Mathematics III
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Lecture - 4
Series Solution of Homogeneous Linear Differential Equations

Dear viewers, the title of my turn today is Series Solution of Homogeneous Linear Differential Equations. We know that a homogeneous linear differential equation can be solved by elementary methods and it is solutions are known functions from calculus like e to the power x, sin x, cos x, etcetera. However if such an equation has variable coefficients, then it cannot easily we solved by elementary methods.

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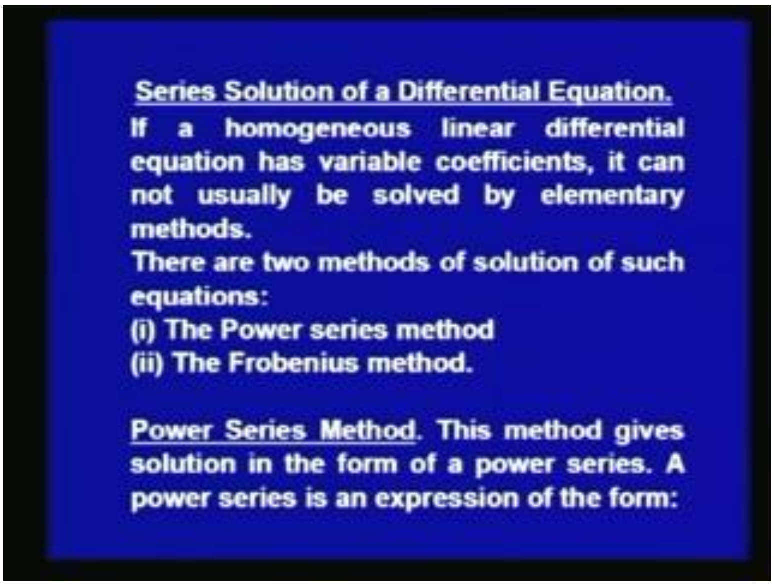

The Legendre's equation, hyper geometric equation and Bessel's equation are very important types of such homogeneous linear differential equations. Since, their quite useful in applied mathematics, we shall discuss the solution of Legendre's equation and Bessel's equation. First we take up the case of Legendre's equation there are two methods of finding solutions of such equations. The first is the power series method and the second one is the extension of the power series method that is the Frobenius method

Let us, first discuss the power series method this method gives the solution of the differential equation in the form of a power series. A power series can be used for computing the values of the solution for exploring is properties and for driving other kinds of differentiation of the solution, a power series is an expression of the form sigma m equal to 0 to infinity.

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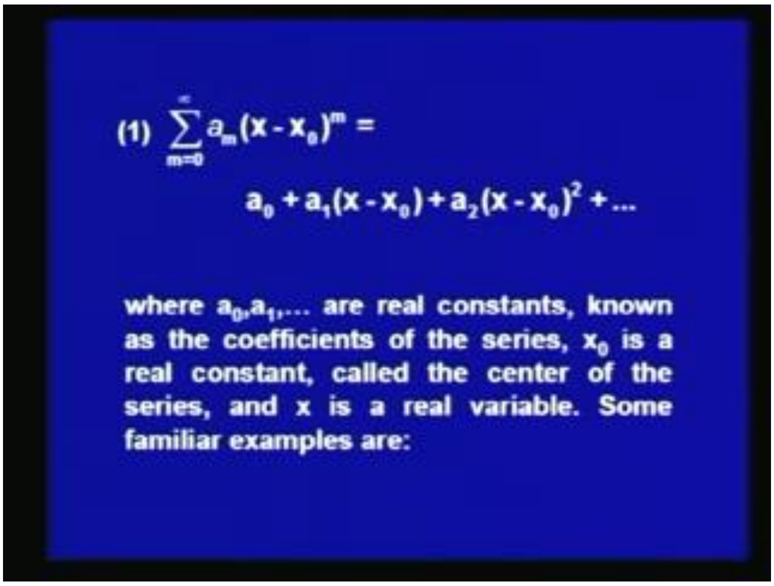

A to m for x minus x naught to the power m, which may be expanded as a naught plus a 1 x minus x naught plus a 2 x minus x naught whole square and so on. Where a naught a 1 and so on are real constants known as the coefficients of the series x naught is a real constant and called the center of the series x is a real variable.

