Mathematics III
Prof. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 6
Bessel Functions and Their Properties

Dear viewers, in my last lecture we had seen how to find the general solution of a homogeneous linear differential equation of second order with variable coefficients about a regular singular point. An important differential equation of this type is the Bessel's equation. So in my lecture today we shall be discussing the solutions of the Bessel's equation. The solutions of the Bessel's equations are called as Bessel functions, so we shall study Bessel functions and their properties.

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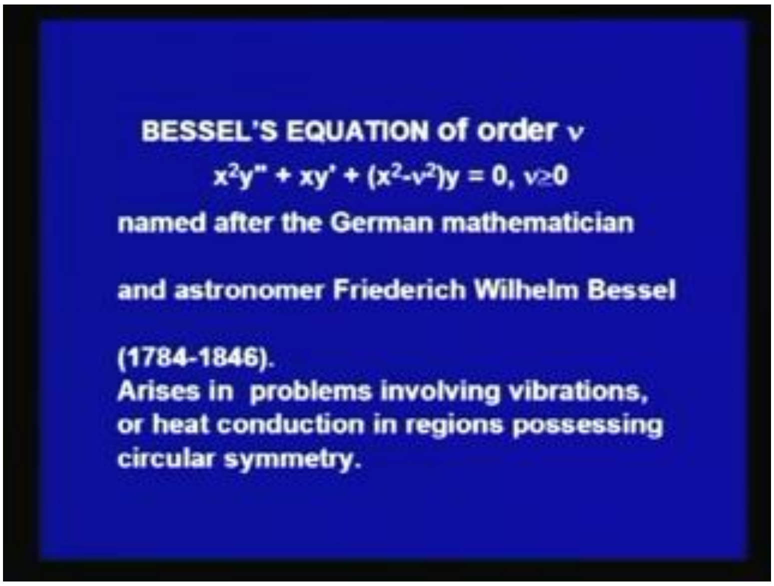

Let us first define the Bessel's equation, a Bessel's equation of order nu is given by x square y double dash plus x y dash plus x square minus nu square into y equal to 0, where nu is greater than or equal to 0. This equation is, so called after the German mathematician and astronomer Friedrich Wilhelm Bessel from 1784 to 1846, now this differential equation arises in problems which involve vibrations are heat conductions in regions that possess circular symmetry.

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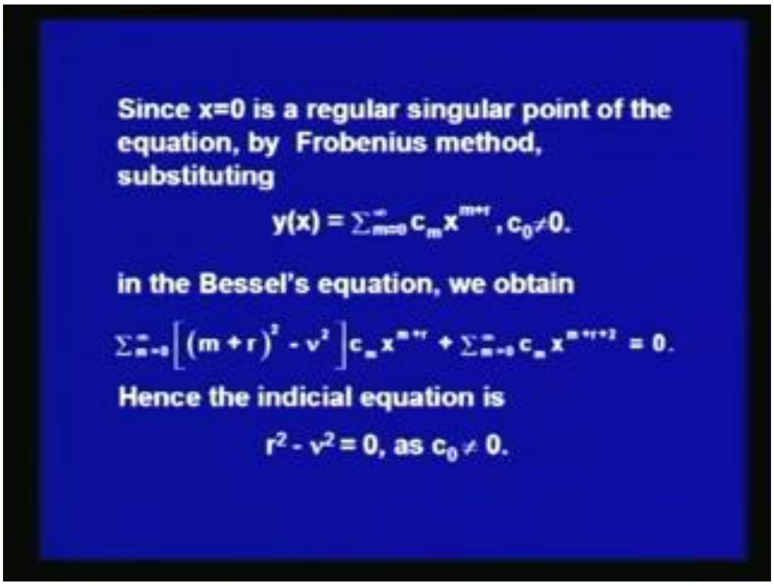

Now, from the differential equation it is clear that x equal to 0 is a regular singular point of the differential equation and, so we can apply Frobenius method to this differential equation. Let us therefore, substitute y equal to sigma m equal to 0 to infinity c m x to the power m plus r, where c naught is not equal to 0 in the Bessel's equation. After simplification we will obtain sigma m equal to 0 to infinity m plus r whole square minus nu square into c m x to the power m plus r plus sigma m equal to 0 to infinity c m x to the power m plus r plus 2 equal to 0.

Now, this equation is an identity therefore, the coefficients of various powers of x can be equated to 0, when we equate it to 0 the coefficient of the least power of x in r that is we equate to 0 the coefficient of x to the power r to 0, we get the indicial equation, the indicial equation is therefore, given by r square minus nu square equal to 0 as c naught is not equal to 0.

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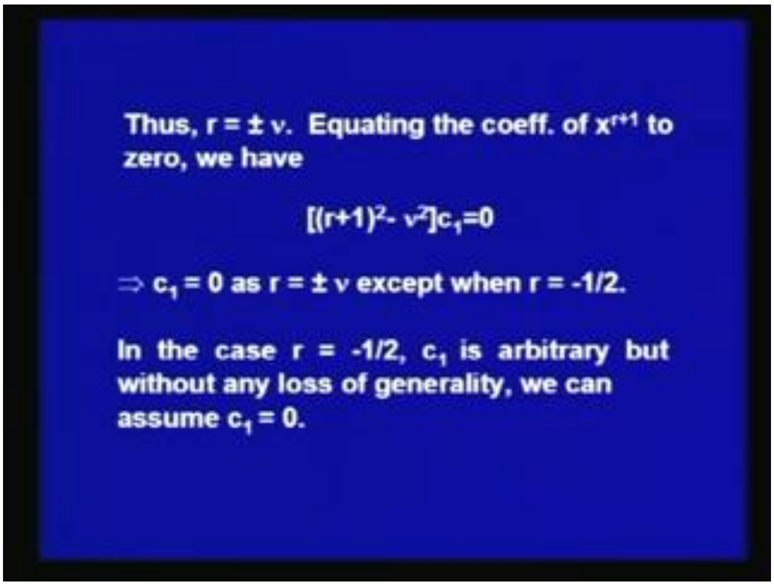

And this gives us the roots of the indicial equation as r equal to plus minus nu, now let us equate the coefficient of next higher power of x that is let us equate the coefficient of x to the power r plus 1 to 0, we shall have r plus 1 whole square minus nu square into c 1 equal to 0. Now, since r is equal to plus minus nu, so r square is equal to nu square and therefore, this equation r plus 1 whole square minus nu square into c 1 equal to 0 implies c 1 equal to 0 in all cases, except one r is equal to minus half.

In the case r is equal to minus half both sides become 0 and therefore, c 1 can take any value, so c 1 is an arbitrary constant. But, then what happens is that when we take c 1 as an arbitrary constant and find the solution corresponding to r equal to minus half, then the terms which correspond to c 1 are observed in the solution, which we find for the other value of r that is r equal to half. And, so no nu part of the solution is obtained corresponding to c 1 being arbitrary and therefore, we shall without any loss of generality we can assume that c 1 is equal to 0 even in this case.

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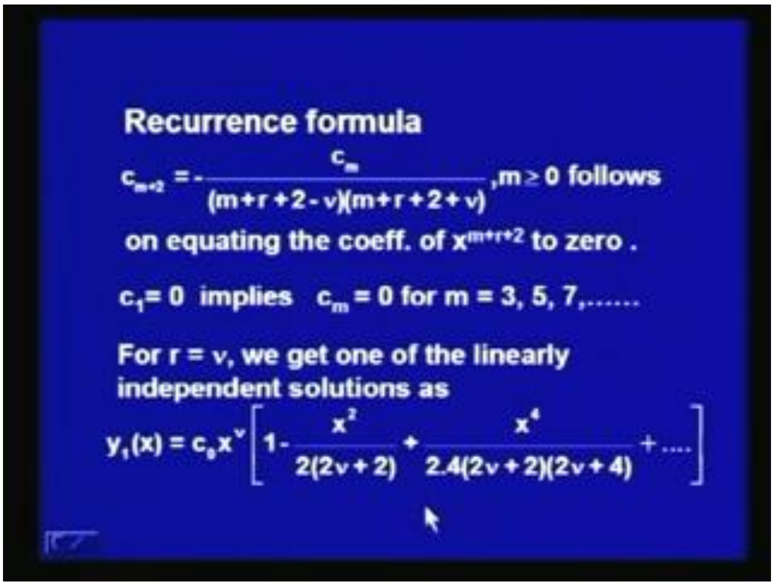

Next, let us equate to 0 the coefficient of x to the power m plus r plus 2 to 0 in the equation, we will have c m plus 2 equal to minus c m over m plus r plus 2 minus nu into m plus r plus 2 plus nu for all m greater than or equal to 0 that is m equal to 0, 1, 2, 3 and so on. Now, c 1 is equal to 0 implies that c m is from this recurrence relation it follows that when c 1 is 0, c 3 is 0 and c 3 is 0 implies c 5 is 0. So, c m c 1 equal to 0 implies that c m is equal to 0 for m equal to 3, 5, 7 and so on.

Thus for r equal to nu we can get one of the linearly independent solutions as y 1 x equal to c naught x to the power nu into 1 minus x square upon 2 into 2 nu plus 2 plus x to the power 4 upon 2 into 4 2 nu plus 2 into 2 nu plus 4 and so on. Where, the values of c 2 and c 4 and so on have been obtained from the recurrence relation c m plus 2 equal to minus c m over m plus r plus 2 minus nu into m plus r plus 2 plus nu.

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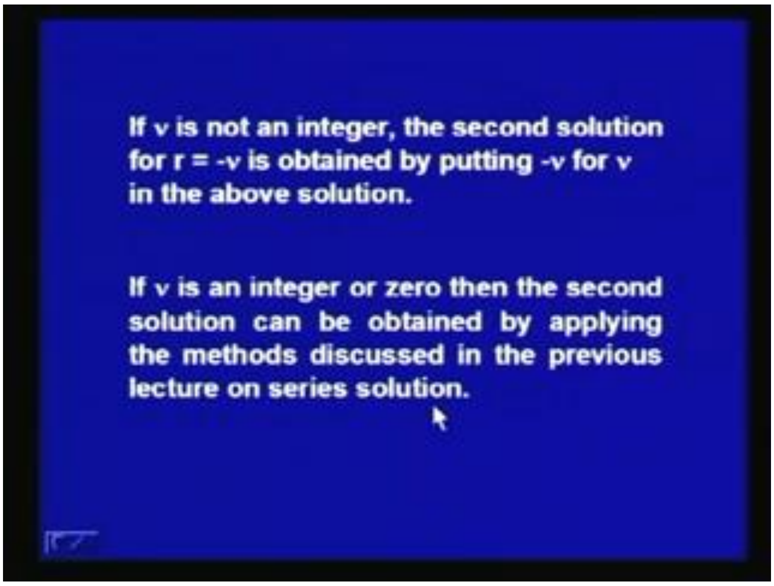

Now, if nu is not an integer the second solution for r equal to minus nu is obtained by putting minus nu for nu in the above solution. However, if nu is an integer or 0 then the second solution will have to be obtained by the methods, which we have discussed in the previous lecture on series solution that is if nu is 0 then the two roots of the indicial equation will be both 0. So, we shall apply the case of the indicial equation having equal roots, if nu is an integer we shall apply the case third, where we had discussed the case of indicial equation having roots which differ by an integer.

