Mathematics III
Prof. Dr. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 5
Series Solution of Homogeneous Linear Differential Equations (Contd..)

Dear viewers, my talk today is in continuation to my previous talk on Series Solution of Homogeneous Linear Differential Equations. We know that homogeneous linear differential equations with constant coefficients can be solved by algebraic methods and the solutions are known functions of calculus like, sin x, cos x, exponential x, and so on. But in case where the coefficients of the homogeneous linear differential equations are not constants, but functions of x the solutions may not be non elementary functions.

So, the examples of Bessel's equation, Legendre's equation, hyper geometric equations fall in this category. In our last lecture we had discussed the solution of Legendre's equation by power series method. In our today's lecture, we shall discuss the general method for finding the solution of a homogeneous linear differential equation in a power series in a ((Refer Time: 01:33)) by the method is will be known as frobenius method.

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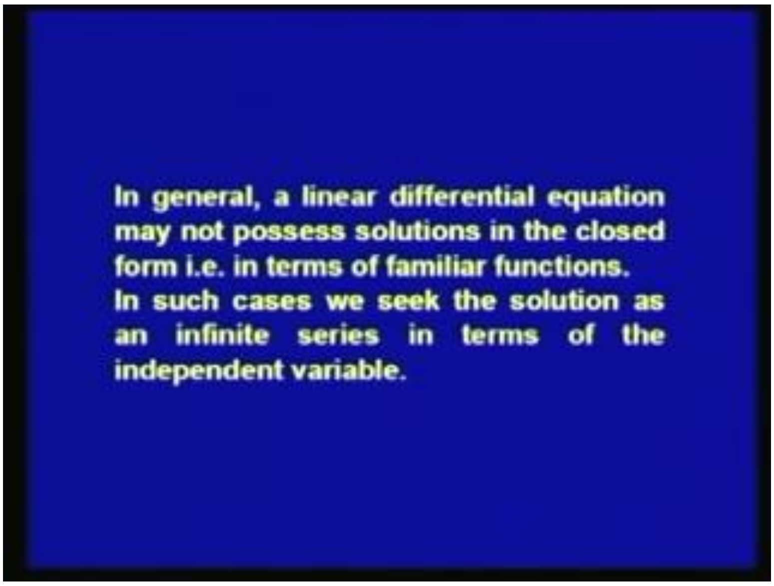

So, we know that linear differential equation may not possess solutions in the closed form that is in terms of familiar functions, in such cases we seek the solution as an infinite series in terms of the independent variable.

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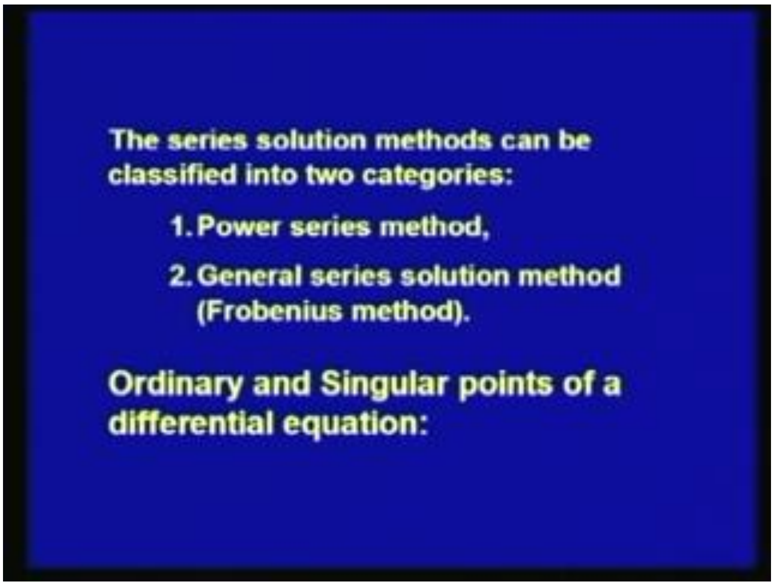

The series solution can be classified into two categories, power series method, general series solution method. In our last lecture we had discussed the power series method for finding the solution of a homogeneous linear differential equation. The general series solution method is an extension of the power series method, it is known as frobenius method. Let us first discuss a the some important points like, ordinary and singular points of a differential equation.

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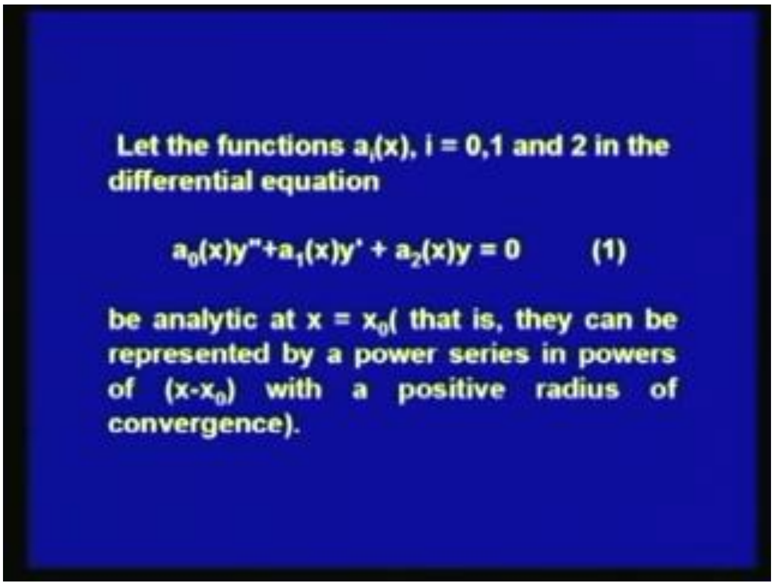

Let the functions a i x i equal to 0 1 and 2 in the differential equation a naught x into y double dash plus a 1 x into y dash plus a 2 x into y equal to 0 where, y is a function of x be analytic at a point x equal to x naught. By analyticity at x equal to x naught we mean that the function can be represented by a power series in the powers of x minus x naught with the positive radius of convergence.

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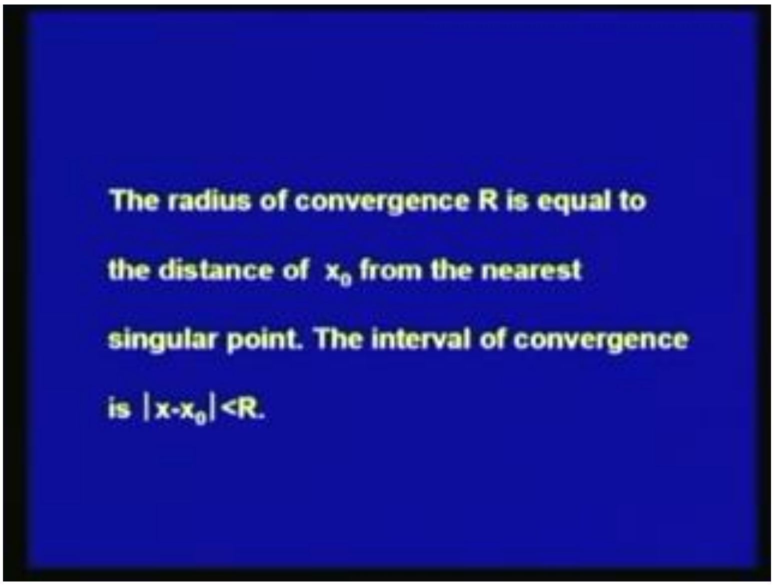

The radius of convergence R is equal to the distance of x naught from the nearest singular point of the function f, the interval of convergence is given by mod of x minus x naught less than R.

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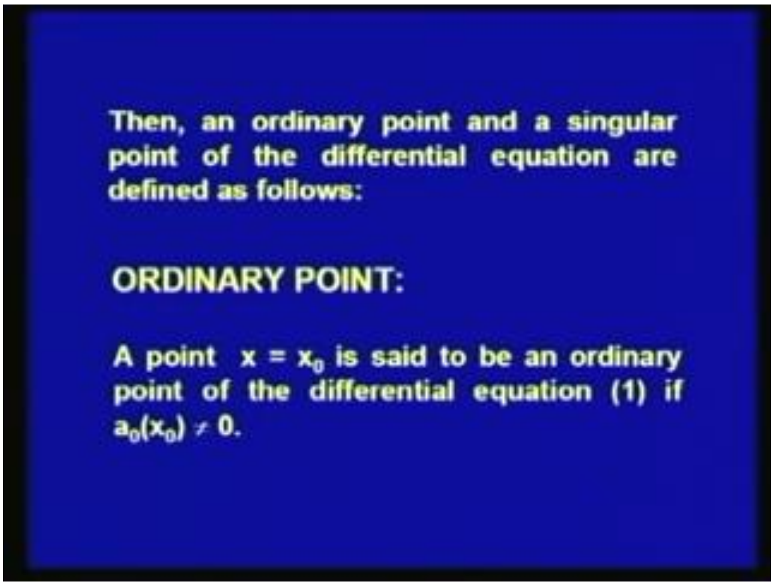

Now, let us define an ordinary point and a singular point of the differential equation, an ordinary point is defined as a point x equal to x naught for which a naught x naught is not equal to 0, a point x equal to x naught is said to be an ordinary point of the differential equation 1, if a naught does not vanish at x equal to...

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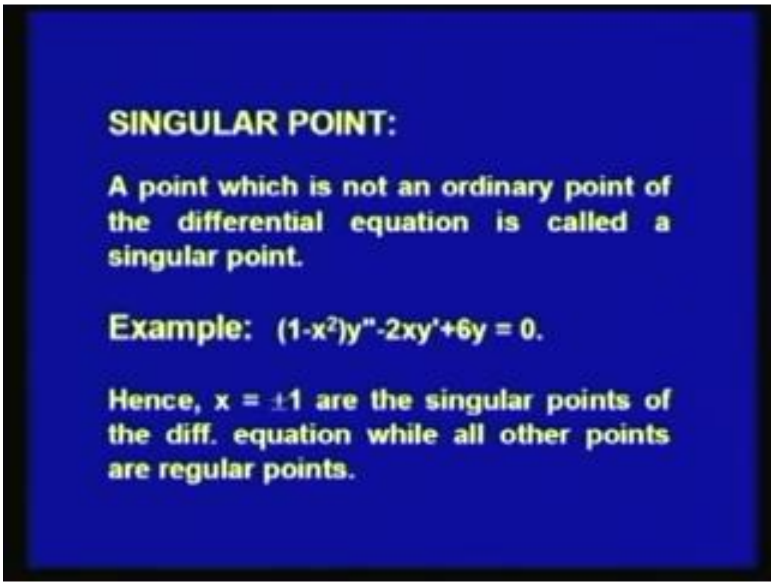

A point which is not an ordinary point of the differential equation is called a singular point. For example, let us take the differential equation 1 minus x square into y double dash minus 2 x y dash plus 6 y equal to 0, so if you compare it with equation 1 here a naught x is equal to 1 minus x square a 1 x is minus 2 x and a 2 x equal to 6. And when we put a naught x equal to 0. We find that x equal to plus minus 1, therefore x equal to plus minus 1 are the singular points of this differential equation while all other points are regular points.

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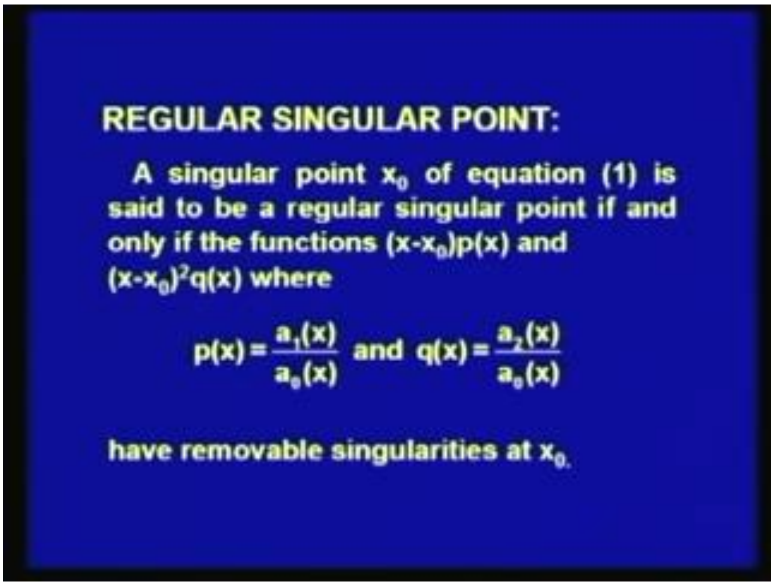

A singular point x naught of equation 1 is said to be a regular singular point if and only if the functions x minus x naught into p x and x minus x naught into q x, where p x is given by a 1 x over a naught x and q x is given by a 2 x over a naught x have removable singularities at x naught.

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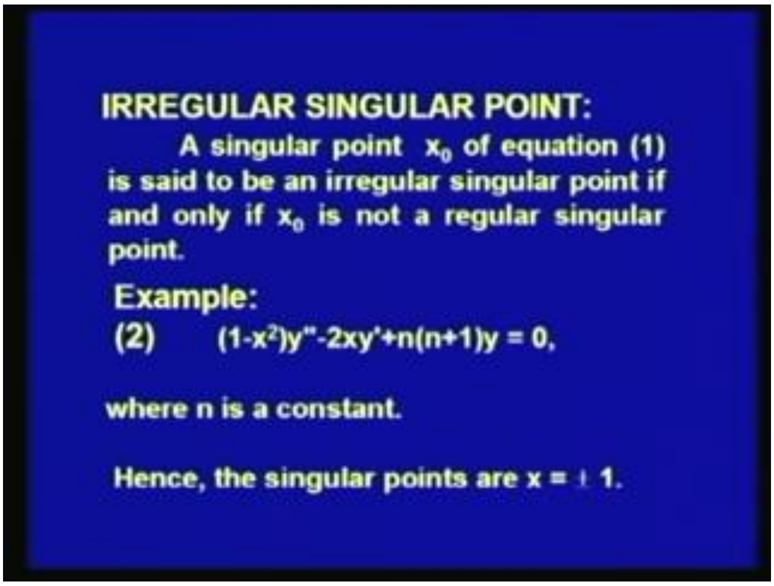

Let us now define an irregular singular point, a singular point x naught of equation 1 is said to be an irregular singular point if and only if x naught is not a regular singular point. For example, let us consider the differential equation 1 minus x square into y double dash minus 2 x y dash plus n into n plus 1 y equal to 0 where n is a constant, here the singular points are x equal to plus minus 1.