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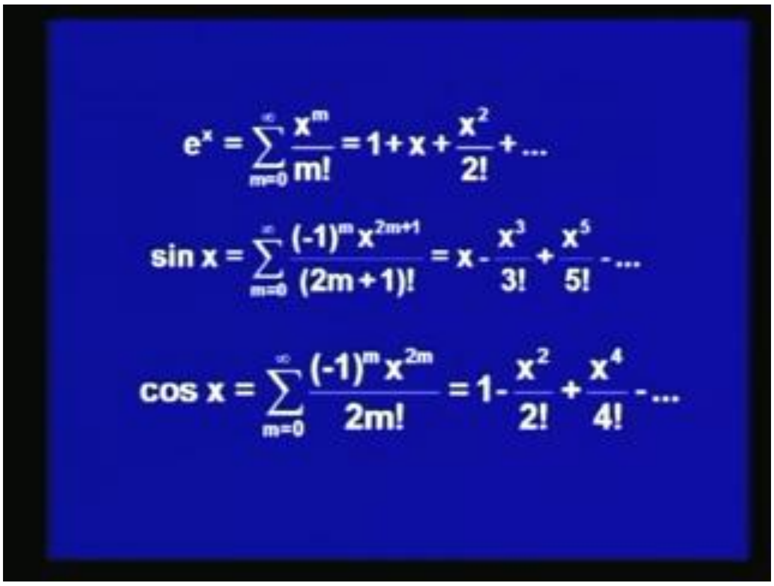

Some familiar examples are e to the power x, which is sigma m equal to 0 to infinity x to the power m over x m factorial, we may write it as 1 plus x plus x square by 2 factorial and so on. Another example is sin x, which is m equal to 0 to infinity minus 1 to the power m x to the power 2 m plus 1 over 2 m plus 1 factorial, which when expanded gives us the series x minus x cube by 3 factorial plus x to the power 5 by 5 factorial and so on.

And then we can take the cos x function which is sigma m equal to 0 to infinity minus 1 to the power m, x to the power 2 m over 2 m factorial or we may write it as 1 minus x square by 2 factorial plus x to the power 4 by 4 factorial and so on. The nth partial sum of the power series given by equation 1 is defined as s n x equal to a naught plus a 1 x minus x naught to the power 1 plus and so on n x minus x naught to the power n.

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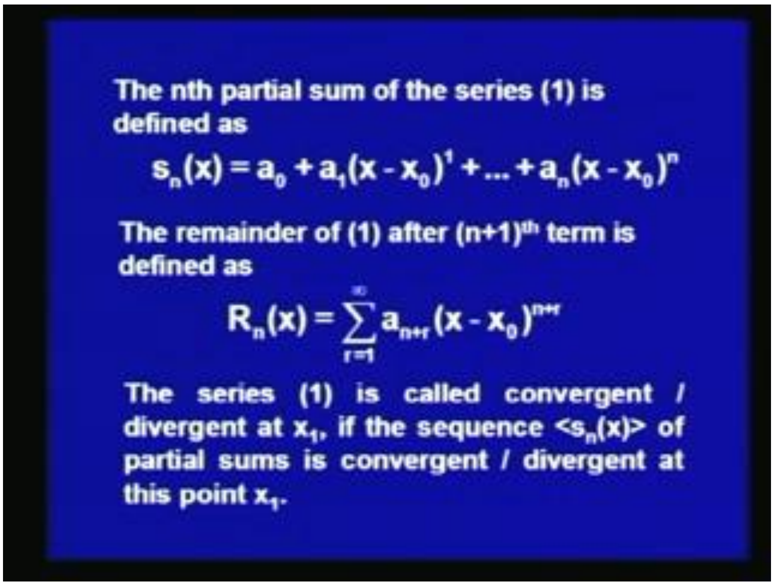

The remainder of the power series given by equation 1 after n plus 1th term is defined as R n X, which is equal to sigma R equal to 1 to infinity a n plus R x minus x naught to the power n plus R. That is the first term will be n plus 1 x minus x naught to the power n plus 1 then we will have n plus 2 x minus x naught to the power n plus 2 and so on. These power series given by equation 1 is said to be convergent or divergent at the point x equal to x 1. If the sequence of partial sums S n X is convergent or divergent at the point x equal to x 1, if the sequence of partial sums S n X is convergent or divergent at the point x equal to x 1.