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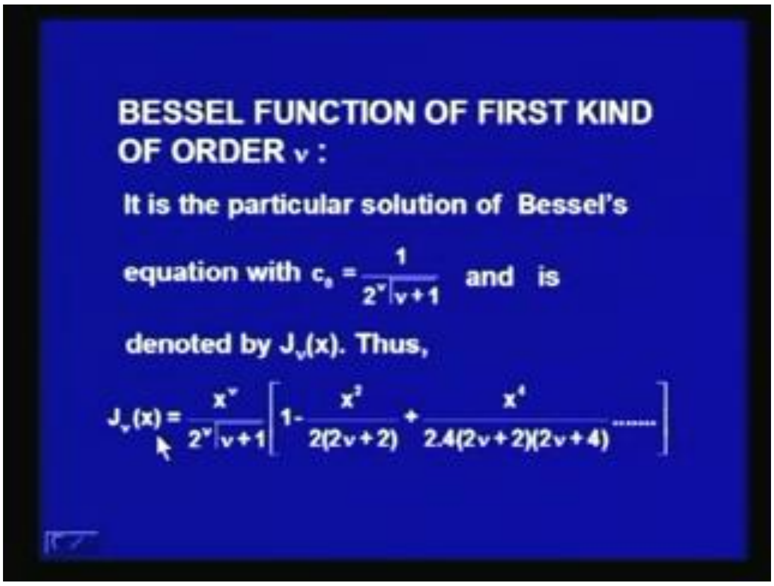

Let us, now discuss Bessel function of first kind of order nu, Bessel function of first kind of order nu is the particular solution of Bessel's equation, where we take c naught equal to 1 over 2 to the power nu into gamma nu plus 1, it is denoted by J nu x. So, thus J nu x becomes x to the power nu over 2 to the power nu into gamma nu plus 1 1 minus x square over 2 into 2 nu plus 2 plus x to the power 4 2 into 4 2 nu plus 2 2 nu plus 4 and so on. This J nu x we have obtained by replacing c naught in y 1 x y 1 over 2 to the power nu gamma nu plus 1. Now, this choice of c naught equal to 1 over 2 to the power nu gamma nu plus 1 has be made in order to get complicated in order to get a simplified expression for J nu x.

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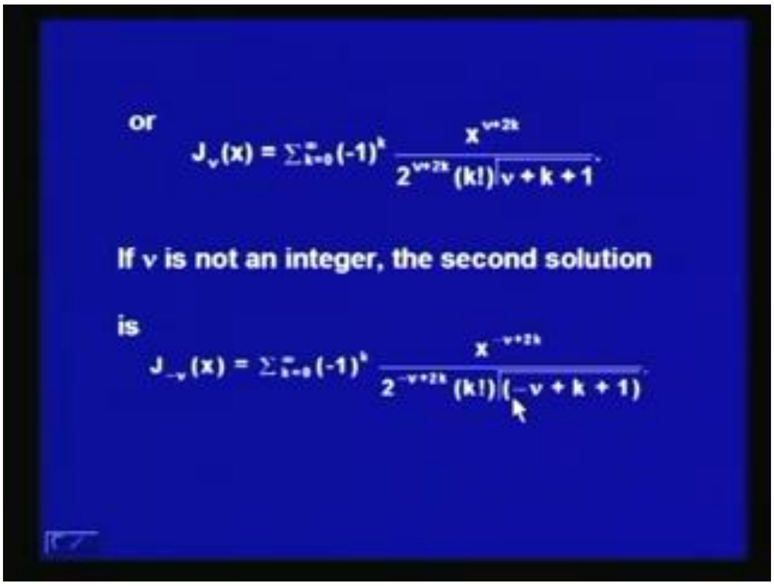

As we see now with this choice of c naught x, we can write J nu x in the form of summation as sigma k equal to 0 to infinity minus 1 to the power k x to the power nu plus 2 k over 2 to the power nu plus 2 k. Now, this can be combined and we may write as x over 2 raised to the power nu plus 2 k into 1 over then 1 over k factorial and 1 over gamma nu plus k plus 1. If nu is not an integer, then the second solution can be obtained by replacing nu by minus nu here. So, say J minus nu x will be equal to sigma k equal to 0 to infinity minus 1 to the power k x to the power minus nu plus 2 k over 2 to the power nu plus 2 k into k factorial gamma minus nu plus k plus 1.

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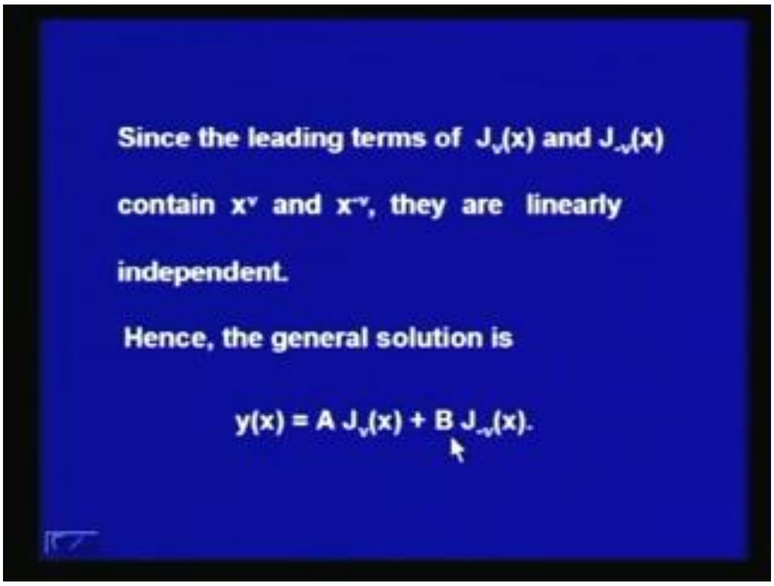

Now, in the expressions for J nu x and J minus nu x, we can see that the leading terms are containing x to the power nu and x to the power minus nu. Therefore, the two solutions J nu x and J minus nu x of the Bessel's equation are linearly independent, that is one is not a scalar multiple of the other. And hence, we can write the general solution of the Bessel's equation for the case where nu is not an integer as y x equal to A times J nu x plus B times J minus nu x.

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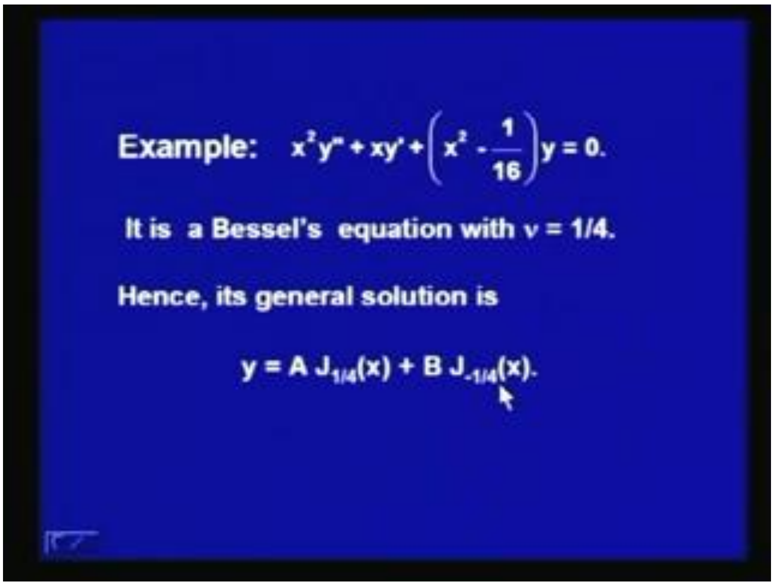

Let us study an example, x square y double dash plus x y dash plus x square minus 1 over 16 y equal to 0. It is a Bessel's equation, where we have taken nu equal to 1 by 4 if you compare this equation with the standard form of the Bessel's equation, you can see that nu is equal to 1 by 4 here. So, nu is not an integer and therefore, the two solutions J nu x and J minus nu x of this Bessel's equation will be linearly independent of each other and therefore, hence we may write its general solution as y equal to A times J 1 by 4 x plus B J minus 1 by 4 x.

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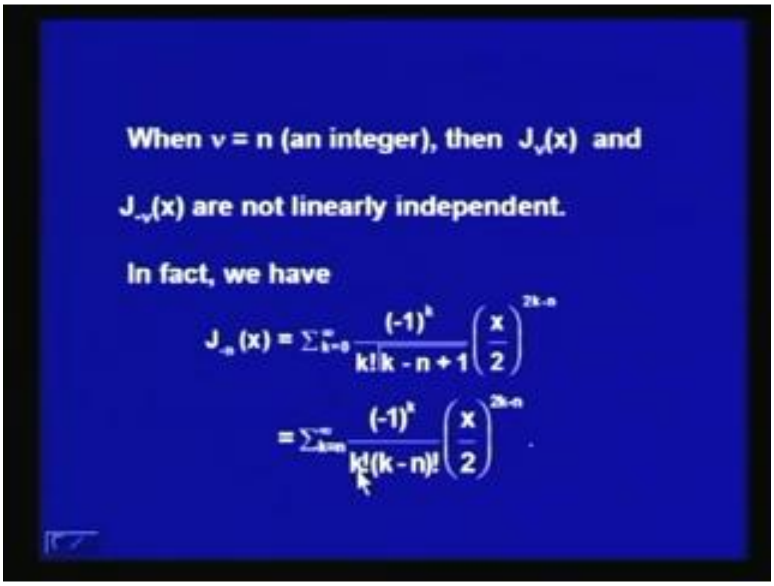

When nu is equal to n let us say an integer then J nu x and J minus nu x are not linearly independent, we are going to show that, in fact J minus n x is equal to minus 1 to the power n J n x. Let us find the value of J minus n x, J minus n x can be written as sigma k equal to 0 to infinity minus 1 to the power k over k factorial gamma k minus n plus 1 x by 2 raised to the 2 k minus n, replacing nu y minus n in the expression for J nu x.

Now, here gamma k minus n plus 1 can be replaced by k minus n factorial because, k is taking values 0, 1, 2, 3 and so on it is second integral values n also v as is an integer. So, k minus n plus 1 is an integer and, so gamma k minus n plus 1 will be equal to k minus n factorial, but then k will start with n onwards k will go from n to infinity.

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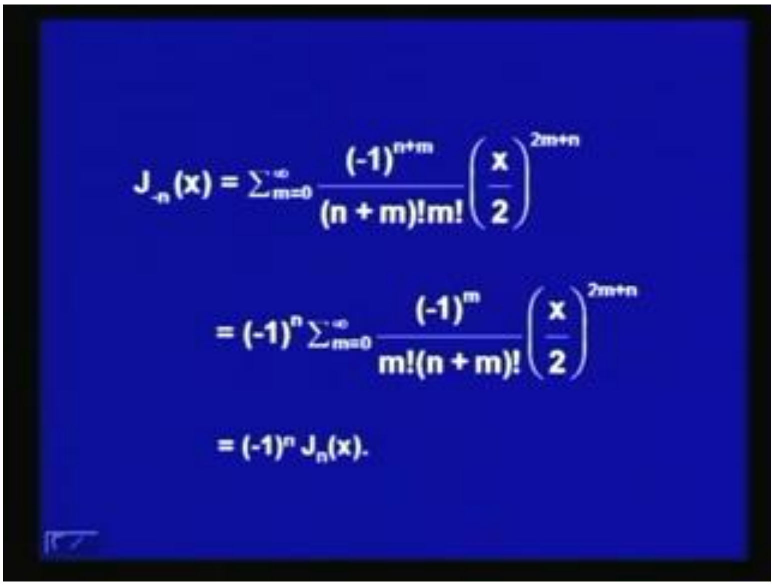

So, replacing k by k minus n by m we will have the value of J minus n x as sigma m equal to 0 to infinity minus 1 to the power n plus m over n plus m factorial into m factorial x by 2 rest to the power 2 m plus n, which is equal to minus 1 to the power n into sigma m equal to 0 to infinity minus 1 to the power m over m factorial n plus m factorial x by 2 raised to the 2 m plus n.