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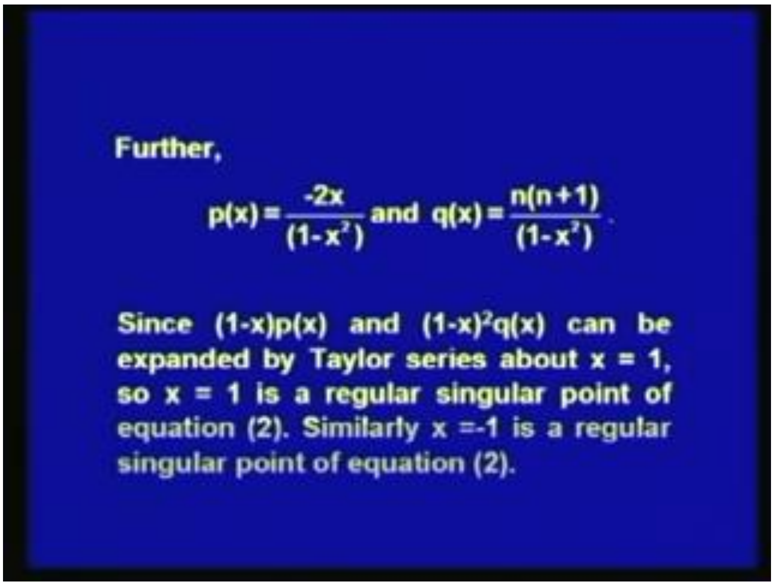

Further, p x is equal to minus 2 x over 1 minus x square and q x is n over n plus 1 1 minus x square. Now, let us multiply p x by 1 minus x and q x by 1 minus x whole square, we note that where the functions relative functions have removable singularities at x equal to 1 and therefore, they can be expanded by Taylor series about x equal to 1. So, x equal to 1 is a regular singular point of the equation differential equation 2 and similarly we can see that x is equal to minus 1 is also a regular singular point of the equation 2.

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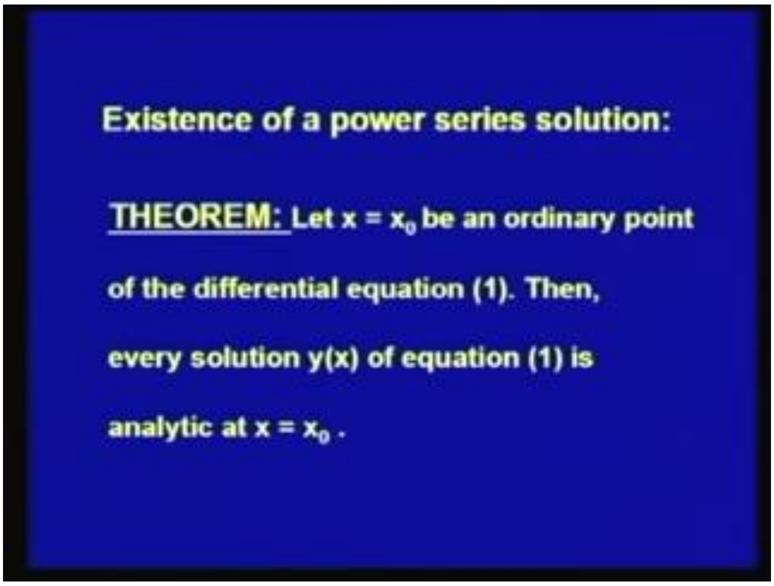

Now, let us look at the theorem on the existence of a power series solution, said this theorem we had used in our previous lecture on power series solution this ((Refer Time: 06:11)) shows that let x equal to x naught be an ordinary point of the differential equation 1. Then every solution y x of the equation 1 is analytic at x equal to x naught that is every solution y x of the equation 1 can be represented by a power series in the powers of x minus x naught with the positive radius of convergence.

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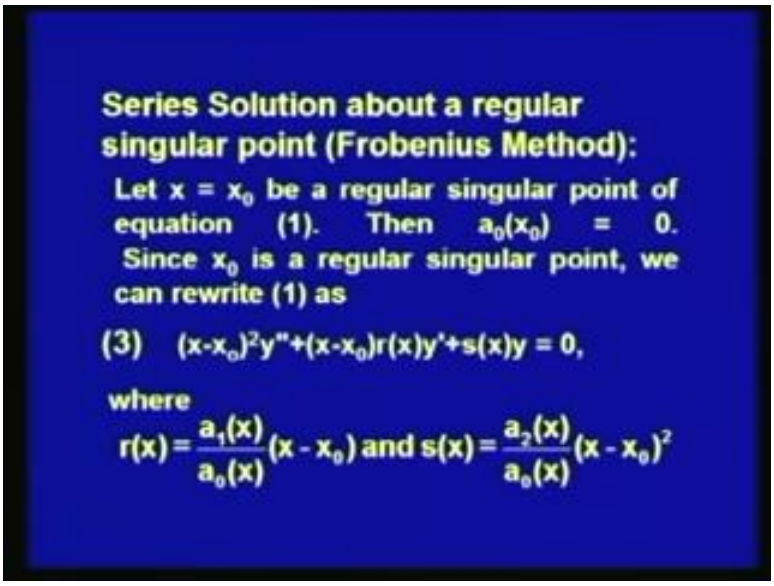

Let us, now discuss series solution about a regular singular point, this method is called as frobenius method. So, let x equal to x naught be a regular singular point of the differential equation 1, then by definition of a singular point a naught x naught is equal to 0. Since, x naught is a regular singular point we can write the equation differential equation 1 as x minus x naught whole square into y double dash plus x minus x naught into r x into y dash plus s x into y equal to 0. Where, r x is a 1 x over a naught x into x minus x naught and s x is equal to a 2 x over a naught x into x minus x naught whole square.

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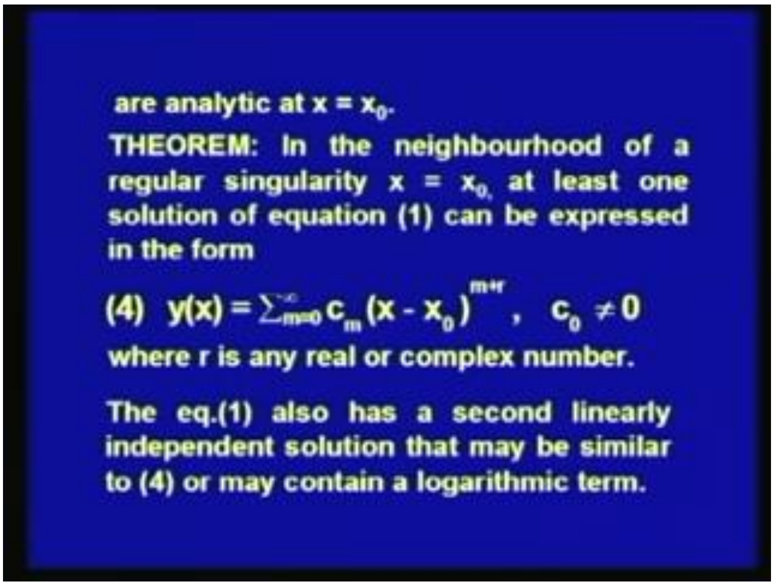

So, they are analytic at x equal to x naught because, x equal to x naught is a regular singular point. Now, the frobenius method actually follows from this theorem, this theorem is given by frobenius it says that in the neighbourhood of a regular singularity x equal to x naught that is in the neighbourhood of a regular singular point at x equal to x naught. At least 1 solution of the differential equation 1 can be expressed in the form y x equal to sigma m equal to 0 to infinity c m x minus x naught to the power m plus r where c naught is not equal to 0 and r is any real or complex number.

The equation 1 also have a second linearly independent solution that may be similar to the solution 4 or it may contain a logarithmic term. The power r of x minus x naught here distinguishes it from the power series solution because, here r need not be a non negative integer if it is a non negative integer then it been reduced to a power series, the series 4 will reduce to a power series. And therefore, we can say that this frobenius method is an extension of the power series method.

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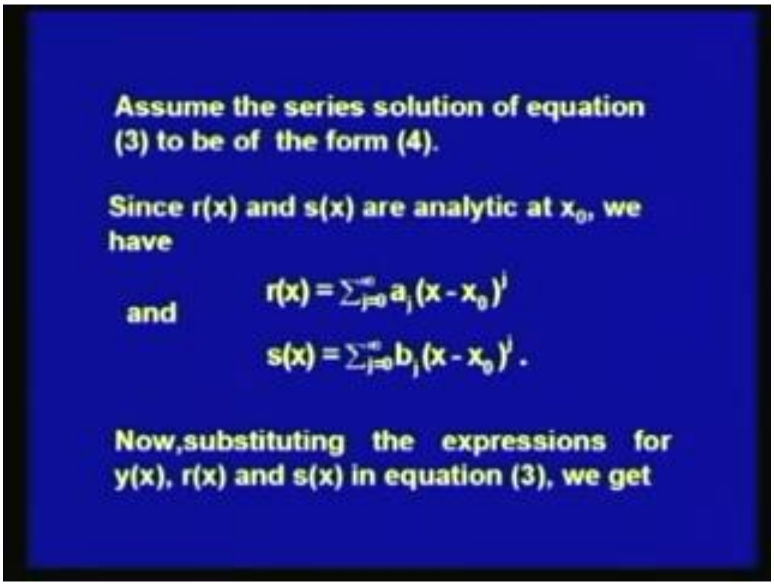

So, now assume the series solution of equation 3 to be of the form 4, since r x and s x are analytic at x equal to x naught, we can write their power series, the series of r x lets write at sigma j equal to 0 to infinity a j x minus x naught to the power j and s x we may write as sigma j equal to 0 to infinity b j x minus x naught to the power j. Then let us now substitute the expressions for y x, r x and s x in the equation number 3.

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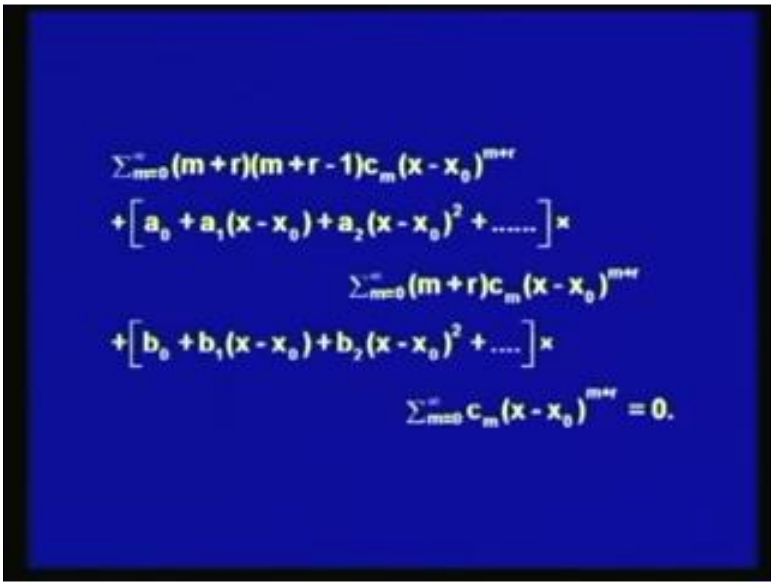

Then we shall have sigma m equal to 0 to infinity m plus r m plus r minus 1 c m x minus x naught to the power m plus r. Because, the first term in that equation is x minus x naught whole square into y double dash, so when you differentiate by twice and multiply by x minus x naught whole square, you get the power of x minus x naught as m plus r e r. And then in the next term we put the power series for the function r x that is a naught plus a 1 into x minus x naught plus a 2 x naught x naught whole square and so on and multiply y x minus x naught into y dash.

When you write y dash then power of x minus x naught will reduce by 1, but when we multiply by x minus x naught it will be remain x naught to the power m plus r. So, in the second term we have the series sigma m equal to 0 to infinity m plus r into c m into x minus x naught to the power m plus r. And then in the last term we have s x into y, so power series for s x we write b naught plus b 1 x minus x naught plus b 2 x minus x naught whole square and so on and multiply by y, that is sigma m equal to 0 to infinity c m x minus x naught to the power m plus r equal to 0.

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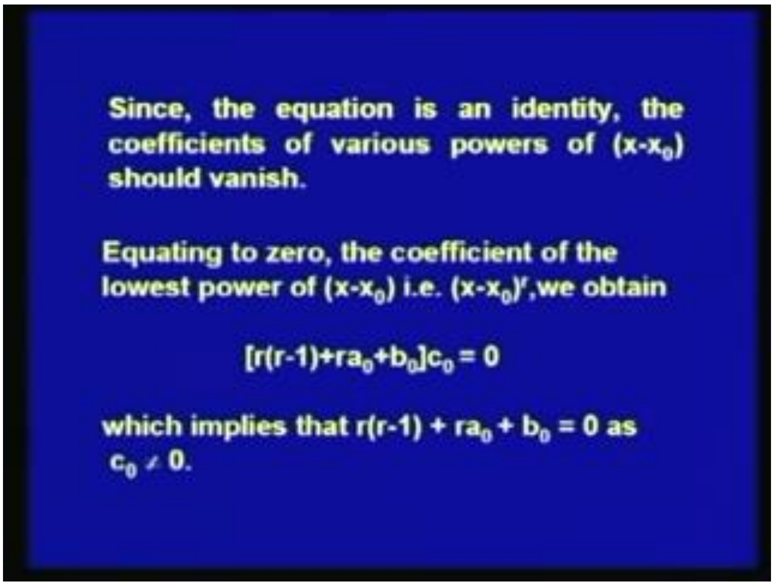

Since, the equation is an identity, the coefficients of various powers of x minus x naught should vanish. Now, let us equate to 0 the coefficient of the lowest power of x minus x naught, which occurs in this equation and the lowest power of x minus x naught is r. So, let us equate the coefficient of x minus x naught to the power r equal to 0 we will obtain r into r minus 1 plus r a naught plus b naught into c naught equal to 0, which implies that r into r minus 1 plus r a naught plus b naught is equal to 0 as c naught is not equal to 0.

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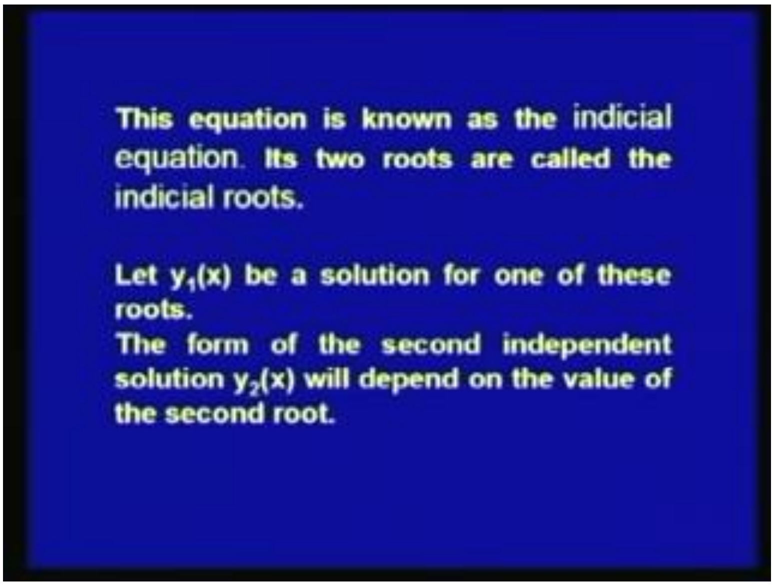

This equation is known as the indicial equation and it is two roots are called the indicial roots. Now, the indicial equation in the frobenius method has a great significance, because the roots of the indicial equation tells us the form of the second independent solution of the differential equation. So, let us say that y 1 x be a solution for one of these roots of the indicial equation, then the form of the second independent solution say y 2 x of the differential equation will depend on the value of the second root.