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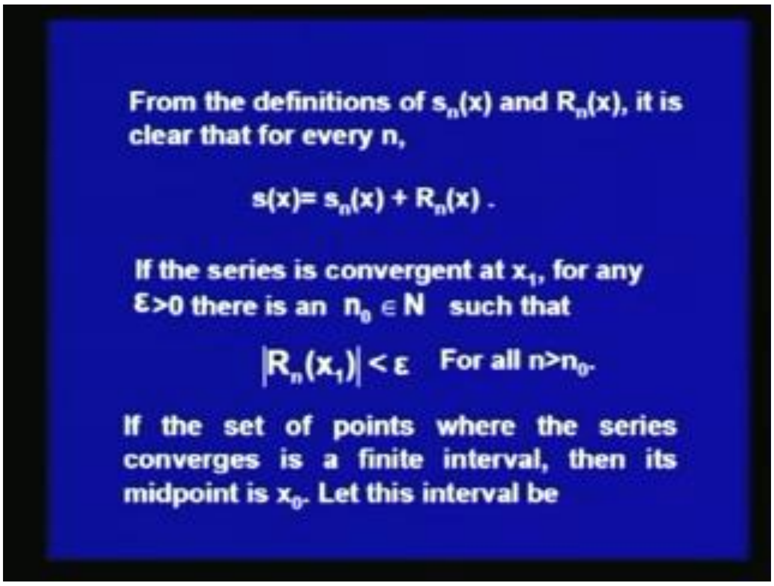

From the definitions of S n X and R n X, it is clear that for every n, s x is equal to S n X plus R n X. If the series is convergent at x 1, then you can say that for any Epsilon greater than 0 there exist an integer and not belonging to n such that modulus of R n X 1 is less than Epsilon for all n greater than n naught. If the set of points, where the series converges is a finite interval, then it is mid point is x naught.

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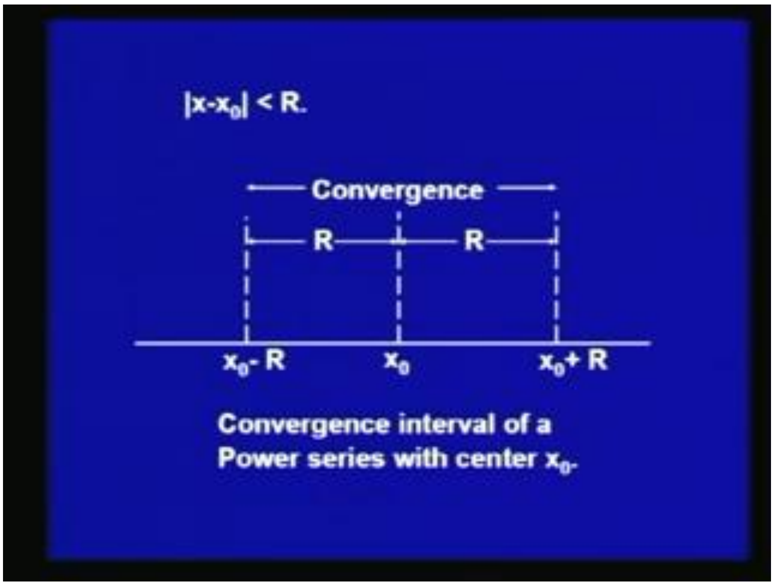

Let this interval be mod of x minus x naught less than R, in this picture we have shown this convergence interval of a power series with center at the x naught. The interval is x naught minus R to x naught plus R, where the series converges in midpoint of the interval is at x not.