And which as we know it is minus 1 to the power n into J n x this is the expression for j n x. So, when nu is an integer equal to m, then J minus n x and J n x are not linearly independent one is a scalar multiple of the other it is J minus n x is equal to minus 1 to the power n into J n x. So, when nu is an integer the second solution has to be obtained by the cases that we have discussed in the last lecture.

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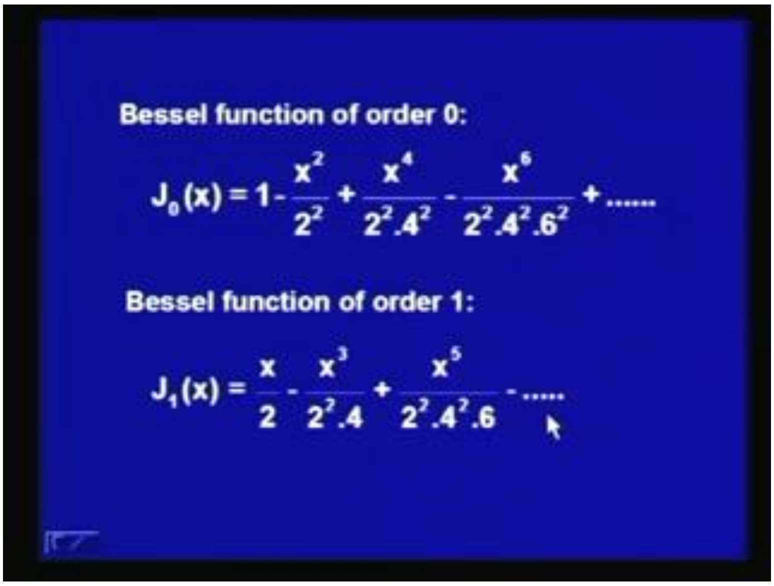

Let us, take an nu equal to 0 we will have the Bessel function of order 0, Bessel function of order 0 is given by 1 minus x square by 2 square plus x to the power 4 by 2 square into 4 square minus x to the power 6 over 2 square 4 square 6 square and so on. This infinite series expression for a J naught x we have obtained from the corresponding expression for J nu x by replacing nu y 0. Next let us take nu equal to 1 we have the expression for J 1 x as x by 2 minus x cube over 2 square into 4, then x to the power 5 over 2 square 4 square into 6 and so on.

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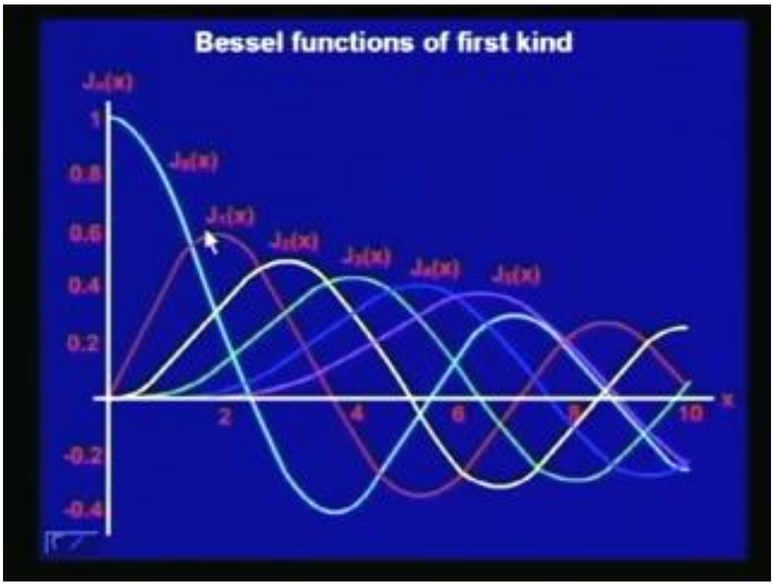

In this picture, we see the graphs of Bessel functions of first kind, this the graph of J naught x, this is the graph of J 1 x, then this the graph of J 2 x, J 3 x, J 4 x, J 5 x and so on. We can see the close resemblance of the graphs of J naught x and J 1 x with the functions cos x and sin x and that is quite interesting, they closely resemble the graphs of cosine x and sin x functions.

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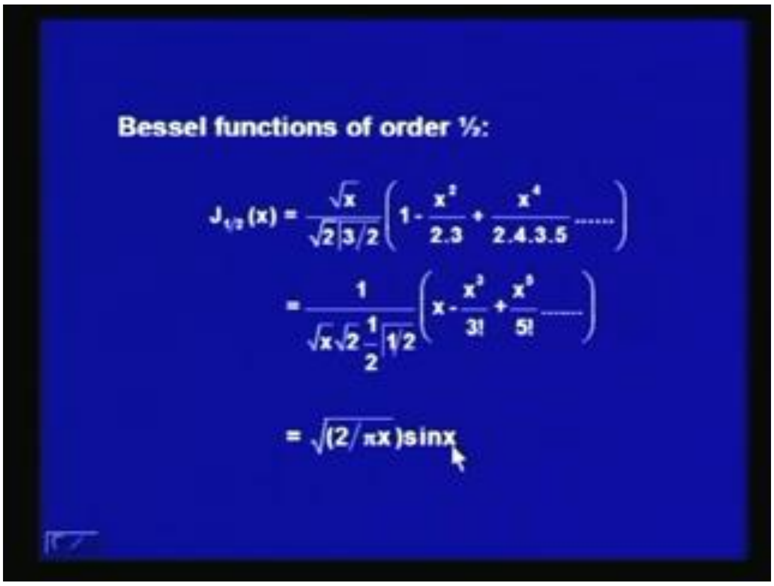

Now, let us study the Bessel functions of order half, we can see that the infinite series expansion of J half x can be put in the closed form that is in terms of the sin x function J half x is equal to root x over root 2 into gamma 3 by 2 1 minus x square over 2 into 3 plus x to the power 4 in over 2 into 4 into 3 into 5 and so on which is equal to 1 over root x into root 2 half gamma half, gamma 3 by 2 is half gamma half gamma half we know is root pi and this becomes after multiplying by root x in the numerator and denominator, this becomes x minus x cube by 3 factorial plus x to the power 5 by 5 factorial and so on. And x minus x cube by 3 factorial plus x to the power 5 by 5 factorial and so on is the ((Refer Time: 14:26)) f sin x function. So, we can write J half x as equal to square root 2 over pi x into sin x.

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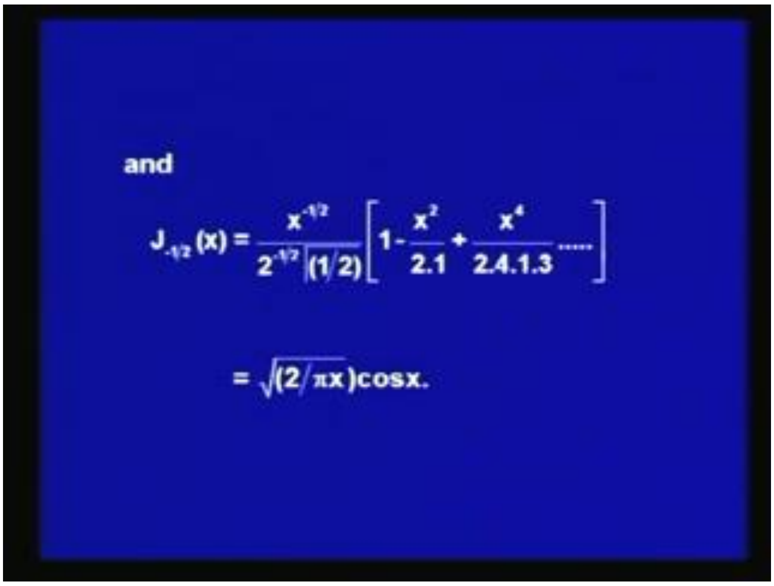

And J minus half x is equal to x to the power minus half over 2 to the power minus half gamma half into 1 minus x square over 2 into 1 plus x to the power 4 over 2 into 4 into 1 into 3, which is equal to square root 2 over pi x into cos x the expression inside the bracket as you can see is the expression of cos x it is 1 minus x square over 2 factorial plus x to the power 4 over 4 factorial and so on. So, J minus half x can be expressed in the closed form as the square root 2 over pi x into cosine x.

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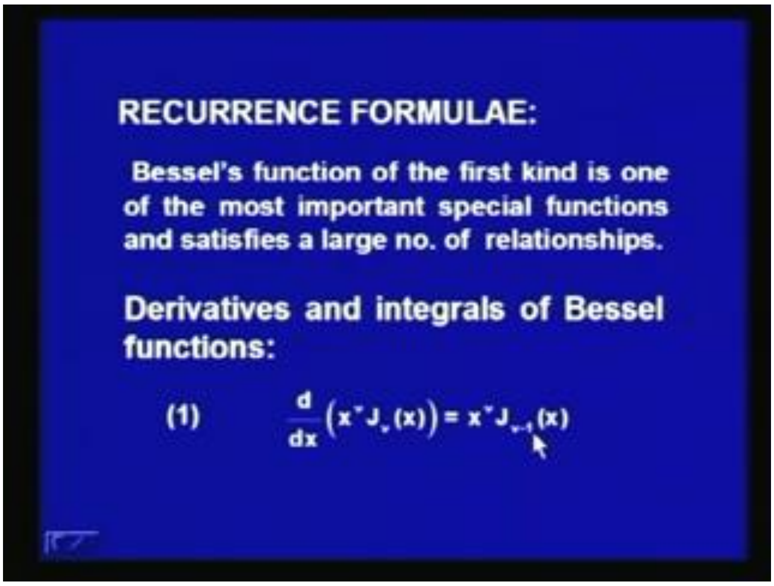

The Bessel functions of first kind possess some very important relationships, we are going to study the relationships, which the Bessel's functions of first kind satisfied. So, let us first see that d over d x of x to the power nu into J nu x is equal to x to the power nu into J nu minus 1 x.