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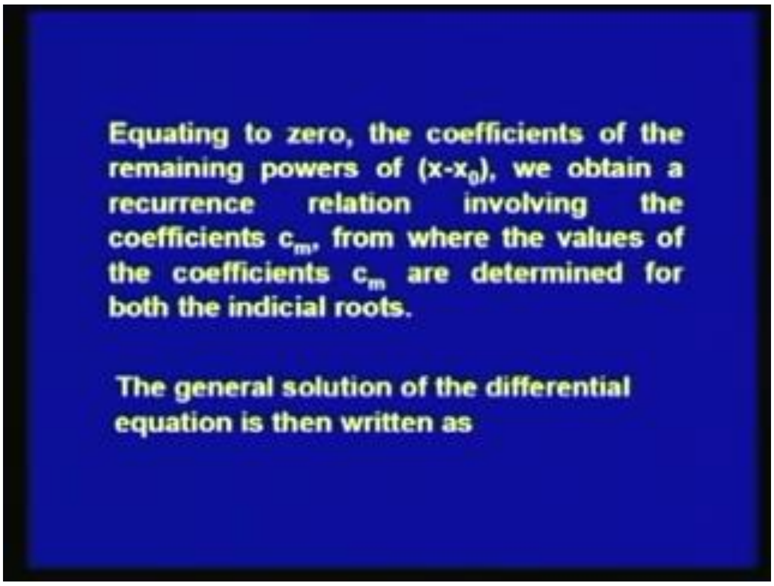

Now, let us equate to zero the coefficients of remaining powers of x minus x naught, they will give us a recurrence relation involving the coefficients c m, from where we can determine the values of the coefficients c m for both the indicial roots.

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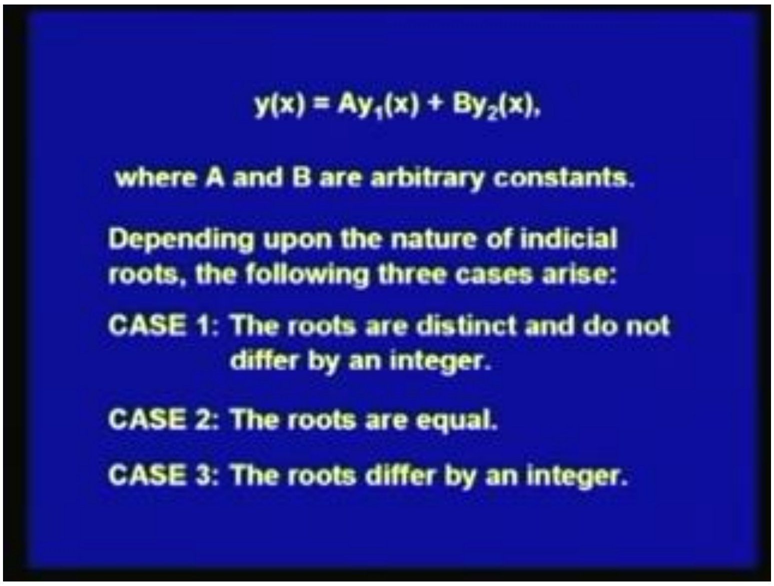

Then the general solution of the differential equation is written as y x equal to A times y 1 x plus B times y 2 x, where A and B are arbitrary constants. Now, depending upon the nature of the indicial roots following three cases arise, the first case is when the roots of the indicial equation are distinct and do not differ by an integer. The second case is the roots of the indicial equation are both equal and the third case is the roots of the indicial equation differ by an integer let us take these cases one by one.

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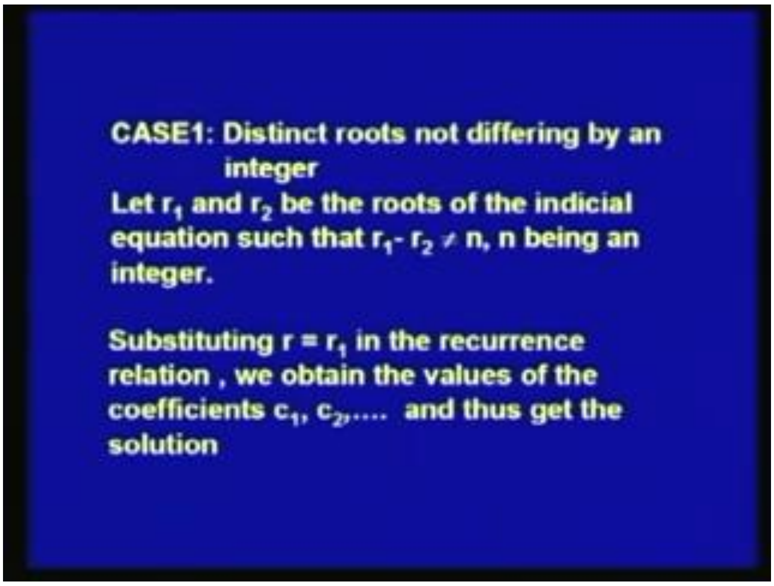

So, let us first discuss the case of the distinct roots not differing by an integer, so let us say that let r 1 and r 2 be the roots of the indicial equation such that r 1 minus r 2 is not equal to n, where n is an integer. So, that we will do is that we will substitute r equal to r 1 in the recurrence relation and we shall obtain the values of the coefficients c 1, c 2 and so on.

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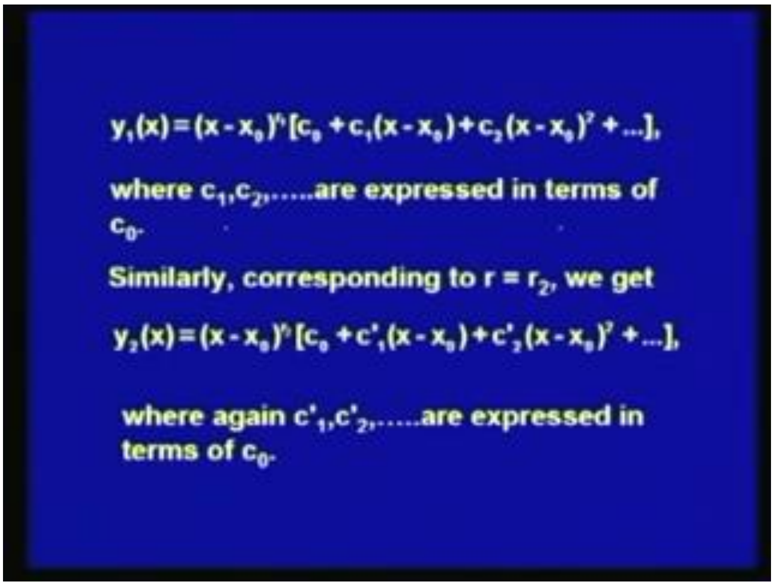

And thus get the first solution of the differential equation y 1 x equal to x minus x naught to the power r 1 into c naught plus c 1 into x minus x naught plus c 2 into x minus x naught whole square and so on, where, c 1, c 2 are expressed in terms of the c naught. Similarly, corresponding to r equal to r 2 we will find the second dissolution y 2 x which is x minus x naught to the power r 2 into c naught plus c 1 dash x minus x naught c 2 dash x minus x naught whole square and so on, where again c 1 dash c 2 dash are expressed in terms of c naught.

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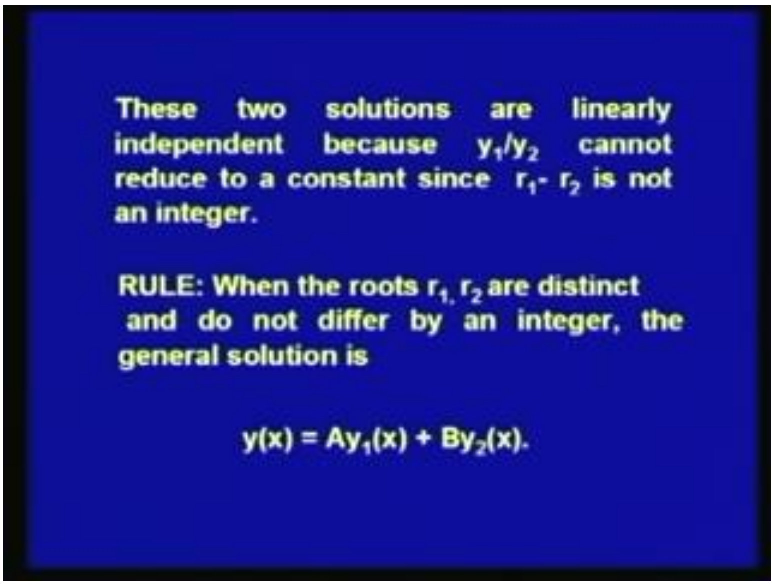

Now, these two solutions y 1 x and y 2 x are both linearly independent because, y 1 over y 2 cannot reduce to a constant, since r 1 minus r 2 is not equal to an integer. So, will then write their linear combination and have the general solution of the differential equation. And thus we have the following rule, when the roots are r 1 and r 2 of the indicial equation are both distinct and do not differ by an integer. The general solution is found by replacing r by r 1 and r by r 2 in the expression y x equal to sigma in c m x minus x naught to the power m plus r, where c m's are found from the recurrence relation and the we write the general solution as y x equal to A times y 1 x plus B times y 2 x.

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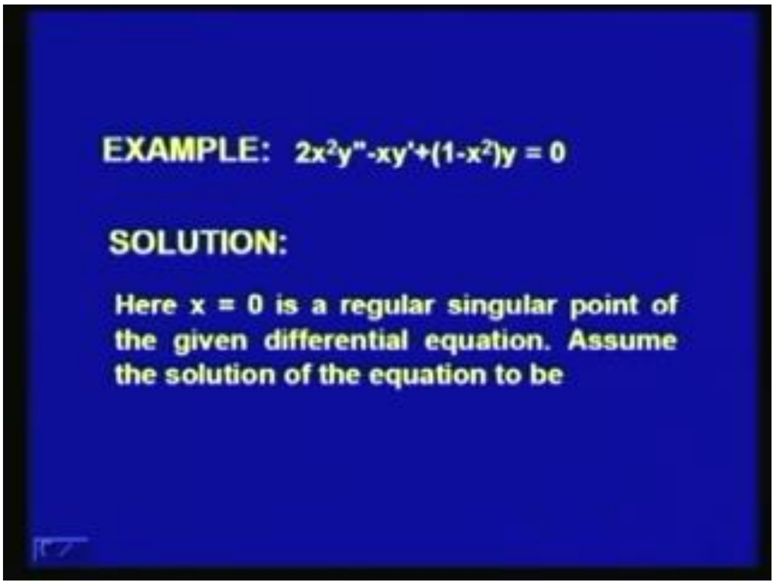

Let us study in example on the case 1, let us consider the differential equation 2 x square y double dash minus x y dash plus 1 minus x square into y equal to 0. Let us, note that x equal to 0 is a regular singular point of the given differential equation, here a naught is equal to 2 x square which vanishes that x equal to 0 and a 1 x over a naught x will be equal to minus 1 upon 2 x a 2 x over a naught x will be 1 minus x square over 2 x square.

So, when we multiply a 1 x over a naught x by x minus x naught that is x and a 2 x over a naught x by x square, then they both have removable similarities at x equal to 0. Therefore, x equal to 0 is a regular singular point of the given differential equation and therefore, we can apply the frobenius method to solve this differential equation of second order.

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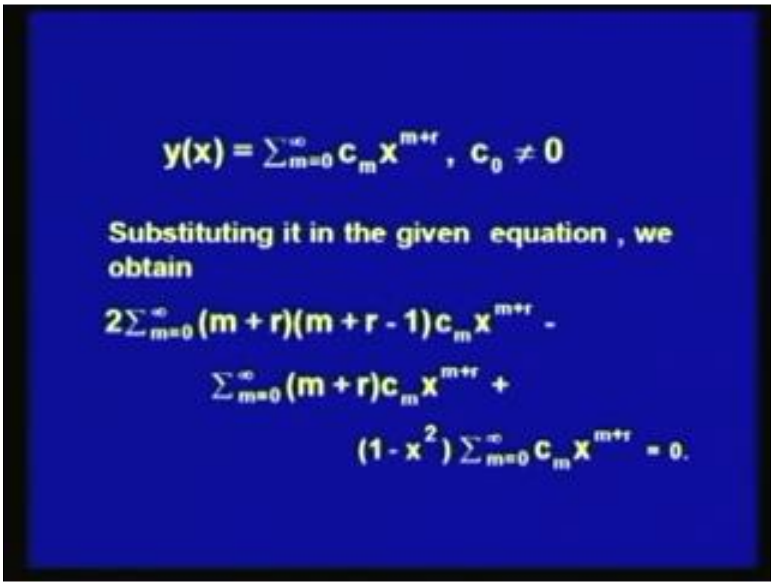

So, let us assume the solution of the differential equation to be y x equal to sigma m equal to 0 to infinity c m x to the power m plus r, where c naught is not equal to 0 and substitute it in the given differential equation. After substituting it in the given differential equation, we shall have two times sigma m equal to 0 to infinity m plus r into m plus r minus 1 c m x to the power m plus r minus sigma m equal to 0 to infinity m plus r c m x to the power m plus r plus 1 minus x square into sigma m equal to 0 to infinity c m x to the power m plus r.

Now, we can note that the lowest power of x here will be r, which we get from the first term and the second term and also from the third term when we multiply sigma c m x to the power m plus r to 1.