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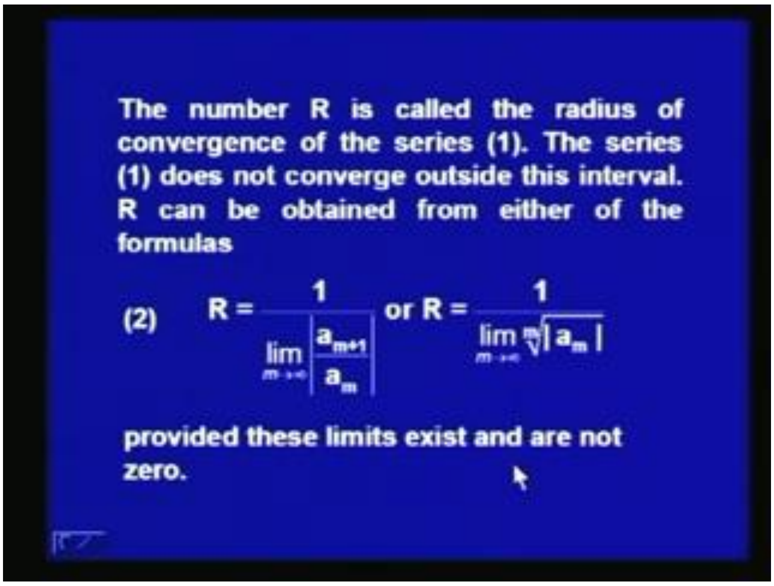

The number R is called the radius of convergence of the power series given by equation 1. The power series does not converge outside this interval; it may or may not converge for mod of x minus x naught equal to R. The value of R can be obtained from one of these two formulas, that is R equal to 1 over limit m tends to infinity, mod of m plus 1 over m or R equal to 1 over limit m tends to infinity mth root of mod of m provided the limits exist and are not 0. If these limits are 0 then the series converges only at the center, sometimes the series convergence for all values of x that is the value of r comes out to be infinity. So, in such cases the series will converge for all values of x, some times the value of R is finite, then the series will converge in the interval mod of x minus x naught less than R.

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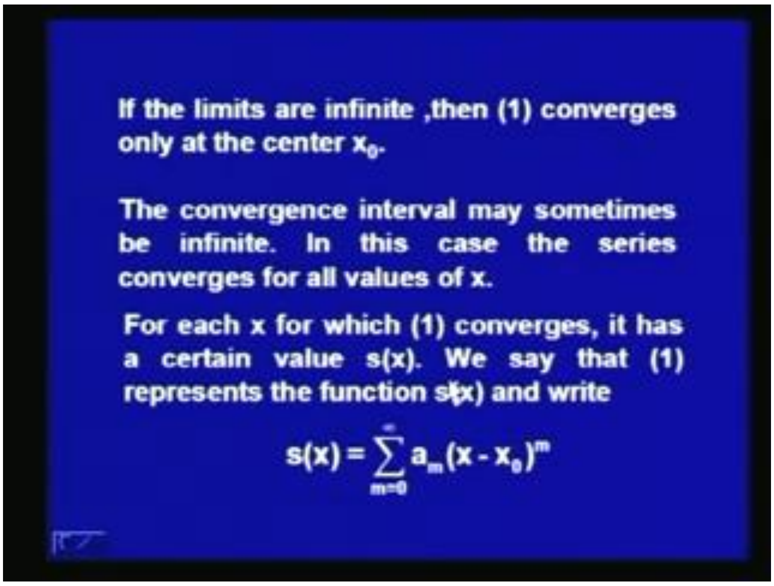

So, when the limits will be infinite will have the convergence only at the center, when the limits will be infinite, we will have the series converging for all values of x. And when, the value of r lies between 0 and infinity, we will have the series convergence for mod of x minus x naught less than R that is in an interval. Now, for each value of x for which the series converges, it has a certain value as x and so we can say that the series 1 represents a function s x. We may therefore, write s x equal to limit sigma m equal to 0 to infinity m x minus x naught raise to the power m.