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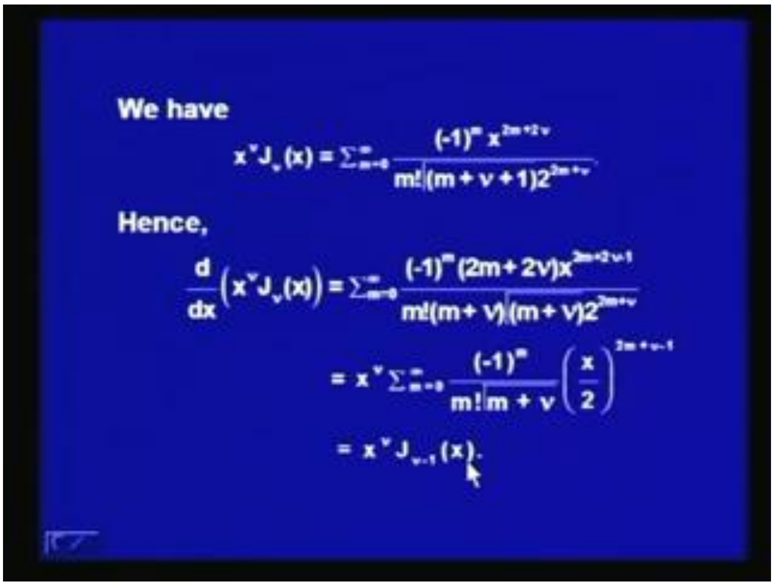

Let us multiply the expression for J nu x by x to the power nu, we shall have x to the power nu into J nu x is equal to sigma m equal to 0 to infinity minus 1 to the power m x to the power 2 m plus 2 nu over m factorial gamma m plus nu plus 1 into 2 to the power 2 m plus nu. Now, let us differentiate this with respect to x, so d over d x of x to the power nu into J nu x will give us sigma m equal to 0 to infinity minus 1 to the power m 2 m plus 2 nu x to the power 2 m plus 2 nu minus 1 over m factorial into m plus nu gamma m plus nu.

Because of the property of gamma function, gamma m plus nu plus 1 can be writtten as m plus nu into gamma m plus nu into 2 to the power 2 m plus nu. Now, we may cancel m plus nu in the numerator and denominator and then we shall have the right hand side equal to x to the power nu into sigma m equal to 0 to infinity minus 1 to the power m over m factorial gamma m plus nu into x by 2 raised to the power 2 m plus nu minus 1, which is equal to x to the power nu into J nu minus 1 x.

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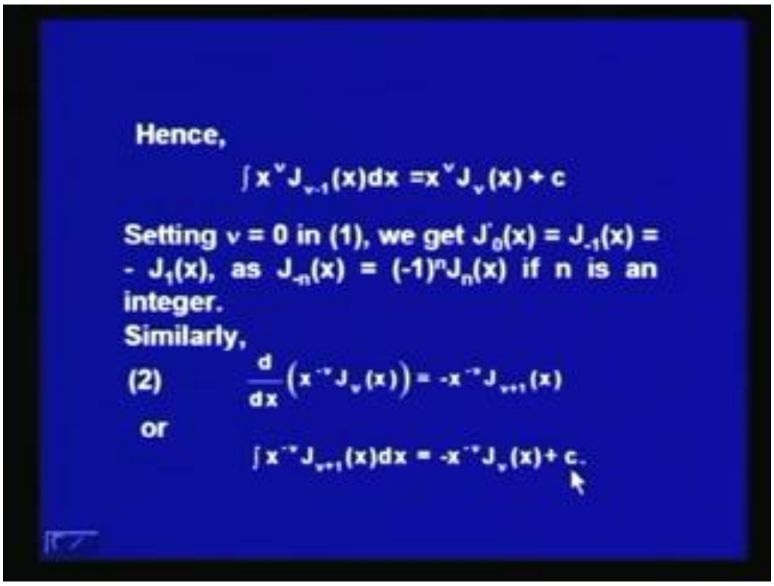

Now, integrating both sides of this relationship we obtain integral of x to the power nu into J nu minus 1 x d x equal to x to the power nu into J nu x plus c. If you set nu equal to 0 in that relationship d over d x of x to the power nu into J nu x equal to x to the power nu into J nu minus 1 x, we obtain that the derivative of J naught x that is J naught dash x is equal to J minus 1 x. And we know that J minus n x is equal to minus 1 to the power n into J n x where n is an integer.

So, J minus 1 x will be equal to minus J 1 x and therefore, J naught dash x is equal to minus J 1 x. Now, the next result that we have is d over d x of x to the power nu minus nu J nu x is equal to minus x to the power minus nu J nu plus 1 x, so here the order of the Bessel function increases by 1 J nu x becomes J nu plus 1 x, in the previous relationship J nu x becomes J nu minus 1 x. So, here when we will integrate we will get integral of x to the power minus nu J nu plus 1 x d x equal to minus x to the power minus nu J nu x plus c.

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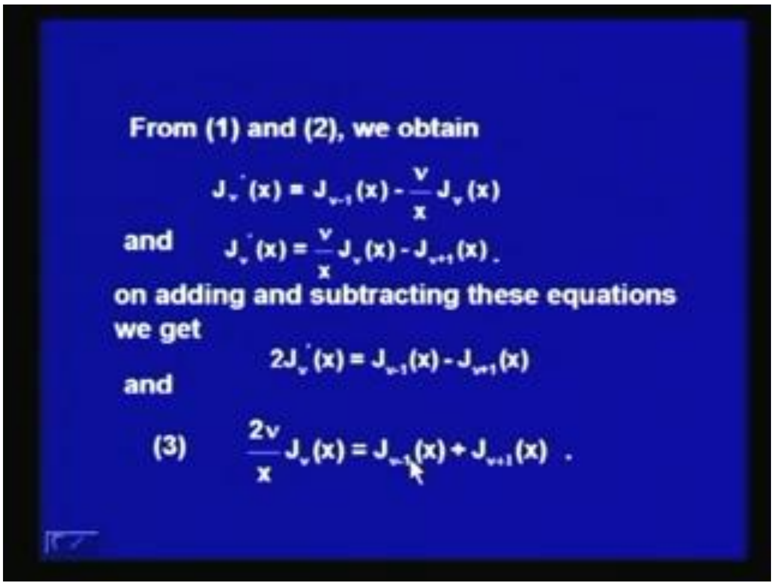

From these two relationships we will obtain J nu dash x the value of J nu dash as x is equal to J nu minus 1 x minus nu by x J nu x and J nu dash x equal to nu by x J nu x minus J nu plus 1 x. Now, adding these two equations and subtracting we shall have the following, if you add these two equations you get 2 J nu dash x equal to J nu minus 1 x minus J nu plus 1 x and if you subtract this equation, this equation from this one you get the left hand side as 0.

And therefore, we will be getting 2 nu y x J nu x minus J nu plus 1 x minus J nu minus 1 x equal to 0, which would give us 2 nu by x into J nu x equal to J nu minus 1 x plus J nu plus 1 x. In three expresses the Bessel function of order nu plus 1 in terms of the Bessel's functions of order nu and nu minus 1 and, so if we know the Bessel functions of lower orders that is J nu minus 1 x and J nu x, then we can get the Bessel function of J nu plus 1 x from this equation and therefore, we call this formula as the recurrence formula.

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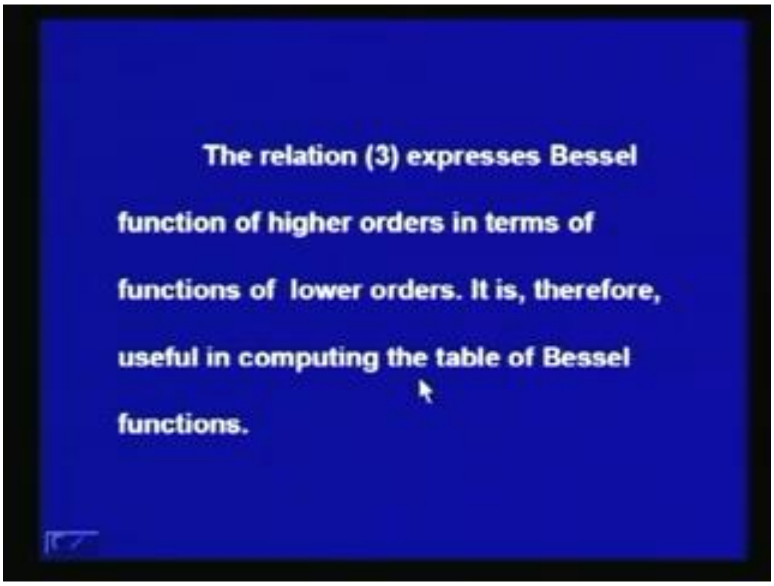

This formula is therefore, useful in computing the table of Bessel functions.

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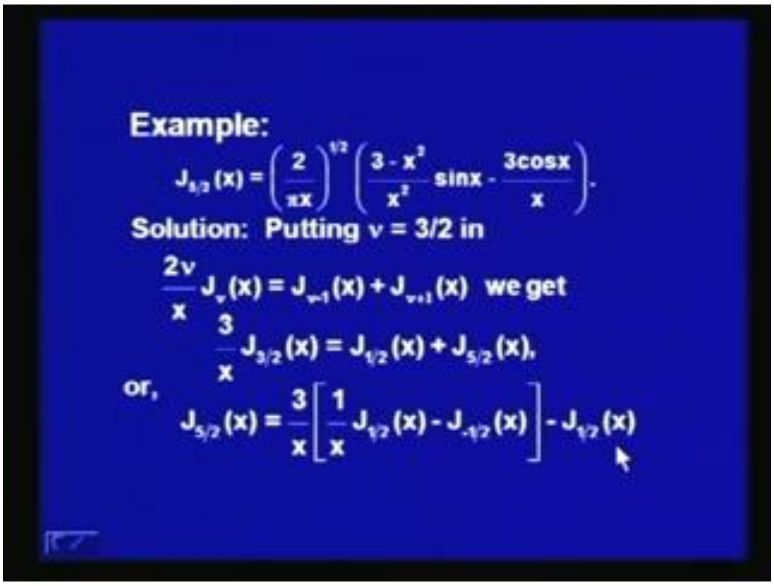

Let us take an example based on this recurrence relation, let us show that J 5 by 2 x equal to 2 over pi x raised to the power half into 3 minus x square over x square into sin x minus 3 cos x over x. So, let us put nu equal to 3 by 2 in the recurrence relation, we will have the recurrence relation as we know is 2 nu by x J nu x equal to J nu minus 1 x plus J nu plus 1 x we have chosen nu equal to 3 by 2.

Because, the J nu plus 1 x will then become J 5 by 2 x and it will be expressible in terms of the Bessel functions of lower order, that is 3 Bessel function of order 3 by 2 and Bessel function of order half. So, making use of this recurrence relation and taking nu equal to 3 by 2, we can see that we get 3 by x J 3 by 2 x equal to J half x plus J 5 by 2 x or we may write J 5 by 2 x equal to 3 over 2 J 3 by 2 minus J half x, the value of J 3 by 2 x will turn out to be from this recurrence relation taking nu equal to half as 1 by x J half x minus J minus half x. So, the value of J 5 by 2 x becomes 3 by x into 1 by x J half x minus j minus half x minus J half x.