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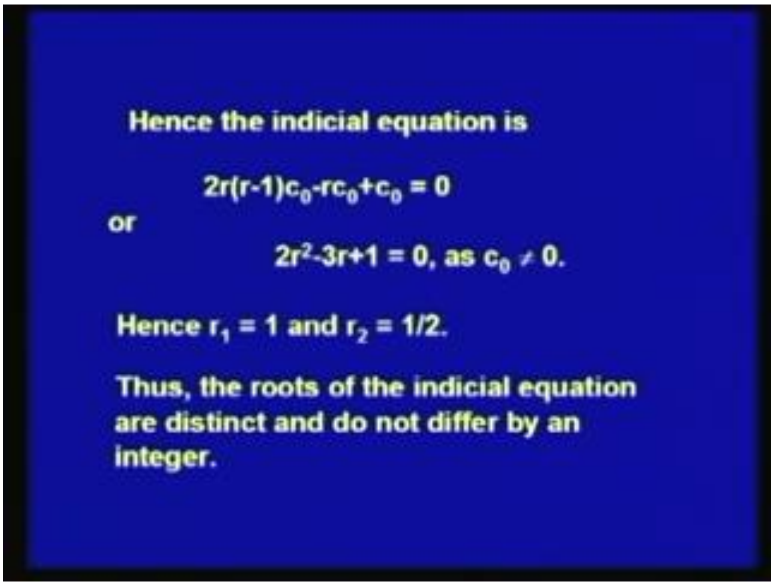

So, we can see that the indicial equation here, we will turn out to be 2 r into r minus 1 c naught minus r c naught plus c naught equal to 0, which we get by equating to 0 the coefficient of the lowest power of x that is r the coefficient of x to the power r we put 0 equal to 0 and get this indicial equation from where we have 2 r square minus 3 r plus 1 equal to 0 as c naught is not equal to 0. And then the roots of this equation are r 1 equal to 1 and r 2 equal to half where, these roots are distinct and do not differ by an integer.

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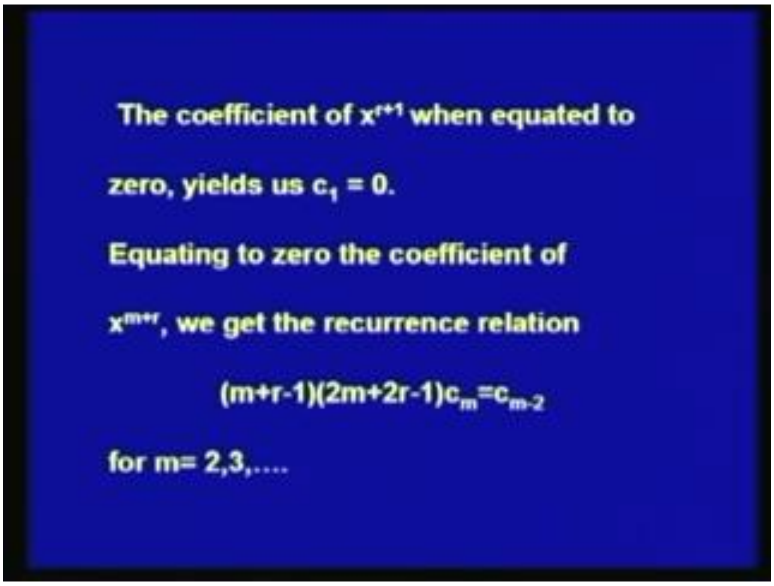

And, so now, next we put the coefficient of the next higher power of x equal to 0, the next coefficient of next higher power of x is gives us c 1 equal to 0. And then we put the coefficient of x to the power m plus r equal to 0 to obtain the recurrence relation, which is m plus r minus 1 into 2 m plus 2 r minus 1 into c m equal to c m minus 2 for m equal to 2 3 and so on, because m is greater than r equal to 2 here.

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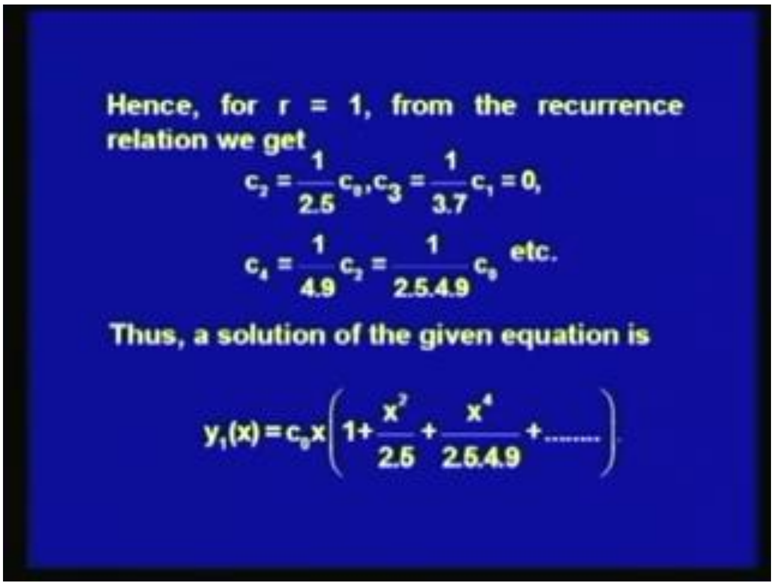

Hence, for r equal to 1 from this recurrence relation we shall have c 2 equal to c naught over 2 into 5 c 3 equal to 1 over 3 into 7 into c 1, but c 1 is equal to 0. So, c 3 is also 0 and c 4 will tell out to be deliver 4 into 9 into c 2 the value of c 2 we substitute here in terms of c naught and get c 4 as 1 over 2 into 5 into 4 into 9 into c naught, etcetera.

Thus a solution of the given equation is y 1 x equal to c naught x into 1 plus x square over 2 into 5 plus x to the power 4 over 2 into 5 into 4 into 9 and so on a 1 equal to 0 c 1 equal to 0 implies c 3, c 5, c 7 all are 0's, so only the coefficients of even powers of x are present in this solution y 1 x.

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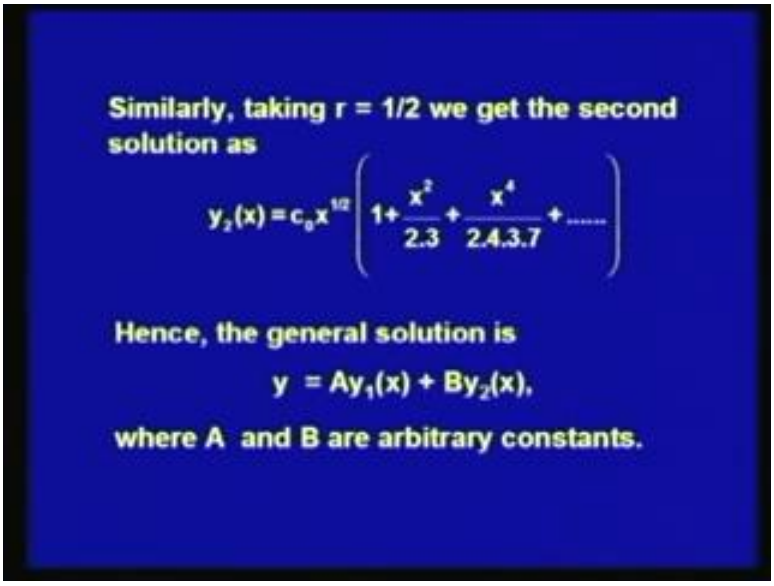

Similarly, taking r equal to half we get the second solution of the differential equation as y 2 x equal to c naught x to the power half into 1 plus x square over 2 into 3 plus x to the power 4 over 2 into 4 into 3 into 7. And then we can write the general solution of the differential equation as y equal to A into y 1 x plus B into y 2 x where A and B are arbitrary constants.

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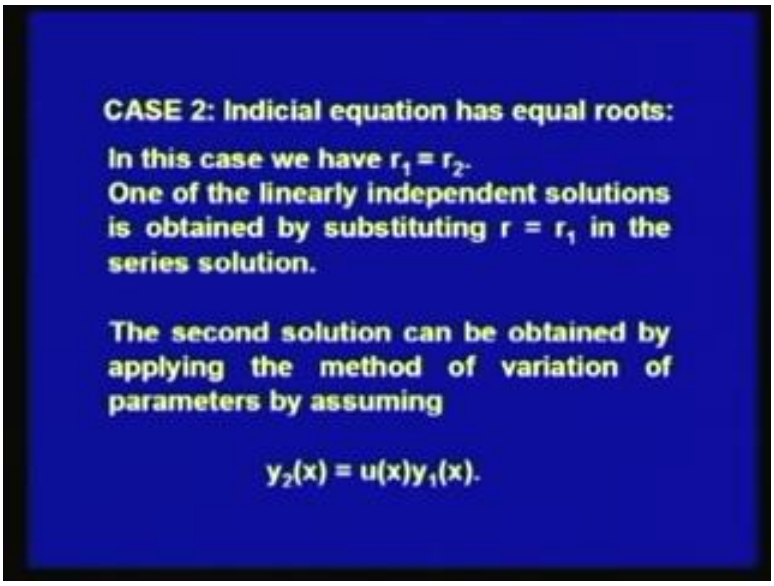

Now, let us next take up the case of the indicial equation having equal roots, so in this case if r 1 and r 2 are the roots of the indicial equation they are equal and so we have r 1 is equal to r 2. Now, in this case 1 of the linearly independent solutions of the differential equation is obtained by substituting r equal to r 1 in the series solution, the second solution can be obtained by applying the method of variation of parameters while assuming y 2 x equal to u x into y 1 x.

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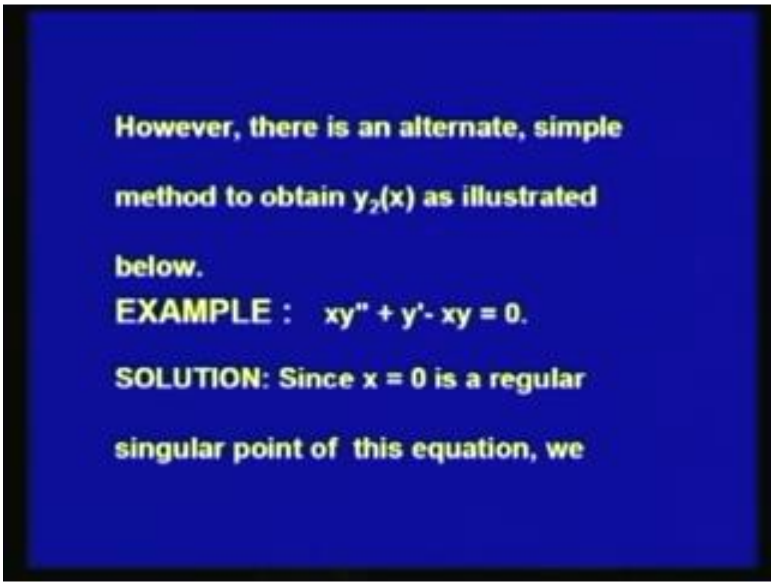

But, there is an alternate simple method to obtain the second independent solution y 2 x, which is illustrated as below. Let us take the example of the differential equation x y double dash plus y dash minus x y equal to 0, you can note here that again x equal to 0 is a regular singular point of this equation, so for various method can be applied.

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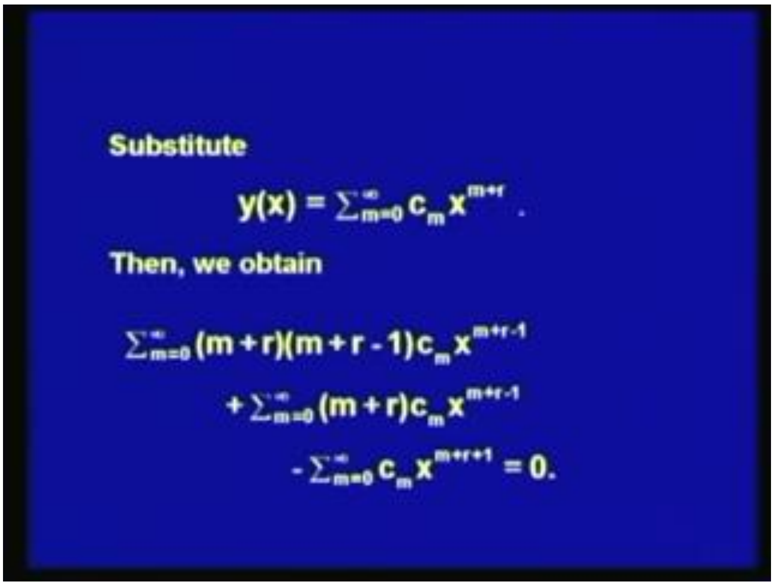

And we can therefore, assume solution of this differential equation to be of the form y x equal to sigma m equal to 0 to infinity c m x to the power m plus r, where c naught is not equal to 0. We substitute these value of y x this expression for y x in the given differential equation to obtain sigma m equal to 0 to infinity m plus r m plus r minus 1 c m x t o the power m plus r minus 1 plus sigma m equal to 0 to infinity m plus r c m x to the power m plus r minus 1 minus sigma m equal to 0 to infinity c m x to the power m plus r plus 1 equal to 0.

And again we will put the coefficient of the lowest power of x to 0 the coefficient of lowest power of x here will be obtained by taking m equal to 0 from the first term when we take m equal to 0 you get x to the power r minus 1, in the second term also you get x to the power r minus 1, but in the third term when we take m equal to 0 you get x to the power r plus 1. So, the least power of x is r minus 1 and the coefficient of x to the power r minus 1 is available from the first and second term only.

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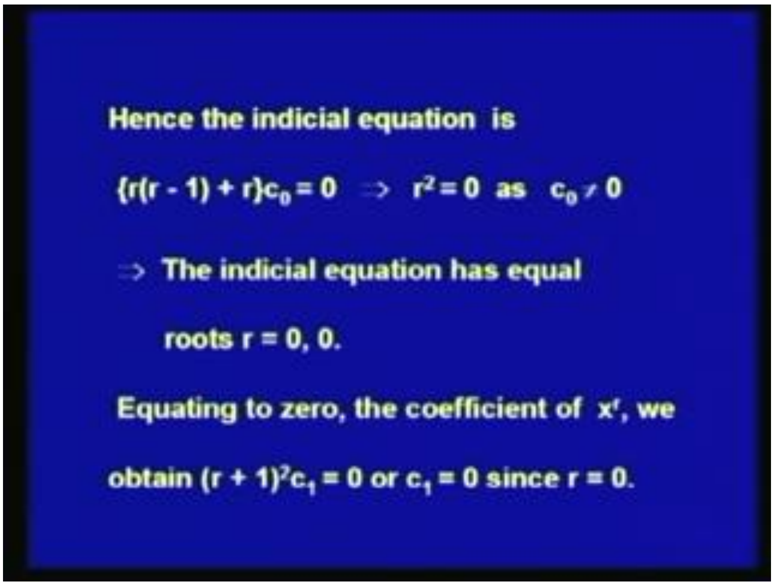

So, the indicial equation therefore is given by the coefficients of x to the power r minus 1 present in the first and second terms, it is given by r into r minus 1 plus r into c naught equal to 0, which implies r square equal to 0 as for our assumption c naught is not equal to 0. And thus we can see that, the indicial equation has two equal roots 0, 0 now next we equate to 0 the coefficient of next higher power of x, that is we equate to 0 the coefficient of x to the power r we this will give us r plus 1 whole square into c 1 equal to 0 since r is equal to 0 therefore, c 1 will be 0.