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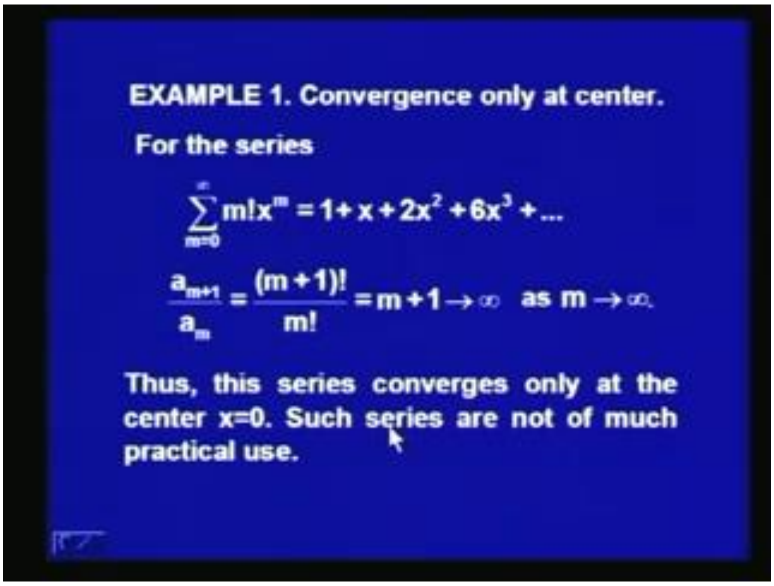

Let us study some examples first we take an example of the series which converges only at the center. Let us, take the example of the series sigma m equal to 0 to infinity m factorial x to the power m, we can write it as 1 plus x plus 2 x square plus 6 x cube and so on. Let us find the value of m plus 1 over m; it is equal to m plus 1, which we can see goes to infinity as m goes to infinity. And therefore, the series converges only at the center, that is x equal to 0, such series are not of much practical interest. Next, we take an example of the series, which converges in a finite interval, let us take the case of a geometric series.

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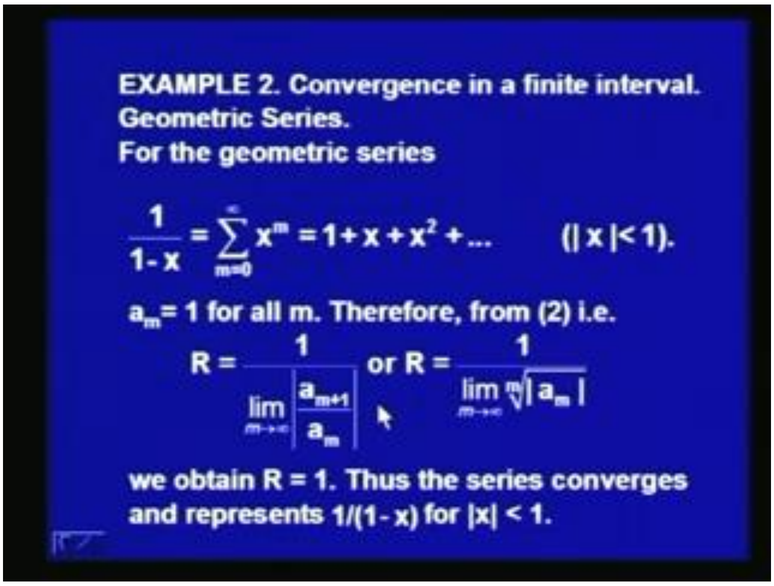

For the geometric series 1 over 1 minus x, which can be expressed as 1 plus x plus x square and so on, it turns out that the region of convergence that is interval of convergence is given by mod of x less than 1, that is open interval minus 1 to 1. Because, here m is equal to 1 for all values of m and therefore, if we apply the formula R equal to 1 over limit m tends to infinity mod of m plus 1 over m or R equal to 1 over limit m tends to infinity mth root of mod of m. You find that, R is equal to 1 and therefore, the series converges for all values of x in the interval minus 1 to 1 and represents the function 1 over 1 minus x.