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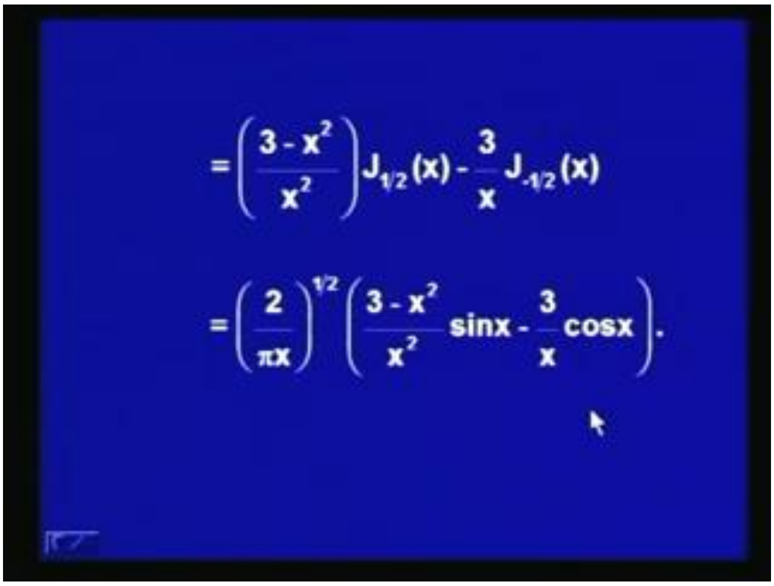

And when we simplify this we get the right hand side as 3 minus x square over x square into J half x minus 3 over x J minus half x. Now, we know the closed forms of J half x and J minus half x, so let us make use of them we will get the right hand side as 2 over pi x raised to the power half multiplied by 3 minus x square over x square into sin x minus 3 over x cos x.

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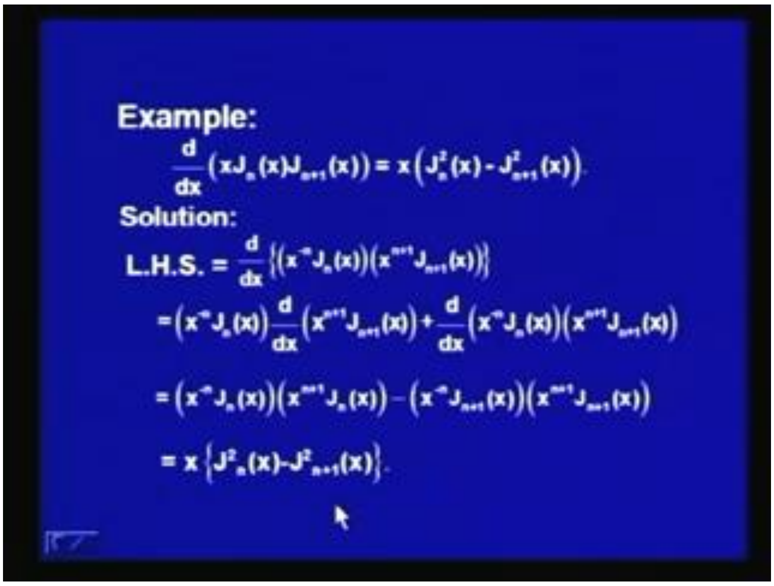

Let us take an example based on the other relationships, which involve the derivatives of Bessel functions. Let us take the example of d over d x of x J n x into J n 1 plus x, we are going to show that it is equal to x into J n square x minus J n plus 1 square x. The left hand side of this example can is expressible as d over d x of x to the power minus n J n x into x to the power n plus 1 J n plus 1 x. This we have done in order to be able to find the derivatives that will arise when we differentiate the product of these two functions of x.

So, x to the power minus n into J n x that is the first function of x multiplied by the derivative of the second function d over d x x to the power n plus 1 J n plus 1 x and then plus derivative of the first function x to the power minus n into J n x into the second function x to the power n plus 1 J n plus 1 x. Now, let us make use of the properties for derivatives of Bessel functions x to the power minus n J n x is same and then the derivative of x to the power n plus 1 J n plus 1 x, this becomes x to the power n plus 1 J n x.

Because, we had the one result as d over d x of x to the power nu into J nu plus 1 x equal to x to the power nu J nu minus 1 x j x to the power nu plus 1 J nu x the derivative of that we had seen was x to the power nu into J nu minus 1 x. So, making use of that the derivative of x to the power n plus 1 J n plus 1 x will be x to the power n plus 1 J n x. And then x to the derivative of x to the power minus n J n x will be minus x to the power minus n J n plus 1 x using the other property of the Bessel function, which involve the derivatives and multiplied by x to the power n plus 1 into J n plus 1 x. Now, let us simplify this it will give us x into J n square x minus J n plus 1 square x that is the right hand side of the given problem.

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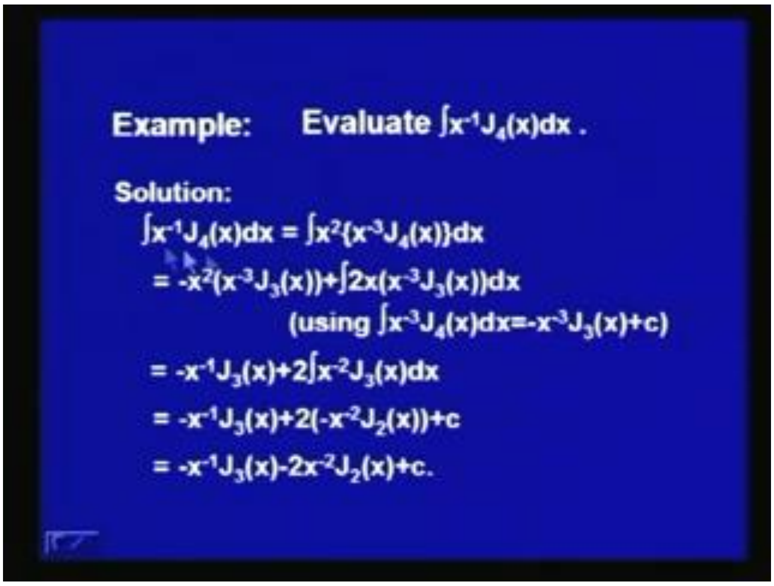

Now, let us evaluate the integral of x to the power minus 1 J 4 x d x, so integral of x to the power minus 1 J 4 x can be then written as integral of x square into x to the power minus 3 J 4 x d x this arrangement also we have done in order to be able to find the integral of x square into x to the power minus 3 J 4 x d x. So, we have first function that is into integral of x to the power minus 3 J 4 x we know, we have seen earlier it is equal to minus x to the power minus 3 J 3 x and then we have plus integral derivative of x square is 2 x then x to the power minus 3 J 3 x d x.

So, we are doing finding the integral y integration by parts and this becomes minus x to the power minus 1 J 3 x plus 2 times integral of x to the power minus 2 J 3 x d x integral of x to the power minus 2 J 3 x we know is equal to minus x to the power minus 2 J 2 x. So, we have the right hand side as minus x to the power minus 1 J 3 x plus 2 times minus x to the power minus 2 J 2 x plus c or we may write it as minus x to the power minus 1 J 3 x minus 2 x to the power minus 2 J 2 x plus c, which is the integral of x to the power minus 1 J 4 x d x.

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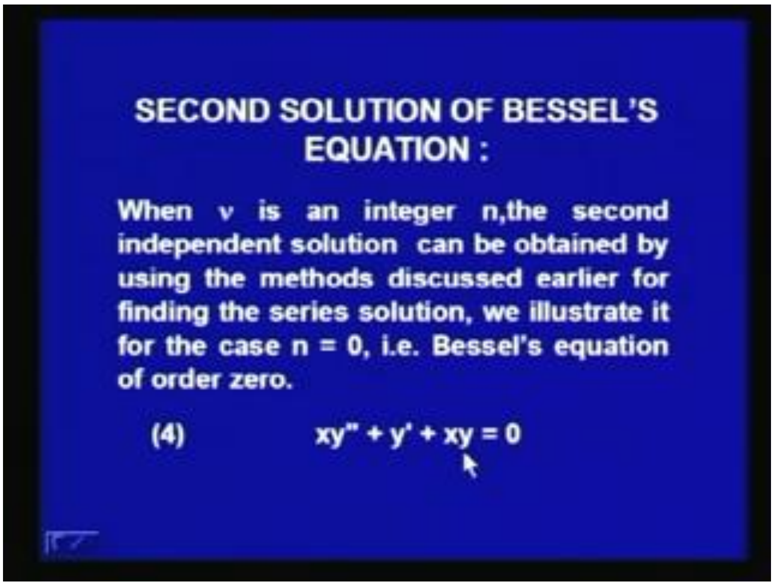

Now, let us study the second solution of Bessel's equation, we had seen that when nu is an integer, the second solution of the Bessel's equation has to be obtained by the methods, which we had discussed in the previous lecture. So, we will be illustrating how to find the second solution of the Bessel's equation for the case nu equal to 0 for the other integral values of nu the second solution of the Bessel's equation can be obtained similarly. Now, when nu is an integer equal to 0 the Bessel's equation of order 0 will be x y double dash plus y dash plus x into y equal to 0, you can see from the Bessel's equation, if you put nu equal to 0 there, then it reduces to x y double dash plus y dash plus x y equal to 0.

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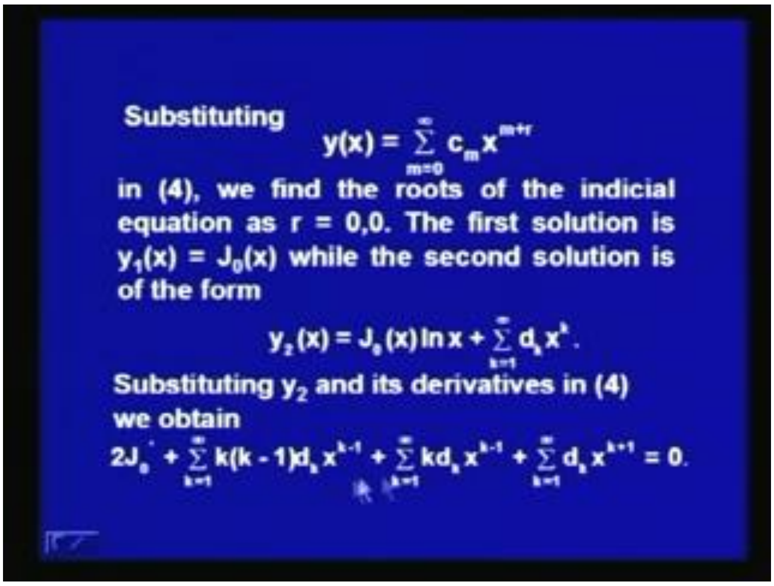

So, let us put again x equal to 0 is a regular singular point here, so let us put y x equal to sigma m equal to 0 to infinity c m x to the power m plus r where c naught is not equal to 0 in this equation, we find that the roots of the indicial equation are 0 comma 0 that is it is the case of indicial equation having equal roots. Now, one solution of the indicial equation, one solution of the Bessel equation can be then obtain it will be y 1 x equal to J naught x. We had seen earlier that corresponding to the indicial equation having roots nu and minus nu one linearly independent solution was of the Bessel's equation were J nu x.

So, in that you put nu equal to 0 you get one solution of the Bessel's equation of order 0 here, as y 1 x equal to J naught x, while the second solution as we know from the previous lecture on series solution can be expressed as y 2 x equal to J naught x into ln x plus sigma k equal to 1 to infinity d k x to the power k. So, substituting this second solution into the and it is derivatives into the Bessel's equation of order 0, we obtain 2 J naught dash plus sigma k equal to 1 to infinity k into k minus 1 d k x to the power k minus 1 plus sigma k equal to 1 to infinity k d k x to the power k minus 1 plus k equal to 1 to infinity d k x to the power k plus 1 equal to 0.