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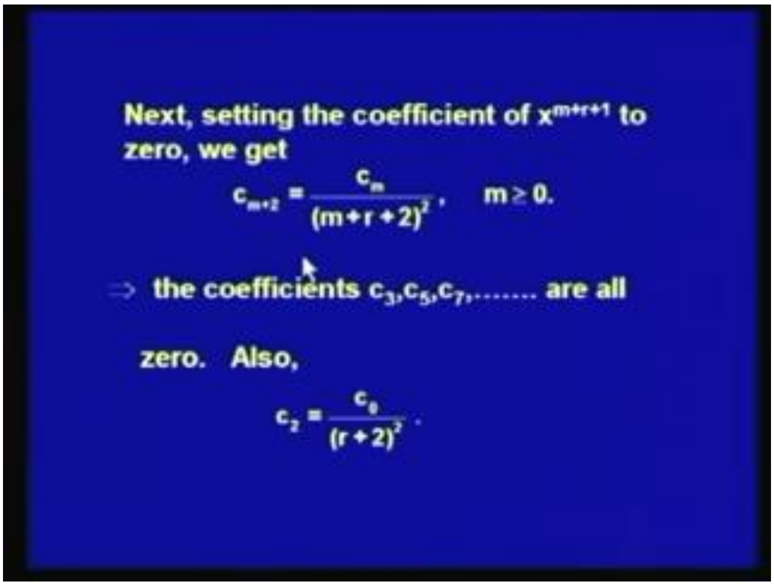

Let us, now substitute set the coefficient of x to the power m plus r plus 1 to 0, this will give us the recurrence relation c m plus 2 equal to c m over m plus r plus 2 whole square, for m equal to 0, 1, 2, 3 and so on. And we have already seen that c m is equal to 0, so and this recurrence relation connects c m plus 2 with c m, so if c 1 is 0 c 3, c 5, c 7 all will be 0's. Now, from this recurrence relation we can see that, if we put m equal to 0 we get the value of c 2 as c naught over r plus 2 whole square.

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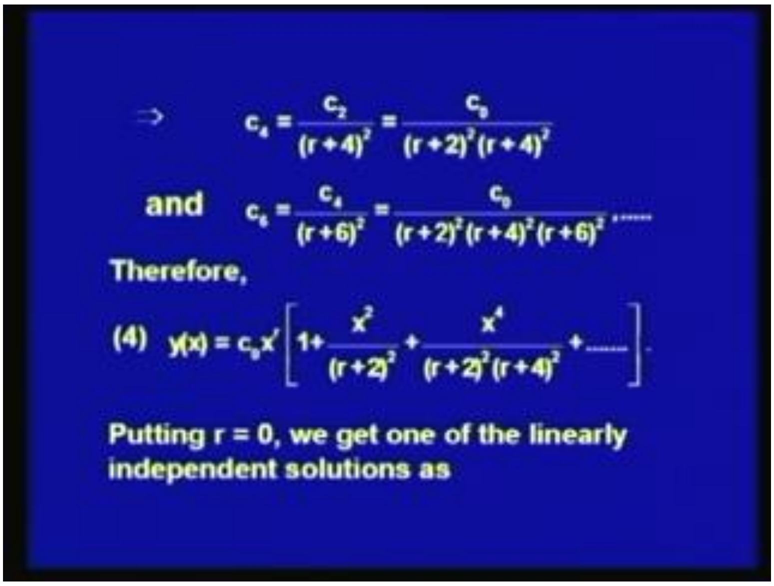

Now, we get the value of c 4 from the recurrence relation c 4 will be equal to c 2 over r plus 4 whole square. And when we put the value of c 2 here, in terms of c naught we get the value of c 4 as c naught over r plus 2 whole square into r plus 4 whole square c 6 similarly will come out to be c naught over r plus 2 whole square into r plus 4 whole square into r plus 6 whole square.

And that the solution of the differential equation can be writ10 as y x equal to c naught x to the power r 1 plus x square over r plus 2 whole square plus x to the power 4 over r plus 2 whole square r plus 4 whole square and so on this satisfies all the recurrence relations.

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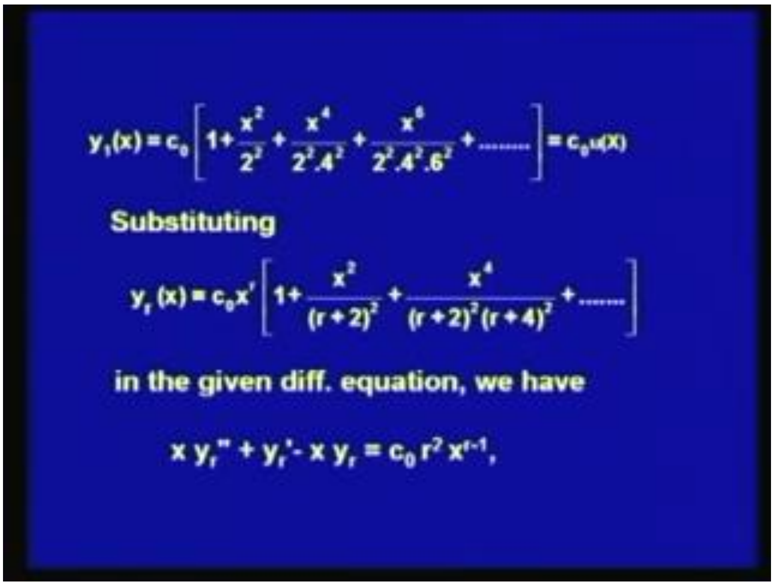

Now, in this if you put r equal to 0 you will get 1 of the linearly independent solutions of the given differential equation as y 1 x equal to c naught into 1 plus x square by 2 square plus x to the power 4 over 2 square 4 square plus x to the power 6 over 2 square 4 square 6 square and so on which we call as c naught into u x. Now, substituting y r x equal to c naught x to the power r into 1 plus x square over r plus 2 whole square plus x to the power of 4 over r r plus 2 whole square into r plus 4 whole square and so on.

In the given differential equation we will have x into y r double dash plus y r dash minus x y r equal to c naught into r square into x to the power r minus 1. Because, this expression for y r x satisfies all the recurrence relations except the indicial equation, so the right hand side of the equation x y r double dash plus y r dash minus x y r is c naught into r square into x to the power r minus 1.

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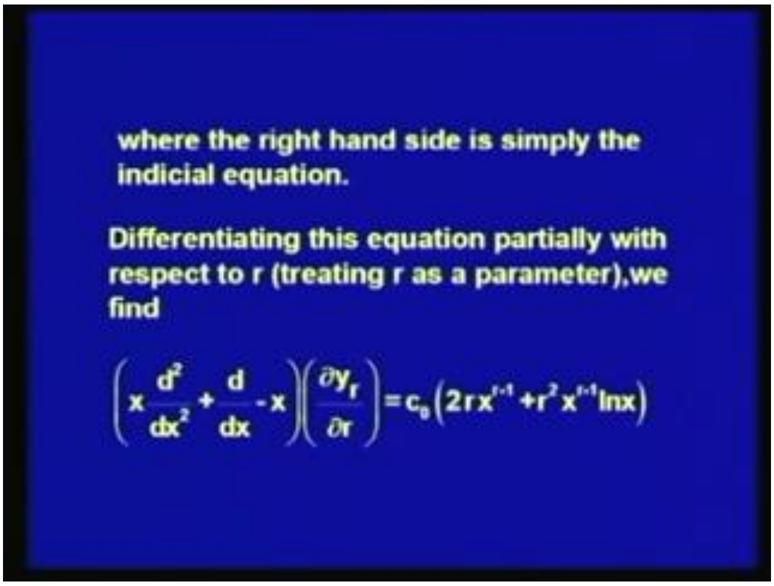

Now, differentiating this equation partially with respect to r, let us treat r as a parameter we shall have x into d square over d x square plus d over d x minus x operating on delta y r over delta r equal to c naught into 2 r x to the power r minus 1 plus r square into x to the power r minus 1 into ln x.

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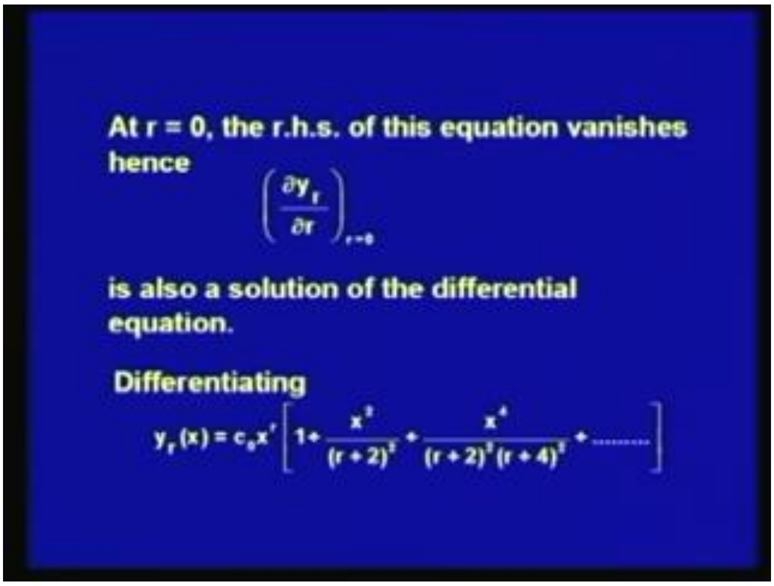

At r equal to 0 we can see here that, the right hand side of this equation vanishes and hence delta y r over delta r at r equal to 0 is also a solution of the differential equation. So, let us differentiate y r x with respect to r partially and put r equal to 0 to get the second independent solution of the given differential equation, so when we differentiate y r x equal to c naught x to the power r into 1 plus x square over r plus 2 whole square plus x to the power 4 over r plus 2 whole square into r plus 4 whole square and so on.

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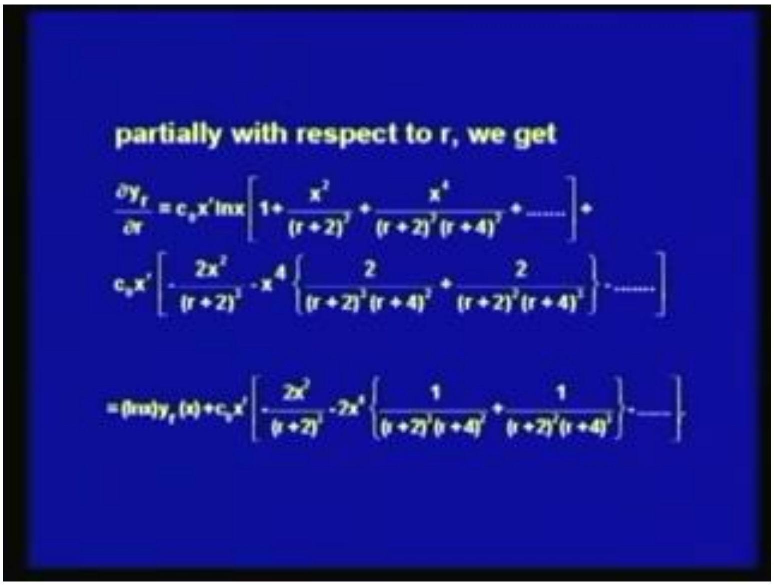

Partially with respect to r we will get delta y r over delta r equal to c naught into x to the power r into ln x into 1 plus x square over r plus 2 whole square plus x to the power 4 over r plus 2 whole square into r plus 4 whole square and so on and plus c naught x to the power r multiplied by minus 2 x square over r plus 2 whole cube minus x to the power 4 into 2 over r plus 2 whole cube into r plus 4 whole square plus 2 over r plus 2 whole cube into r plus 2 over r plus 2 whole square into r plus 4 whole cube and so on.

Which is equal to ln x into y r x plus c naught x to the power r into minus 2 x square over r plus 2 whole cube minus 2 times x to the power 4 over 1 into 1 over r plus 2 whole cube into r plus 4 whole square plus 1 over r plus 2 square into r plus 4 whole cube and so on.

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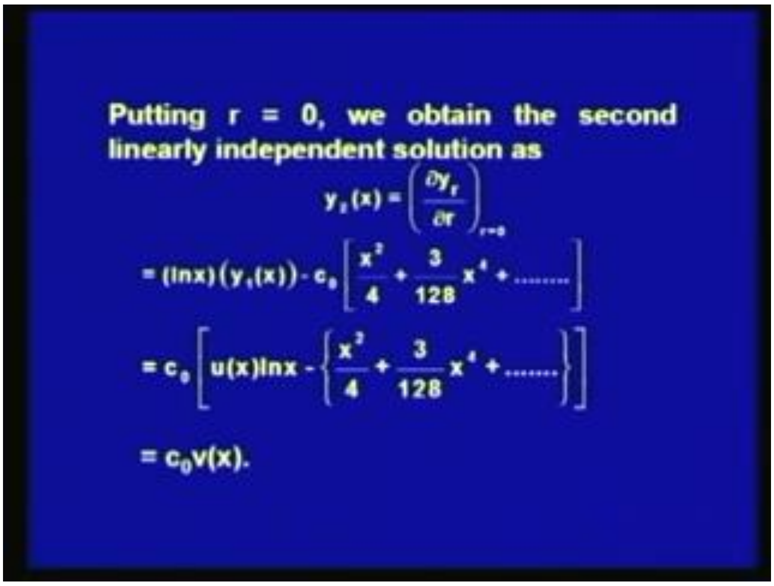

Let us put r equal to 0 in this solution we will get the second linearly independent solution of the given differential equation, which we denote by y 2 x. So, y 2 x is equal to delta y r over delta r let r equal to 0 which will give us ln x into y 1 x minus c naught times x square over 4 plus 3 over 128 into x to the power 4 and so on y 1 x is equal to c naught into u x. So, we get c naught times u x into ln x minus x square by 4 plus 3 over 128 into x to the power 4 and so on which let us call as c naught into v x.

