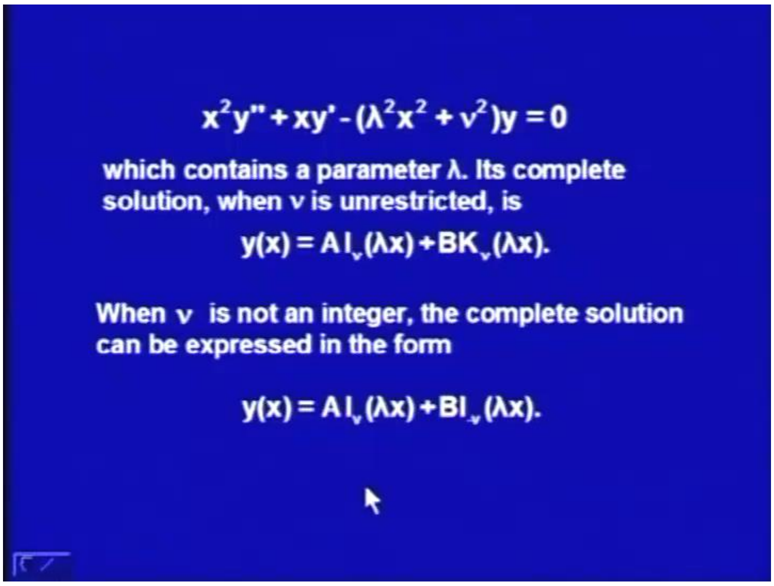

If you put lambda equal to 1 in this equation, we get x square y double dash plus x y dash minus lambda square x square plus nu square y equal to 0 reduces to x square y double dash plus x y dash minus x square plus nu square into y equal to 0, which contains a parameter lambda. It is complete solution is, therefore y x equal to A times I nu lambda x plus B times K nu lambda x, where nu can take integer or non integral value, so there is no restriction on nu.

Now, when nu is not an integer the complete solution may also be expressed as y x equal to A times I nu lambda x plus B times I minus nu lambda x, because we know that I minus nu lambda x is independent of I nu lambda x. So, modified Bessel's equation is the particular case of this equation of lambda equal to 1, whose solution are given by for all values of nu by y x equal to a times I nu lambda x plus B times K nu lambda x. And when nu is not an integer y x is equal to a times I nu lambda x plus B times I minus nu lambda x.

In our next lecture we shall discuss the Laplace transformation, Laplace transformation is very useful in engineering mathematics, the linear differential equation with constant coefficients are solved by using Laplace transformation method. When we solve the linear differential equation with constant coefficients by finding the complementary function and particular integral, then the constants that occur there, their values are obtained after putting the conditions given.

While in the case Laplace transformation it takes in to account the initial conditions while taking the Laplace transformation itself. So, we do not have to find the values of the arbitrary constants after we have found the general solution as the case while solving the linear differential equation with constant coefficient. So, it is very useful for solving linear differential equation with constant coefficients, we can also solve simultaneous system of simultaneous linear equations by using Laplace transformation method. And that is used in the study of electric circuits, then dynamical systems and then, many other problems, so we will be discussing Laplace transformation in our next lecture.

Thank you.