Now, you can see in this equation the terms containing logarithm do not occur, this is because of the fact that J naught x is a solution of the Bessel's equations. So, all the terms which involve logarithm ln x function they banish and therefore, when you substitute y 2 x and it is derivatives in the equation 4 it reduces to this form.

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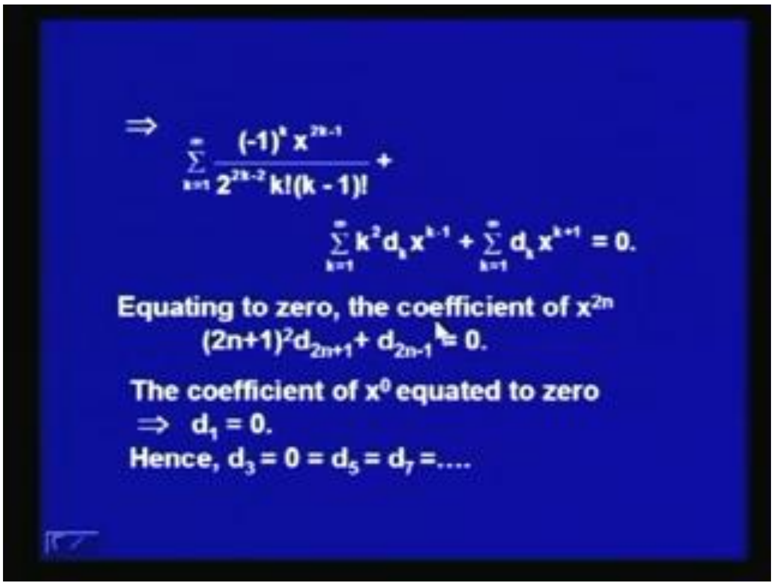

Now, let us find the derivative of J naught x we know the infinite series expensing of J naught x. So, let us differentiate that infinite series with respect to x and hence find J naught dash x and put that is value here, we get the previous equation like this. So, it is an identity and therefore, the coefficients of various powers of x can be equated to 0 in order to find the values of the unknown coefficients d k x.

Now, you can see that in this first term only odd powers of x are occurring, the power of x is 2 k minus 1 k is running from 1 to infinity. So, only odd powers of x are occurring here while here power of x begins with 0 k equal to 1, so power of x begins with 0 and here the powers of x begin with 2. So, when we equate to 0, so what we will do, we will first equate to 0 the coefficients of even powers of x.

Now, when you put the coefficient of x to the power n to 0, the coefficient of x to the power n will not be found here because, it involves only odd powers of x terms. So, coefficient of x to the power n if we want to equate to 0, we will have to find that from this second and third terms. So, coefficient of x to the power 2 n here will be 2 n plus 1 square into d 2 n plus 1, where the coefficient of x to the power 2 n here will be d 2 n minus 1, so we get the equation as 2 n plus 1 square d 2 n plus 1 plus d 2 n minus 1 equal to 0.

Now, next we equate to 0 the coefficient of x to the power 0 which is available only in this term. So, it gives us 1 square into d 1 equal to 0 or you can say d 1 equal to 0. Now, while d 1 is equal to 0 from this equation, we find that d 3 is 0, d 5 is 0, d 7 is 0 and so on. So, when d 1 is 0, d 3, d 5, d 7 all are 0's it follows from this equation, next we equate to 0 the coefficients of odd powers of x, we have already equated to 0 the coefficients of even powers of x and also the power of x as 0.

So, let us now equate to 0 the coefficients of x to the power 2 n plus 1 to 0, but we noticed that when n will be 0 that is if we 1 to equate to 0 the coefficient of x, then the coefficients of x are available in this term, this term, but it is not here. So, let us equate to 0 the coefficients of x 0 separately and the coefficients of x to the power 2 n plus 1 where n will take values 1, 2, 3 and so on from as those terms will be available in all these three terms.

So, let us first equate to 0 the coefficient of x to 0, the coefficient of x if we want to equate to 0 put k equal to 1 here, we will get minus 1 over 2 to the power 0, then we have 1 factorial then we have 0 factorial. So, we will have minus 1 over 1 that is the coefficient of x here and the coefficient of x here will be k equal to 2 we put k equal to 2. So, we get 4 d 2, so 1 plus 4 d 2 equal to 0 that will be the coefficient of x, when we put equal to 0 and the coefficient of x to the power 2 n plus 1 when we want to find, we will have to put here k equal to n plus 1, so that 2 k minus 1 becomes 2 n plus 2 minus 1 that is 2 n plus 1.

So, when you take k equal to n plus 1, then we will have minus 1 to the power n plus 1 over 2 to the power 2 n plus 2 minus 2. So, 2 to the power 2 n over n plus 1 factorial and then n factorial and then here x to the power 2 n plus 1 the coefficient of that will be you put k equal to 2 n plus 2 factorial 2 n plus 2 whole square into d 2 n plus 2 and here it will be, if you want the coefficient of x to the power 2 n plus 1 you have to take k equal to 2 n. So, we will get the coefficient of x to the 2 n plus 1 as d 2 n, so we will put them equal to 0.

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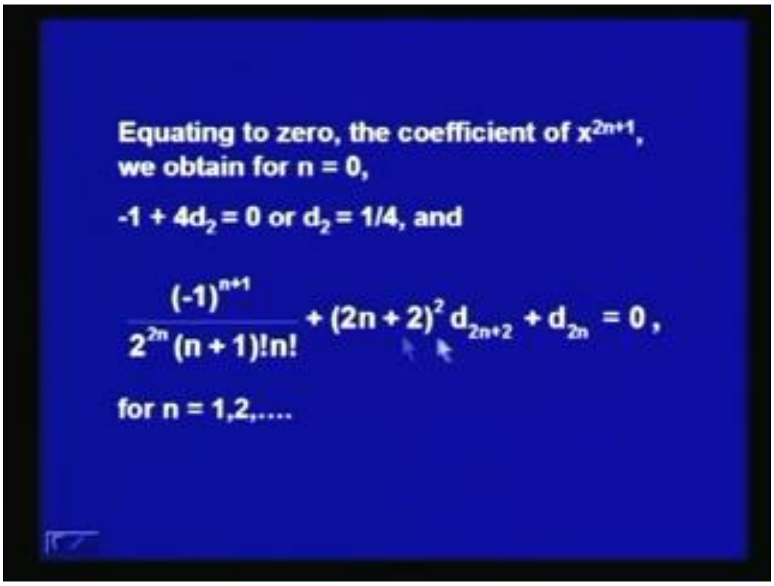

So, when we put the coefficient of x to the power 2 n plus 1 equal to 0 for n equal to 0 as we have seen just now, we will get minus 1 plus 4 d 2 equal to 0, which will give us the value of d 2 as 1 by 4 and the other coefficients of x, that is coefficients of x to the power 2 n plus 1 where n takes values going to on 1, 2, 3 and so on, we have just now seen that this is the equation that we will get.

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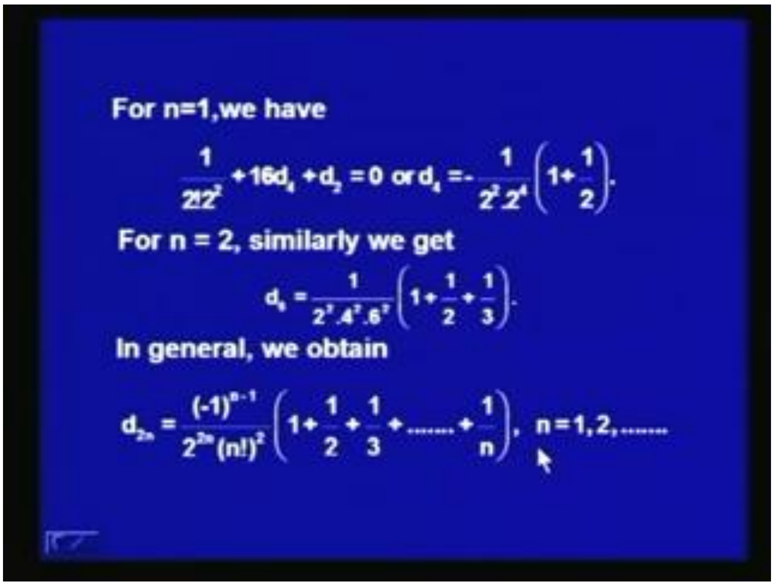

So, from this equation for n equal to 1 we shall have 1 over 2 factorial into 2 square plus 16 d 4 plus d 2 equal to 0 or we will have d 4 equal to minus 1 over 2 square into 2 to the power 4 into 1 plus half making use of the value of d 4 that is 1 by 4. And similarly for n equal to 2 we shall have d 6, d 6 will be equal to 1 over 2 square 4 square 6 square into 1 plus 1 by 2 plus 1 by 3.

In general we will have the value of d 2 n, d 2 n will be equal to minus 1 to the power n minus 1 2 to the power 2 n into n factorial square 1 plus 1 by 2 plus 1 by 3 and so on plus 1 by n, where n takes values 1, 2, 3, and so on. Thus we have obtain the values of all the unknown coefficients degage that occur in our second solution.

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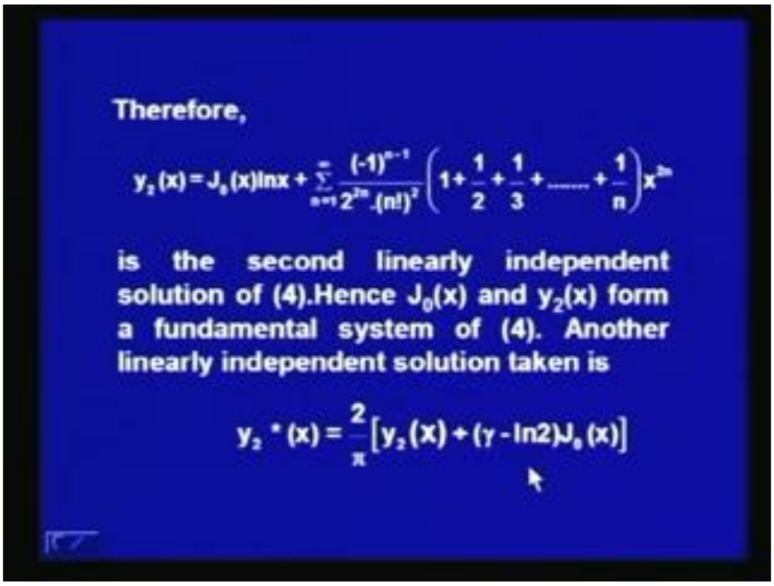

And let us put the values of these unknowns in the second solution we find that y 2 x now becomes J naught x into ln x plus sigma n equal to 1 to infinity minus 1 to the power n minus 1 over 2 to the power 2 n into n factorial square into 1 plus 1 by 2 1 plus 1 by 3 and so on plus 1 by n into x to the power 2 n. Now, these two solutions are linearly independent of each other because, y 2 x involves the logarithmic function. So, they form a fundamental system of the Bessel's equation of order 0.

Now, another linearly independent solution that is usually taken is given by y 2 star x equal to 2 over pi y 2 x plus gamma minus ln 2 into J naught x. Now, you can see that this solution is independent of the first solution that is J naught x, so it could also be taken as the second linearly independent solution of the Bessel's equation. This is a linear combination of the functions J naught x and y 2 x, which we have seen or solutions of the Bessel's equation already.