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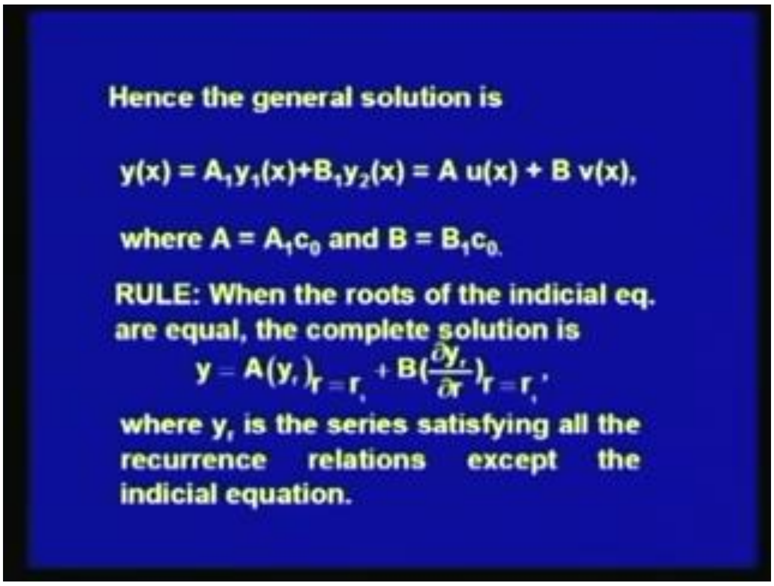

Hence, the general solution of the given differential equation is y x equal to A 1 into y 1 x plus B 1 into y 2 x which is equal to A times u x plus B times v x, where we write A equal to A 1 c naught and B is equal to B 1 c naught. So, thus we have the following rule in the case of equal roots of the indicial equation, when the roots of the indicial equation are equal the complete solution of the differential equation is obtained from y equal to A times y r evaluated at r equal to r 1 plus B times delta y r over delta r evaluated at r equal to r 1, where y r is the series satisfying all the recurrence relations except the indicial equation.

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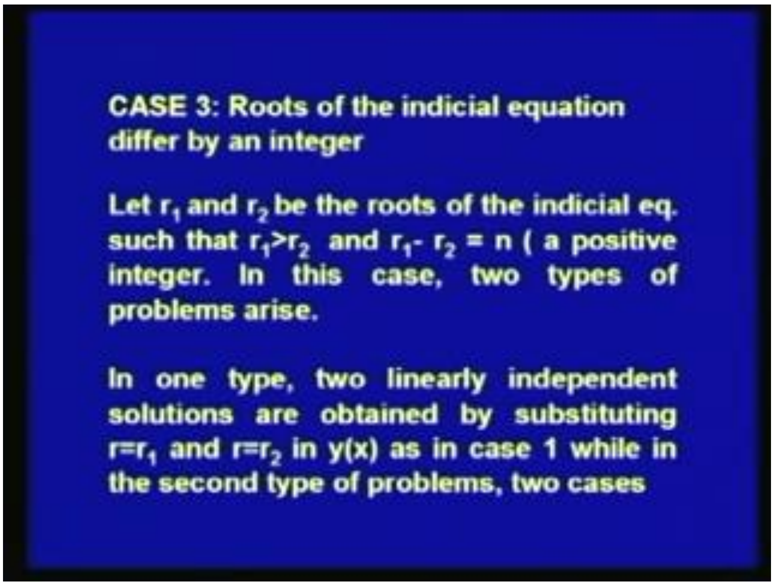

Now, let us study the case 3 where the roots of the indicial equation differ by an integer, so let us say that r 1 and r 2 be the two roots of the indicial equation such that r 1 is greater than r 2 and r 1 and r 2 differ by an integer that is r 1 minus r 2 is equal to n, where n is a positive integer. Now, in this case two types of problems occur in the first type two linearly independent solutions can be obtained by substituting r equal to r 1 and r equal to r 2 in y x as in the case 1 while in the second type of problems two cases arise.

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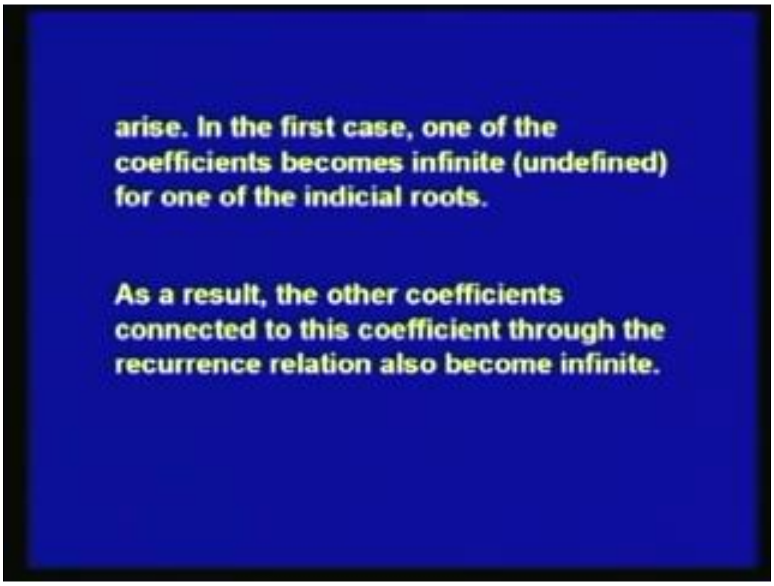

In the first case, one of the coefficients becomes infinite for one of the indicial roots, as a result the other coefficients which are connected to this coefficient through the recurrence relation also become infinite.

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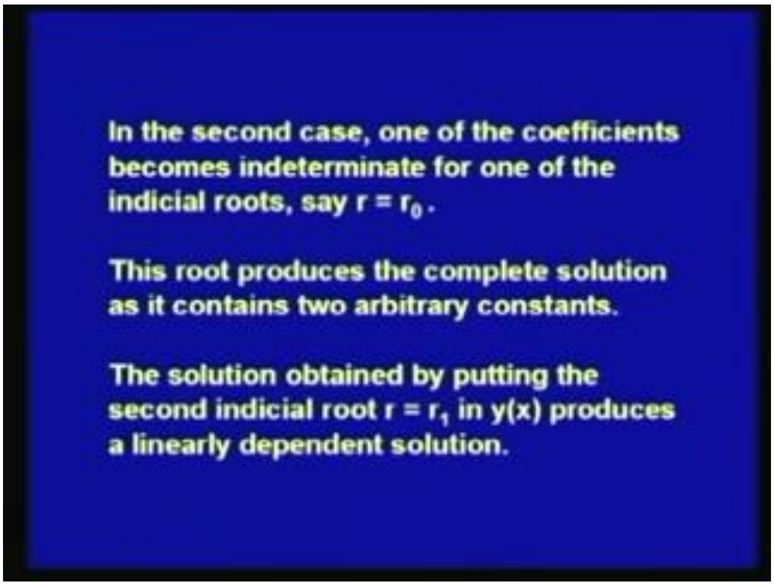

In the second case one of the coefficients becomes indeterminate for one of the indicial roots, say r equal to r naught. Now, this root produces the complete solution because, it contains two arbitrary constants, the solution obtained by putting the second indicial root r equal to r 1 in y x produces a linearly dependent solution, let us now discuss these cases.

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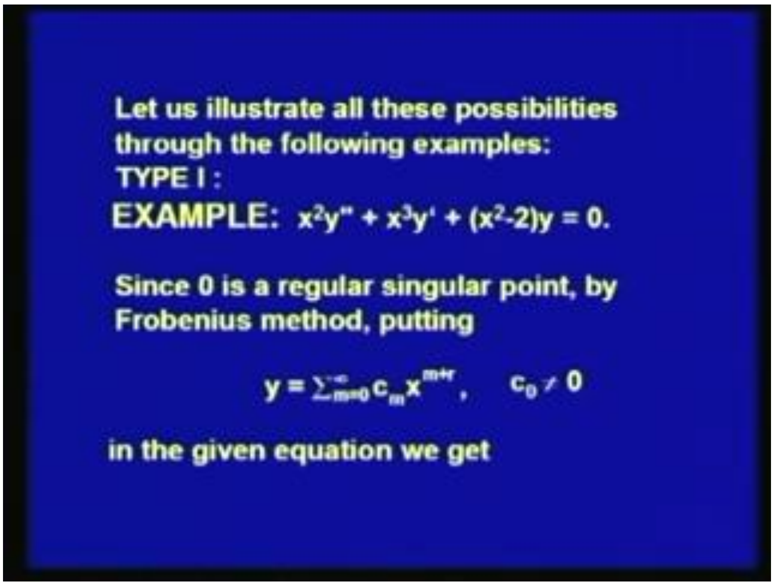

Let us illustrate all these possibilities through the following examples, so let us take the an example of type 1. Let us consider the differential equation x square into y double dash plus x 3 into y dash plus x square minus 2 into y equal to 0, we can see here that again 0 is a regular singular point of this differential equation. So, we can apply frobenius method, so let us put y equal to sigma m equal to 0 to infinity c m x to the power m plus r where c naught is not equal to 0 in the given differential equation.

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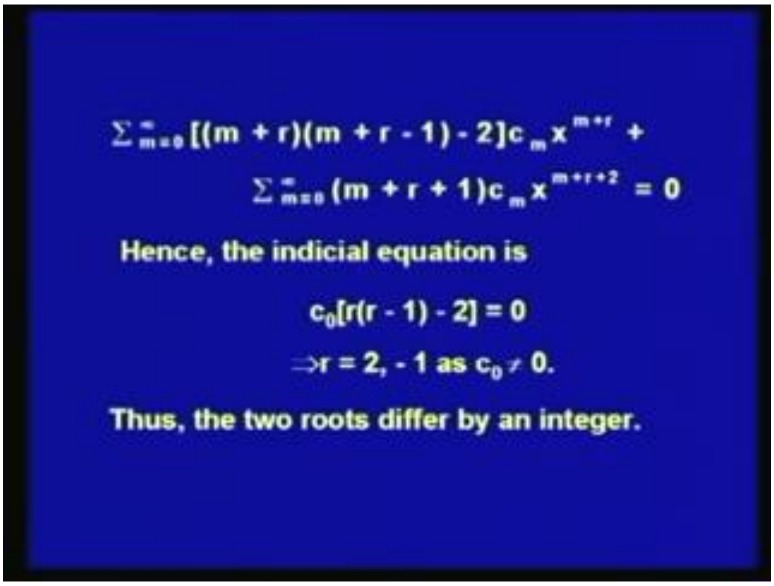

We will get sigma m equal to 0 to infinity m plus r into m plus r minus 1 minus 2 into c m into x to the power m plus r plus sigma m equal to 0 to infinity m plus r plus 1 into c m into x to the power m plus r plus 2 equal to 0. Again we will equate the coefficient of the least power of x to 0, which we will get from the first term, when we put m equal to 0 in the first term you get the power of x as r in the second term then we put m equal to 0 you get the power of x as r plus 2, so the coefficient of the lowest power of x that is the coefficient of x to the power r occurs in the first term.

And that when we equate to 0 we get the indicial equation as c naught into r into r minus 1 minus 2 equal to 0. Since c naught is not equal to 0 the 2 values of r are 2 and minus 1 you can see that both these values of r differ by an integer r 1 is 2 here r 2 is minus 1, so r 1 is greater than r 2 and r 1 minus r 2 is equal to 3 which is a positive integer, so the roots differ by an integer.

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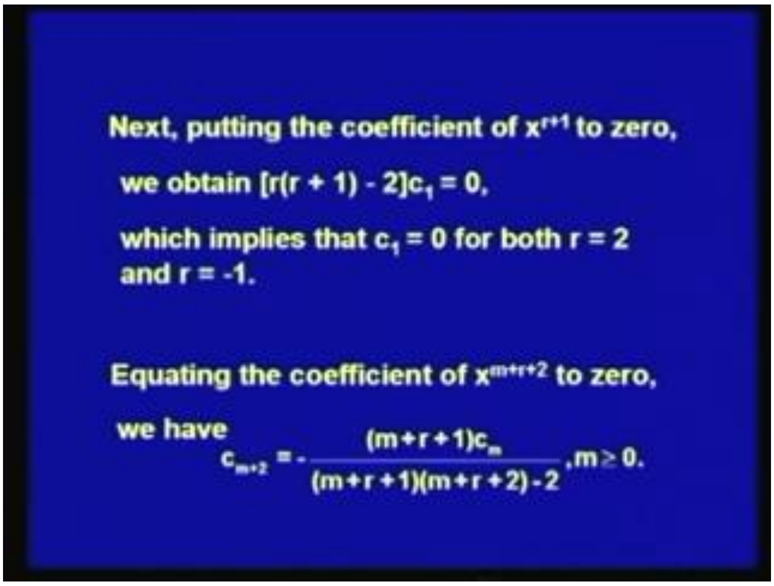

Now, put the coefficient of next higher power of x to 0 that is the coefficient of x to the power r plus 1 when equate it to 0 gives us r into r plus 1 minus 2 into c 1 equal to 0. And if you put r equal to 2 or if you put r equal to minus 1 you can see here from this equation that for both these values of r c 1 turns out to be 0. Next, in other two find the recurrence relation between the coefficients of the series solution, we shall put the coefficient of x to the power m plus r plus 2 to 0, this will give us c m plus 2 equal to minus of m plus r plus 1 into c m over m plus r plus 1 into m plus r plus 2 minus 2 for m equal to 0, 1, 2, 3 and so on or you can say m is greater than or equal to 0. Now, we have seen here that for both values of r that is for r equal to 2 and r equal to minus 1 c 1 is equal to 0 and c m plus 2 is related to c m through this recurrence relation, so c 1 equal to 0 implies that c 3, c 5, c 7 all are 0's.

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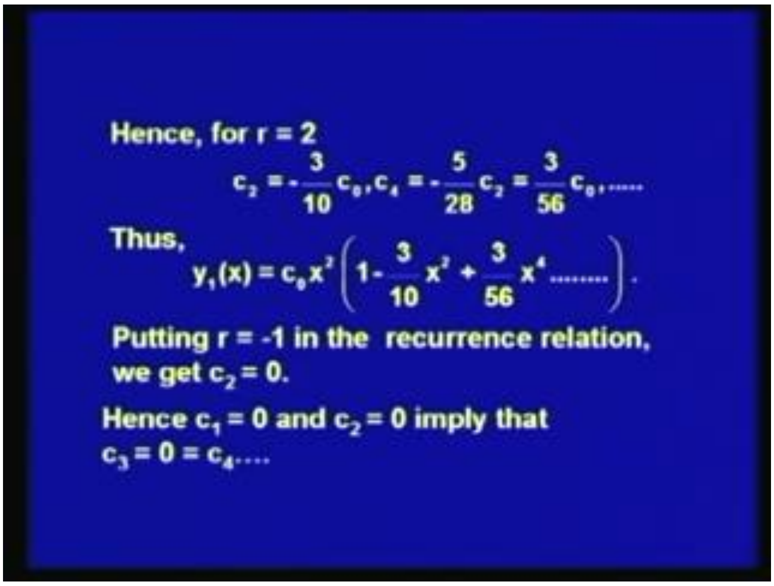

Hence, for r equal to 2 we get the value of c 2 equal to minus 3 over 10 into c naught c 4 is equal to minus 5 over 28 into c 2 which will give us 3 over 56 into c naught when we put the value of c 2 in this. And thus we get the solution when solution of the differential equation y 1 x as c naught x square into 1 minus 3 over 10 into x square plus 3 over 56 into x to the power 4 and so on.