So, it is also a solution of the Bessel's equation and clearly it is independent of J naught x, so it may also be taken as the second linearly independent solution of Bessel's equation.

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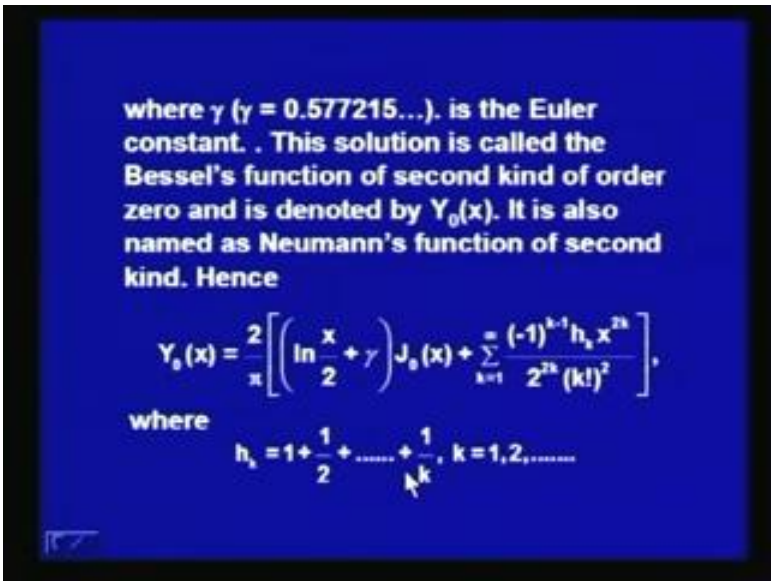

Here, gamma is equal to 0.577215 and so on it is called the Euler's constant, this solution is called the Bessel's function of second kind of order 0 and is denoted by Y naught x. We also call it Neumann's function of second kind, so with this as the second solution Y naught x will then assume the form 2 over pi ln x by 2 plus gamma into J naught x plus sigma k equal to 1 to infinity minus 1 to the power k minus 1 over 2 to the power 2 k k factorial square h k x to the power 2 k. Where we have substituted the value of y 2 x and as we have seen our y 2 star x becomes Y naught x. So, here this h k denotes 1 plus 1 by 2 plus n and so on 1 by k where k takes values 1, 2, 3, and so on.

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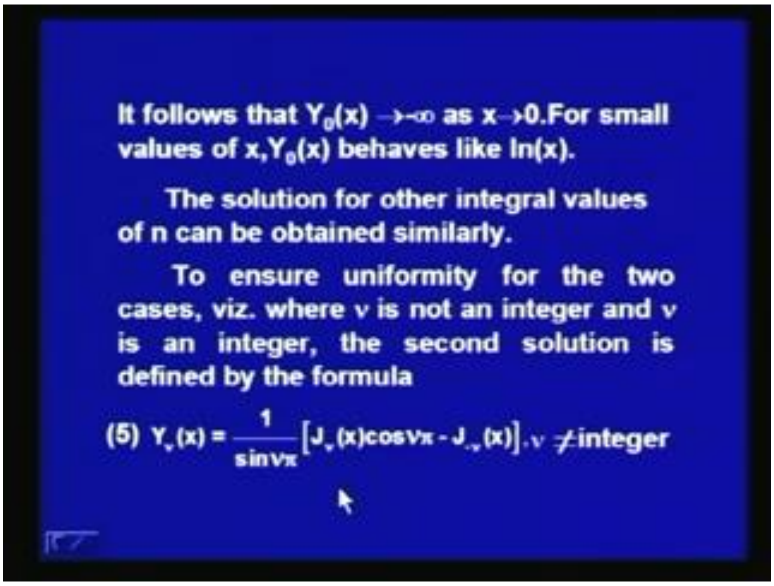

Now, from this expression Y naught x you can see that Y naught x goes to minus infinity as x goes to 0, we know that ln x goes to minus infinity as that is goes to 0. So, Y naught x goes to minus infinity as x goes to 0 and for small values of x Y naught x we have like the logarithmic function ln x. So, this how we find the second solution of the Bessel's equation of order 0, the second solution of the Bessel's equation for other integral values of n can be obtained in a similar manner.

Now, in practise when we make use of the Bessel's equation the two solutions of the Bessel's equation are given by this for uniformity in the two cases, that is when nu is not an integer and when nu is an integer. The second solution is given by the following formula Y nu x equal to 1 over sin nu pi J nu x cos nu pi minus J minus nu x, when nu is not an integer, you see we have seen earlier that when nu is not an integer the two solutions of the Bessel's equation J nu x and J minus nu x are linearly independent.

So, when nu is not an integer this Y nu x being a linear combination of J nu x and J minus nu x is also a solution of the Bessel's equation and therefore, may be taken as the second solution of the Bessel's equation.

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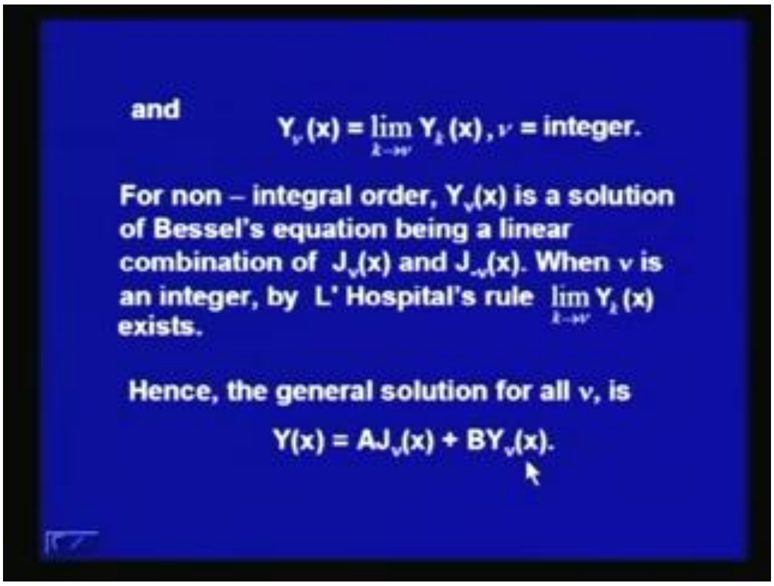

When nu is an integer then the second solution of the Bessel's equation Y nu x is the limit of Y k x as k goes to nu, this Y k x is given by the expression for Y nu x where nu is not an integer. Now, for non-integral order we Y nu x is a solution of Bessel's equation being a linear combination of J nu x and J minus nu x, when nu is an integer we find that by L Hospital's rule, this limit exists the limit of Y k x as k goes to nu it exist. And therefore, the general solution for all nu of the Bessel's equation is given by Y x equal to A into J nu x plus B into Y nu x.

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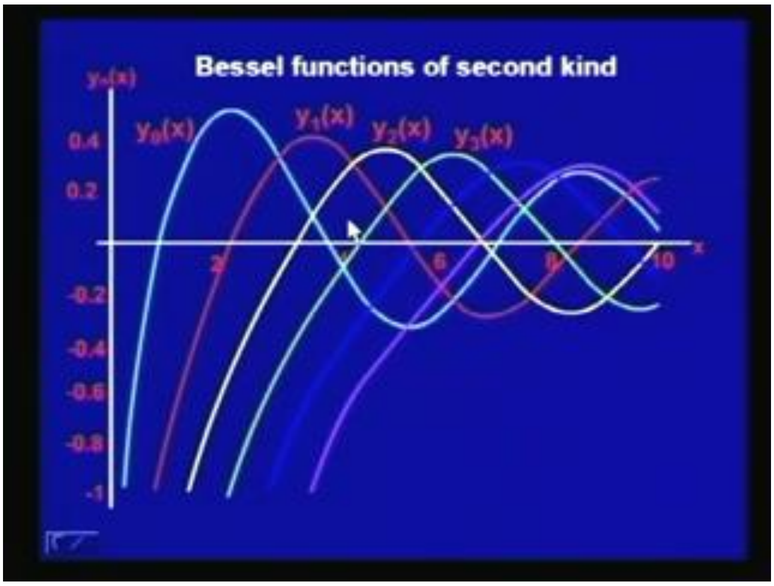

This picture shows us the graphs of Bessel functions of second kind, this is the graph of y naught x, this is the graph of y 1 x, then we have the graph of y 2 x, y 3 x and so on.

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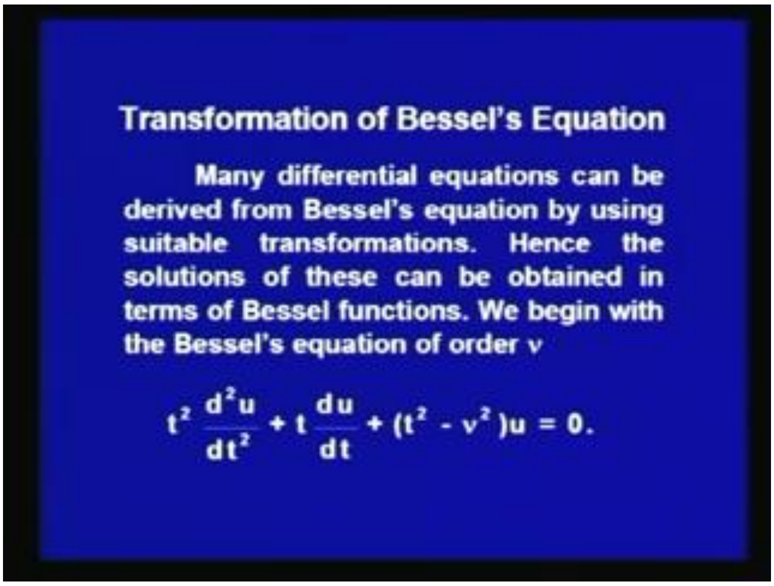

Now, many differential equations can be derived from the Bessel's equations by making use of some transformations. And the solutions of such equations can therefore, be obtained in terms of Bessel functions, we shall begin with the Bessel equation of order nu and arrive at a fairly general linear differential equation of second order whose solution can be expressed in terms of Bessel functions by comparing that differential equation, we will be able to find the solutions of many differential equations that have solution in terms of Bessel functions. So, let us consider the Bessel's equation of order nu t square d square u over d t square plus t times du by d t plus t square minus nu square into u equal to 0.

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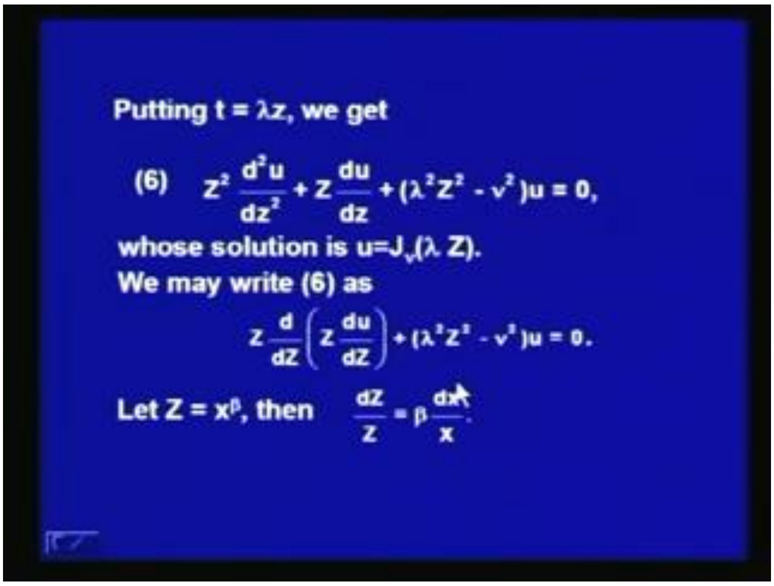

The first substitution that we make in this Bessel's equation of order nu is t equal to lambda z that is we change from the independent variable t to the independent variable z lambda is a constant here. So, when we put t equal to lambda z in the Bessel's equation of order nu, we get the equation Z square d square u by d Z square plus Z du by d Z plus lambda square Z square minus nu square into u equal to 0, now we know that a solution of the Bessel's equation of order nu is J nu t.