Now, next let us put the other value of r that is r equal to minus 1 in the recurrence relation, we can see that from the recurrence relation c 2 becomes equal to 0 for this value of r. Thus, now we have c 1 equal to 0 and c 2 equal to 0 because, c 1 equal to 0 we got for both values of r, r equal to 2 as well as r equal to minus 1 and c 2 comes out to be 0 for r equal to minus 1. So, in the case of r equal to minus 1 both c 1 and c 2 are 0 and thus c 3, c 4, c 5, c 6 all are 0's. And so we get the second solution of the differential equation y 2 x as c naught over x for r equal to minus 1.

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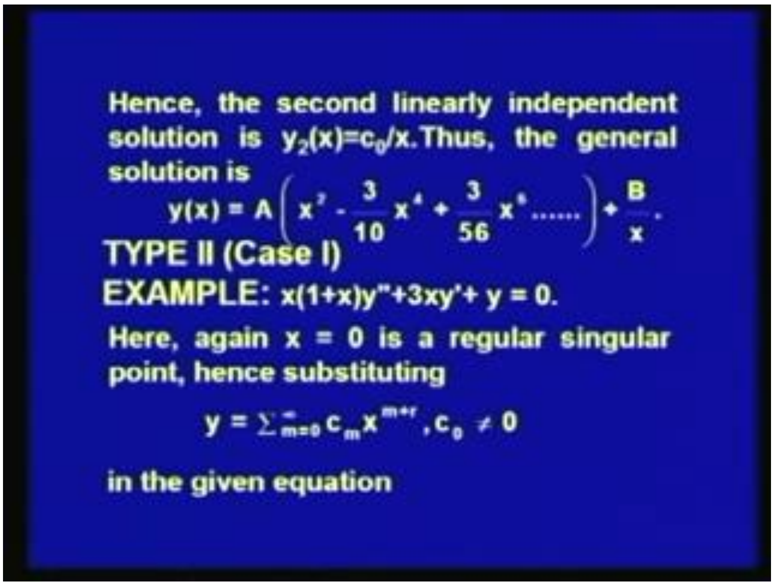

And thus we can write the general solution of the differential equation as y x equal to a times x square minus 3 over 10 x to the power 4 plus 3 over 56 x to the power 6 and so on plus b over x, you can see that the both the solutions y 1 x and y 2 x are clearly linearly independent. Now, let us look at the type 2 I mean of the second type of differential equations, we will first consider the case 1, where del of the coefficients of the differential equations becomes infinite at indicial root.

So, how we will take an such problems, so let us take an example of the differential equation x into 1 plus x y double dash plus 3 x y dash plus y equal to 0, we can see here that again x equal to 0 is a regular singular point, hence we can apply the frobenius method. So, let us substitute y equal to sigma m equal to 0 to infinity c m x to the power m plus r where c naught is not equal to 0 in the given differential equation.

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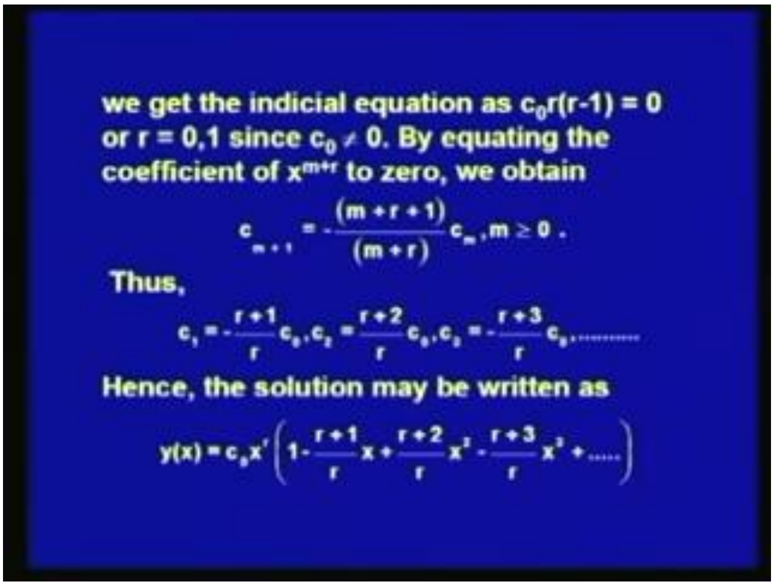

We get the indicial equation as c naught into r into r minus 1 equal to 0, which gives us r equal to 0 and 1 since, c naught is not equal to 0 by equating to 0 the coefficient of the least power of x. And by equating to 0 the coefficient of x to the power m plus r we obtain the recurrence relation, the recurrence relation is c m plus 1 is equal to minus into m plus r plus 1 over m plus r into c m where m is greater than or equal to 0.

And thus the values of c 1, c 2, c 3 can be determined in terms of c naught from this recurrence relation c 1 comes out to be minus r plus 1 over r into c naught c 2 r plus 2 over r into c naught c 3 is minus r plus 3 over r into c naught and when we put the values of those coefficients c m's in the expression for y x, we will have the solution as y x equal to c naught x to the power r into 1 minus r plus 1 over r into x plus r plus 2 over r into x square minus r plus 3 over r into x cube and so on.

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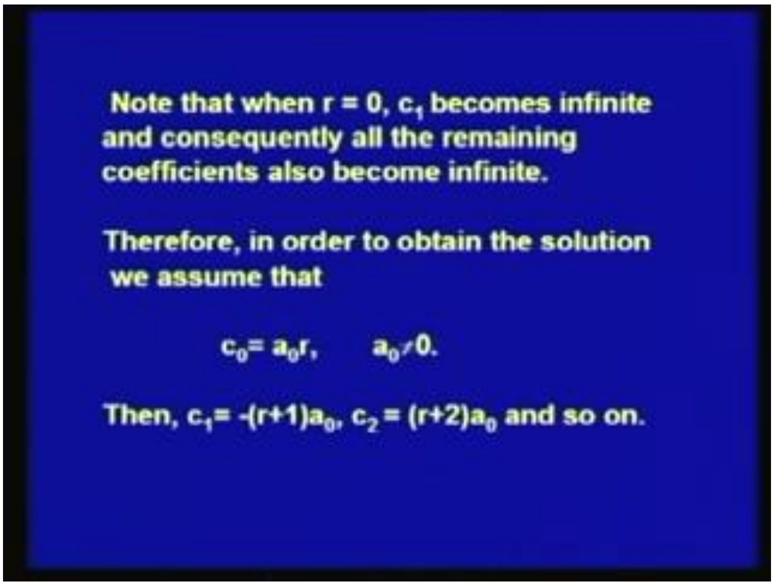

Now, let us note that when r is equal to 0, c 1 becomes infinite here, it becomes infinity and consequently all the remaining coefficients also become infinite because, they are related to c 1 and hence to c naught. So, therefore, in order to obtain the solution we assume that c naught is equal to a naught into r in order to overcome the similarity at r equal to 0, since c naught is an arbitrary constant we can choose it in any manner, so we choose c naught as a naught into r, where a naught is not equal to 0. Then the values of c 1, c 2 will be c 1 is equal to minus r plus 1 into a naught c 2 will be r plus 2 into a naught and so on.

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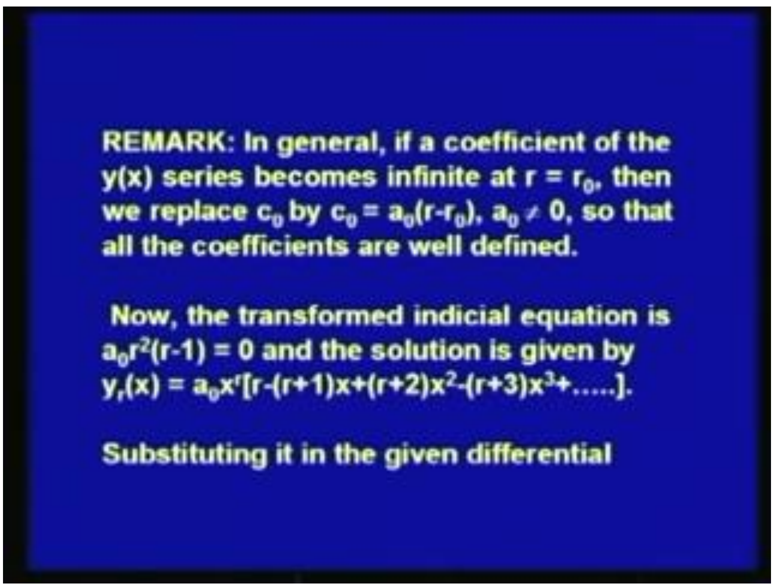

In general we will make a remark here in general if a coefficient of the y x series becomes infinite at a certain value of r. So, r is equal to r naught, then in order to overcome the similarity at r naught, we replace c naught by a naught into r minus r naught, where a naught is not equal to 0.

So, that all the coefficients are well defined and with this choice of c naught equal to a naught into r, it turns out that the transformed indicial equation becomes a naught into r square into r minus 1 equal to 0. And the solution transforms into y r x equal to a naught into x to the power r into r minus r plus 1 into x plus r plus 2 into x square minus r plus 3 into x cube and so on substituting this series for y r x in the given differential equation.

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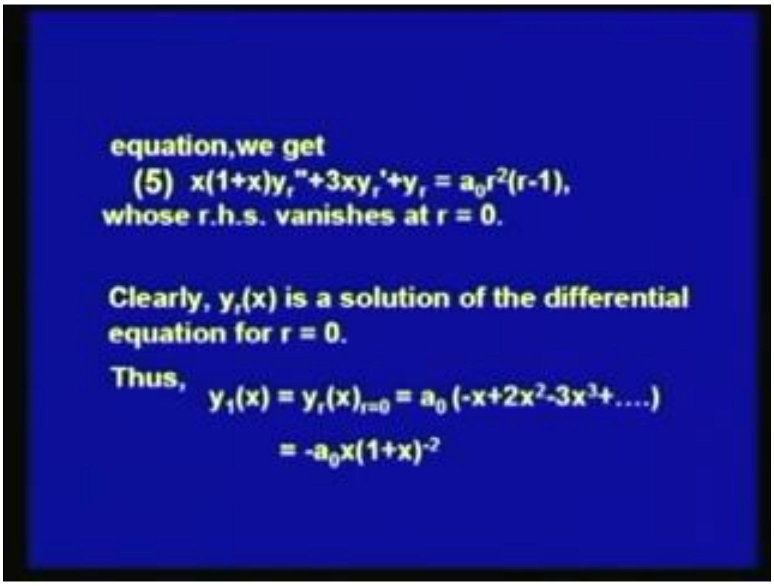

We get x into 1 plus x y r double dash plus 3 x y r dash plus y r equal to the indicial equation a naught into r square into r minus 1. Because, y r satisfies all the recurrence relations, except the indicial equation, an indicial equation which was c naught into r into r minus 1 equal to 0 changed into c a naught r square into r minus 1 after we had chosen c naught equal to a naught into r.

So, we can see here that the right hand side of this equation vanishes at r equal to 0 and thus y r x is a solution of the given differential equation for r equal to 0. And if we put r equal to 0 in the expression for y r x, then we get 1 solution of the differential equation as a naught into minus x plus 2 x square minus 3 x cube and so on which may be written as minus a naught into x into 1 plus x to the power minus 2 because, the infinite power series in the bracket is an expression of 1 plus x to the power minus 2.

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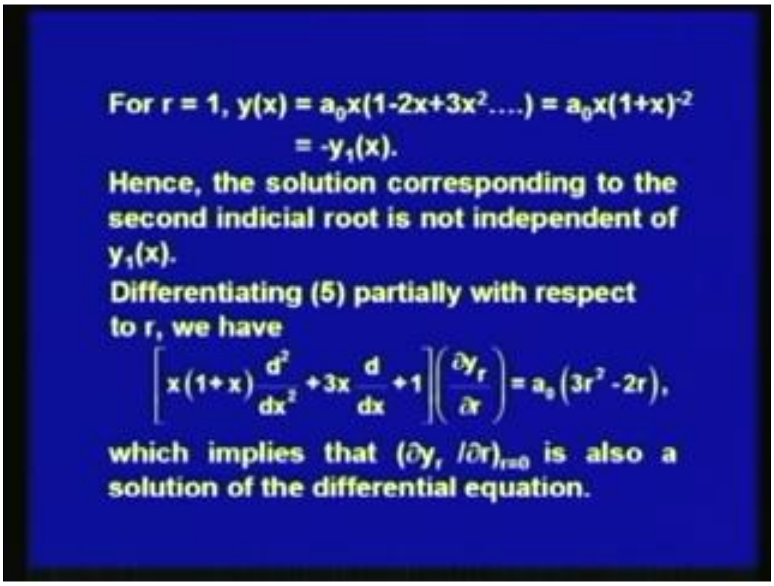

For r equal to 1, if we find the value of y x from y r x it turns out that y x is equal to a naught x into 1 minus 2 x plus 3 x square and so on, which is a naught x into 1 plus x to the power minus 2 and which is nothing but, minus y 1 x. So, these second solution if we find from y r x by putting r equal to 1 straight away, it gives us a linearly dependent solution it is nothing but, negative of the solution, first solution that is a negative of y 1 x.

Thus to find linearly independent solution, second linearly independent solution of the given differential equation will have to do something else. So, what we will do is the solution corresponding to the second indicial root is not independent of y 1 x by putting r equal to 1 in y r x straight away. So, what we will do, we will differentiate the equation 5 partially with respect to r.

And we shall then have x into 1 plus x into d square over d x square plus 3 x into d over d x plus 1 operated on delta y r over delta r equal to a naught into 3 r square minus 2 r, if we put r equal to 0 in the right hand side of this equation it vanishes. So, we can see that delta y r over delta r at r equal to 0 is also a solution of the given differential equation.