So, and we have put t equal to lambda Z, so a solution of this equation 6, this transformed equation is therefore, J nu lambda Z we can we may also note that the first two terms of this equation 6 can be expressed as Z into d over d Z of Z du by d Z. Because, when you differentiate Z into du by d Z with respect to Z what you get is d u by d Z plus Z d square u by d Z square and when you multiply Z to that you get Z square d square u by d Z square plus Z du by d Z.

So, equation 6 can be written in an alternate form as Z d over d Z of Z du by d Z plus lambda square Z square minus nu square into u equal to 0. Let us now make another substitution to this differential equation of second order, let us put Z equal to x to the power beta, where beta is a constant. When we take the logarithm on both sides we get log Z equal to beta log x and when we differentiate on both sides what we get is d Z over Z equal to beta d x over x. So, making this substitution Z equal to x to the power beta and hence d Z over Z equal to beta d x over x in this equation.

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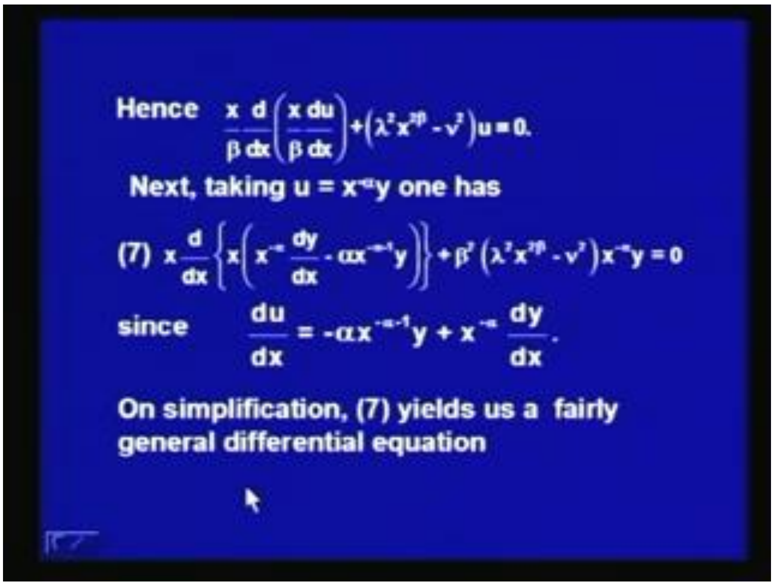

We will get the transformed equation as x over beta into d over d x of x over beta d u by d x plus lambda square x to the power 2 beta minus nu square into equal to 0. Let us, now put u equal to x to the power minus alpha into y in this differential equation, so now, at the these twp transformations that we have done to the Bessel’s equation, there we change the independent variables first from t to z and then from z to x, now we are changing the dependent variable from u to y.

So, u let us put u equal to x to the power minus alpha into y, in this differential equation what we shall have x d over d x in this can be combined and you get beta square. So, beta square be are multiplying to this whole equation, this beta square is coming up here and so we get by x into d over d x then x over beta d u by d x changes into x times d u by d x. So, d over d x of u will be x to the power minus alpha d y by d x minus alpha times x to the power minus alpha minus 1 into y plus beta square into lambda square x to the power 2 beta minus nu square into x to the power minus alpha into y equal to 0. There we have written the derivative of u like this, so the value of d over d x we have put here, now on simplification this equation 7 yields us a fairly general differential equation of second order.

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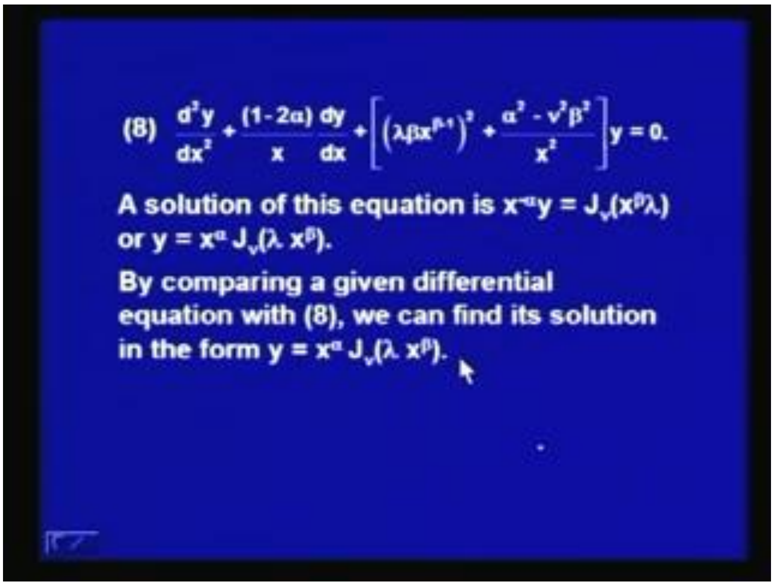

It may be written as d square y over d x square plus 1 minus 2 alpha over x into d y by d x plus lambda beta x to the power beta minus 1 whole square plus alpha square minus nu square beta square over x square into y equal to 0 or it may also be written as if you multiply by x square in the whole of this equation you get x square d square y by d x square plus 1 minus 2 alpha into d y by d x plus lambda square beta square x to the 2 beta plus alpha square minus nu square beta square into y equal to 0.

So, it may also in that form, now a solution of this equation will there for be we had earlier u equal to J nu lambda x to the power beta. But, then we adds up change the independent variable from u to y by making this substitution u equal x to the power minus alpha into y. So, a solution of this equation is therefore, x to the power minus alpha into y equal to J nu lambda x to the power beta by multiplying by x to the power alpha both sides, we get a solution of this equation as y equal to x to the power alpha J nu lambda x to the power beta.

So, whenever we are given a differential equation whose solution is to be expressed in the form of the Bessel’s functions, we can compare that differential equation with this fairly general differential equation of second order and find the values of the constants alpha beta, this alpha, beta, lambda and nu. And we can then find the solution of the Bessel’s the given differential equation in terms of the Bessel functions, so now, let us take an example based this article.

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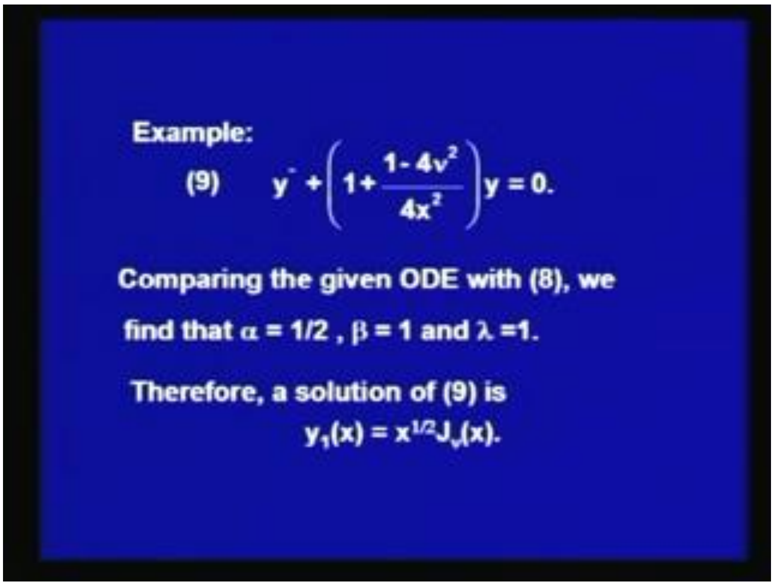

Let us consider the differential equation y double dash plus 1 plus 1 minus 4 nu square over 4 x square into y equal to 0, when we compare this differential equation with the transformed Bessel's equation that is the equation number 8, we find that alpha is equal to half, beta is equal to 1 and lambda is equal to 1. And therefore, a solution of this equation 9, may be expressed as y 1 x equal to x to the power half into J nu x, where nu can take integral or non-integral values here.

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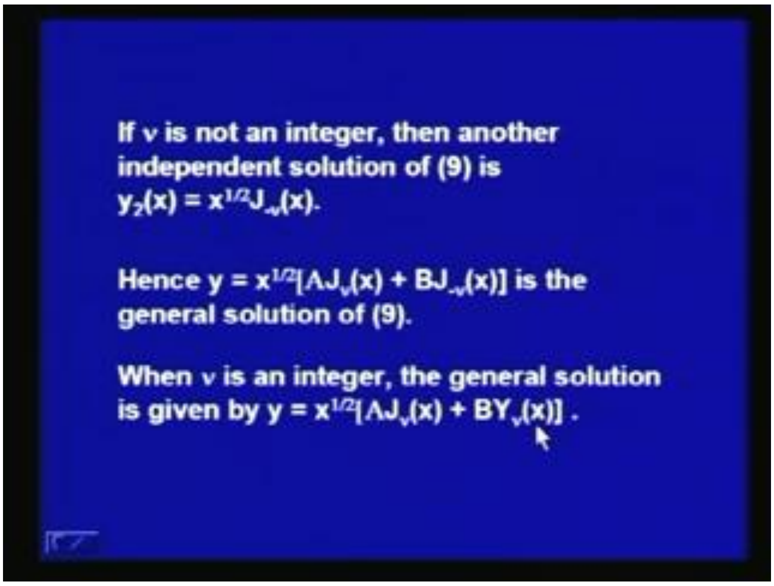

So, if the differential equations such that nu is not an integer, then another independent solution of the equation 9 can be written as y 2 x equal to x to the power half into J minus nu x. And hence we may write the general solution of equation 9 as y equal to x to the power half into A J nu x plus B J minus nu x, now if nu is an integer then the second solution of the Bessel's equation will have to be obtained as we have discussed earlier for nu equal to 0 and for other integral values of nu we can similarly find the solution.

So, the second solution where the equation 9 will have to are we obtained using those methods and so then y will be equal to x to the power half A into J nu x plus B into Y nu x, we have represented the second solution of the Bessel's equation by Y nu x. This lecture today we have discussed the solutions of the Bessel's equation, which be called as Bessel function of first kind that is J nu x and the Bessel function of the second kind that is Y nu x.

And also seen how we can find the solution in terms of Bessel function of a fairly general differential equation, which we obtained by making some transformations to the Bessel's equation of order nu. In our next lecture, we shall be discussing the orthogonality of Bessel functions, then how to expand a function, which is continuous over a say an interval 0 a and has oscillations in that interval in terms of Bessel functions, we shall discuss the generating function. And we shall also discuss the representation of Bessel function of first kind in terms of an integral, which we call as the Bessel integral. So, in that the Bessel function of order nu will consider where nu will be an integer, so all these we will be discussing in our next lecture on Bessel's functions and their properties.