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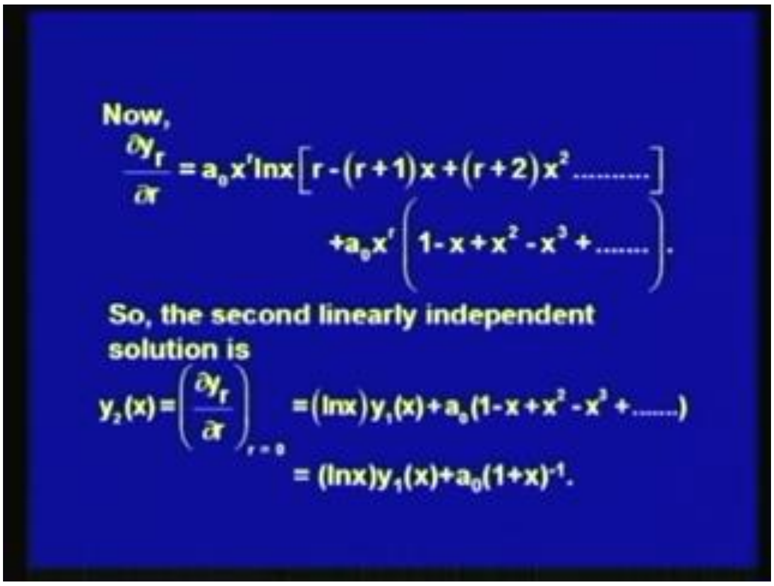

Now, let us differentiate partially the series for y r x that it will give us delta y r over delta r equal to a naught x to the power r into ln x multiplied by r minus r plus 1 into x plus r plus 2 into x square and so on plus a naught x to the power r into 1 minus x plus x square minus x cube plus and so on. And so when we put r equal to 0 in this expression for delta y over delta r it will lead us to second linearly independent solution of the given differential equation that is y 2 x, y 2 x is now equal to delta y r over delta r at r equal to 0 which is ln x into y 1 x plus a naught into 1 minus x plus x square minus x cube and so on, which is equal to ln x into y 1 x plus a naught times 1 plus x to the power minus 1 when plus x to the power minus 1. We know, we have been we all know that it is equal to 1 minus x plus x square minus x cube and so on through aided mod of x is less than 1.

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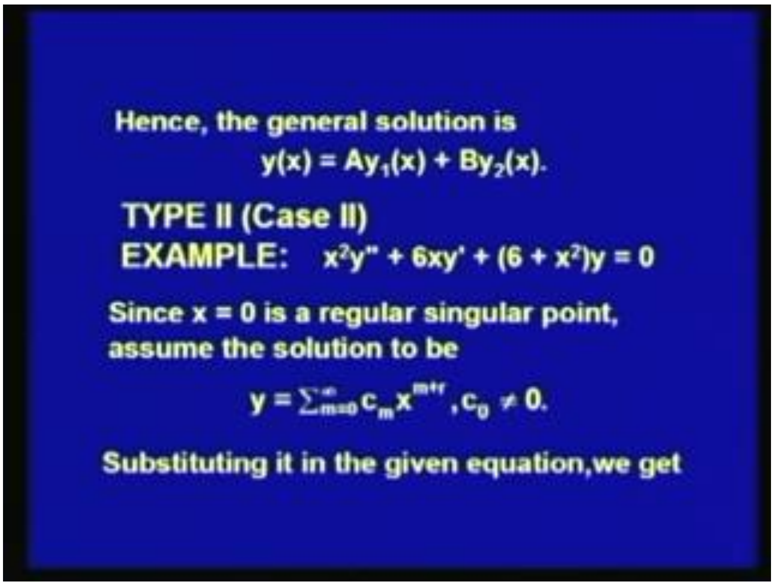

Hence, the general solution of the given differential equation we can write it is y x equal to A times y 1 x plus y 2 x. Now, let us study the second case of second type 2 in where we had said that the indicial roots when solution of the differential equation is such that one of the coefficients of the solution becomes indeterminate. So, how to deal with such type of differential equations, so let us consider the differential equation x square y double dash plus 6 x y dash plus 6 plus x square into y equal to 0.

Now, here again x equal to 0 is a regular singular point of the given differential equation, so we can apply the frobenius method. Let us, assume the solution of the differential equation to be y equal to sigma m equal to 0 to infinity c m x to the power m plus r, where c naught is not equal to 0.

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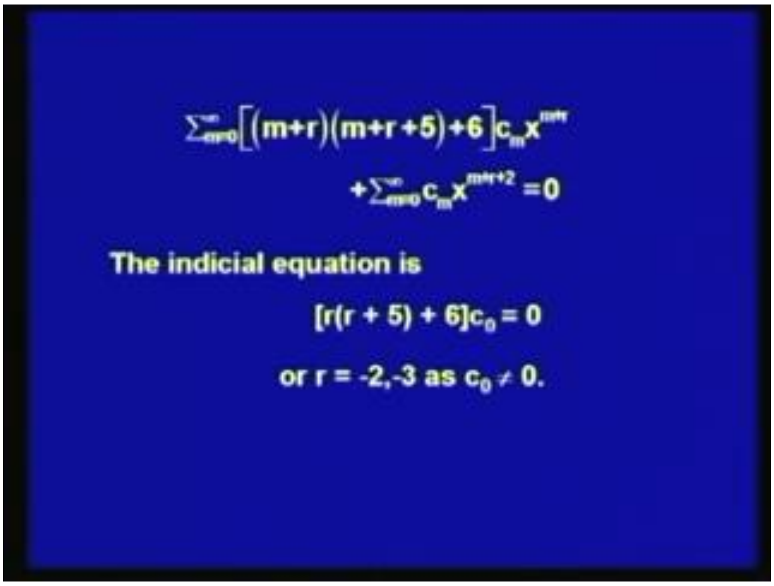

Let us substitute it in the given equation, we shall have sigma m equal to 0 to infinity m plus r into m plus r plus 5 plus 6 into c m x to the power m plus r plus sigma m equal to 0 to infinity c m into x to the power m plus r plus 2 equal to 0. Again, we will put the coefficient of the lowest power of x to 0 the lowest power of x that occurs in this equation is r. So, we put the coefficient of x to the power r to 0 we will get r into r plus 5 plus 6 into c naught equal to 0, which gives us the 2 values of r as minus 2 and minus 3, clearly minus 2 is greater than minus 3 and the difference of minus 2 and minus 3 is 1, which is an integer.

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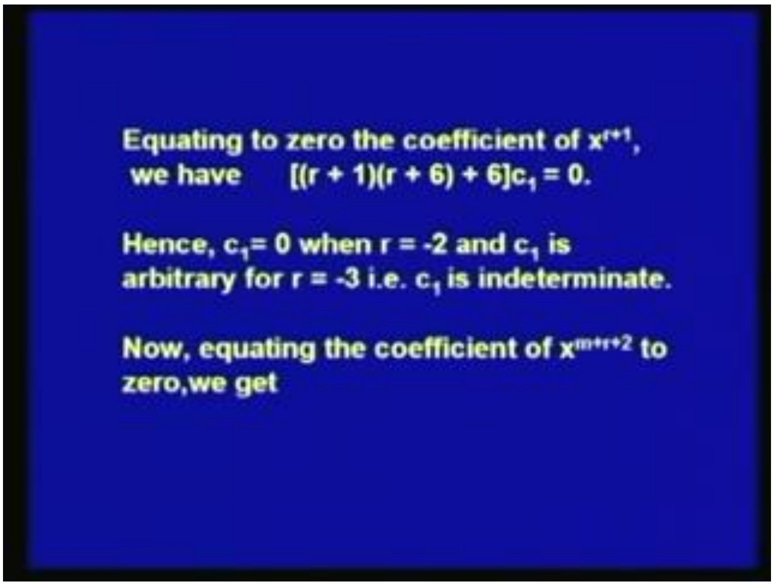

Now, let us equate to 0 the coefficient of the next higher power of x, that is we put the coefficient of x to the power r plus 1 to 0 we have r plus 1 into r plus 6 plus 6 into c 1 equal to 0. So, when we put r equal to minus 2 in this equation, we note that c 1 turns out to be 0, but if you put r equal to minus 3 in this what we get is 0 into c 1 equal to 0, so c 1 becomes indeterminate c 1 is arbitrary it can take any value. So, what we do in this case let us equate the coefficient of x to the power m plus r plus 2 to 0.

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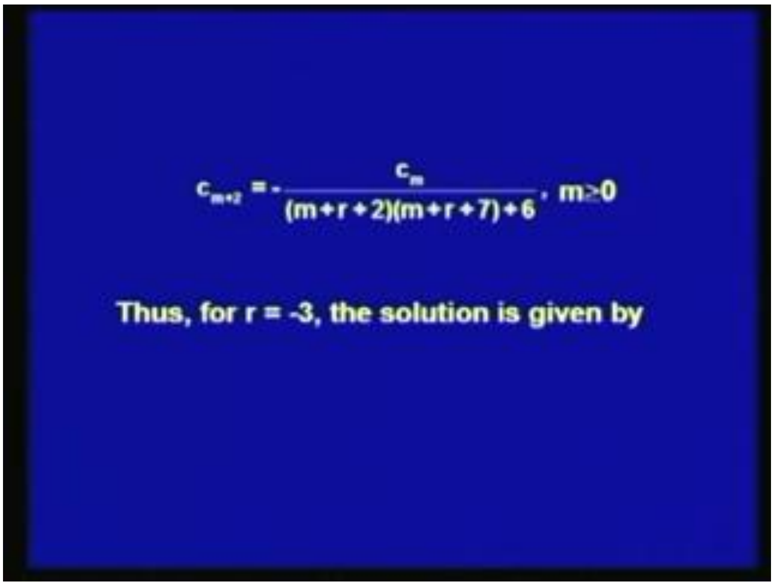

We get the recurrence relation c m plus 2 equal to minus c m over m plus r plus 2 into m plus r plus 7 plus 6 for m greater than or equal to 0 or we can say m equal to 0, 1, 2, 3 and so on.

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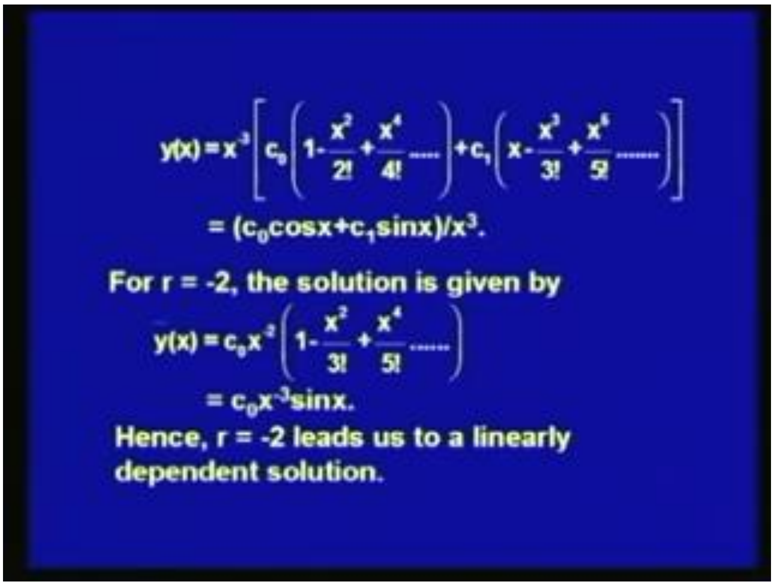

Now let us put r equal to minus 3, here the solution is given by y x equal to x to the power minus 3 into c naught times 1 minus x square by 2 factorial plus x to the power 4 over 4 factorial and so on plus c 1 times x minus x cube by 3 factorial plus x to the power 6 over 5 factorial and so on. See, when the this is because c 1 is arbitrary, so now, therefore, y x contains two arbitrary constants c naught and c 1.

Now, we know the series ((Refer Time: 46:30)) of sin x and cos x, we know that sin x is x minus x cube by 3 factorial plus x to the power 5 by 5 factorial and so on. And the ((Refer Time: 46:39)) series expansion for cos x is 1 minus x square by 2 factorial plus x to the power 4 by 4 factorial and so on. So, when the cubes of that will have the value of y x as c naught into cos x plus c 1 into sin x over x cube.

Now, when we calculate the solution for r equal to minus 2 it turns out that y x is equal to c naught into x to the power minus 2 into 1 minus x square by 3 factorial plus x to the power 4 over 5 factorial and so on, which is also equal to c naught into x to the power minus 3 into sin x and this clearly shows that r equal to minus 2 leads us to a linearly dependent solution, this is not independent of the solution for r equal to minus 3. And thus the indicial root r equal to minus 3 gives us the complete solution of the given differential equation.

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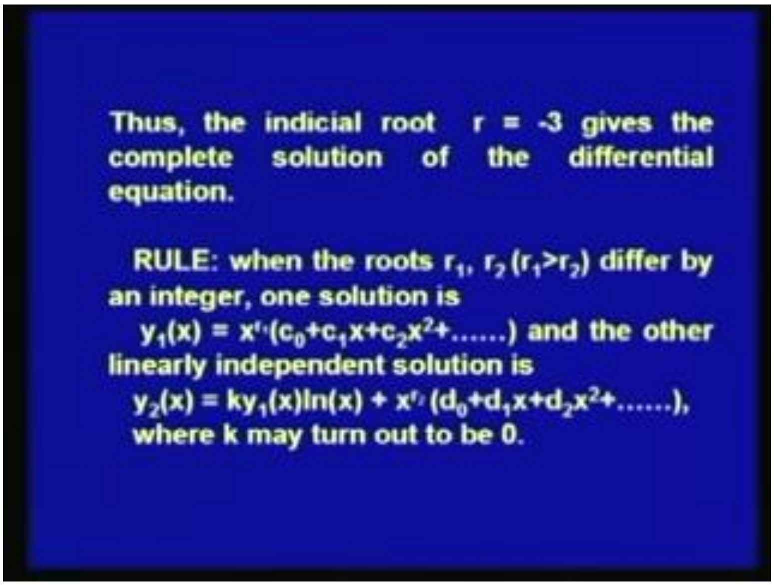

So, thus we have the following rule for dealing with such type of differential equations that is the differential equations, which come under the type 2, when the roots r 1 and r 2 of the differential equation of the indicial equation differ by an integer. Then one solution of the differential equation is given by y 1 x equal to x to the power r 1 into c naught plus c 1 x plus c 2 x square and so on. And the other linearly independent solution is given by y 2 x equal to k times y 1 x into ln x plus x to the power r 2 into d naught plus d 1 x plus d 2 x square and so on.

Now, when we dealt with there were two cases in type 2, in first case we had seen that the solutions for r 1 and r 2 can be obtained directly. And then in the second case we had seen that in type 1 we had seen that r 1 the solution for r 1, r 2 can be obtained directly. So, in that case k was equal to 0 while in the type 2 we had dealt with 2 cases, in the first case we had seen that k is not equal to 0 because, we had to differentiate partially the series for y r x with respect to r.

So, that second solution contained the logarithmic term in suppose k was not equal to 0, but in type 2 it turned out that k is equal to 0 because, the second solution was not, but linearly dependent over to the first solution. And the first solution itself had two arbitrary constants c naught and c 1, so it had given as the general solution. And thus for dealing with the differential equations, where the roots of the indicial equation differ by an integer one solution will be x to the power r 1 into c naught plus c 1 x plus c 2 x square and so on.

While the other solution could be k times y 1 x into ln x plus x to the power r 2 into d naught plus d 1 x plus d 2 x square and so on where k may turn out to be 0. We will discuss the solution of Bessel's equation in our next talk, where we will use the frobenius method to find the series solution of the Bessel's equation. And we will discuss the properties of the Bessel's equation, we will also discuss orthogonality of Bessel functions and then the generating function of the Bessel's function and so on, so all those things will be discuss in our next lecture.

Thank you.