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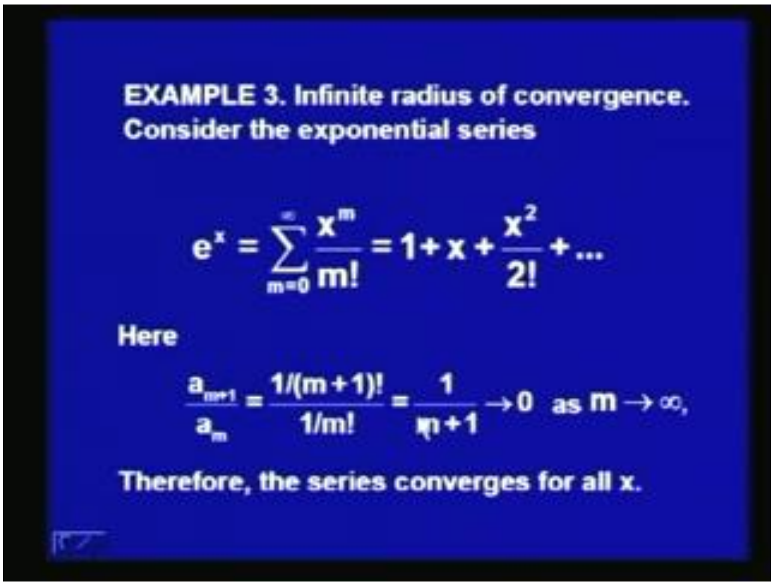

Next, we take an example of the series, which has an infinity radius of convergence that is it converges for all values of x. Let us, take the example of the function e to the power x, which can be written as 1 plus x plus x square by 2 factorial and so on, if you find here m plus 1 over m it turns out to be 1 over m plus 1 which we see goes to 0 as m goes to infinity. And therefore, r is equal to infinity in this case and so the series converges for all values of x.

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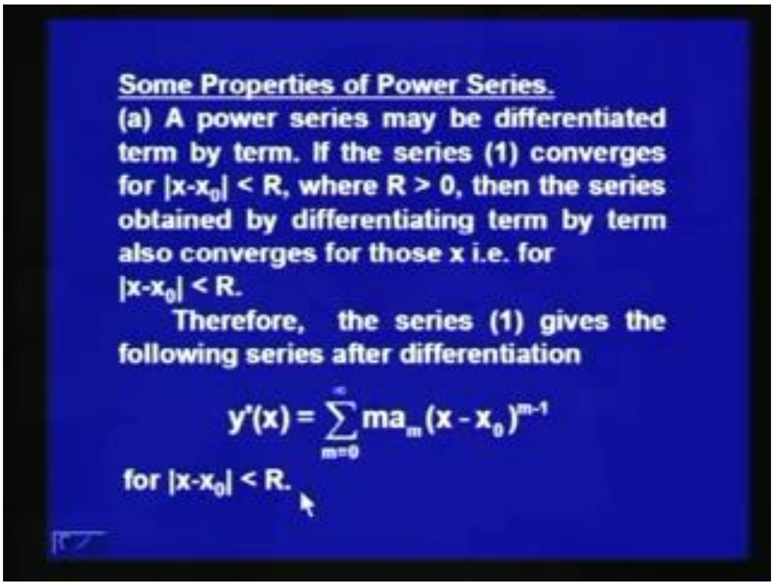

Now, let us study some properties of the power series, a power series may be differentiated term by term, that is if the series 1 converges for mod of x minus x naught less than R, where r is greater than 0. Then, the series obtained by differentiating term by term, also converges for those values of x that is for mod of x minus x naught less than R.

And, you can see that when you differentiate the series given by 1, you get the series sigma m equal to 0 to infinity ma m x minus x naught to the power m minus 1, which is y dash x. And the series converges for mod of x minus x naught less than R, the region of convergence of the original power series.

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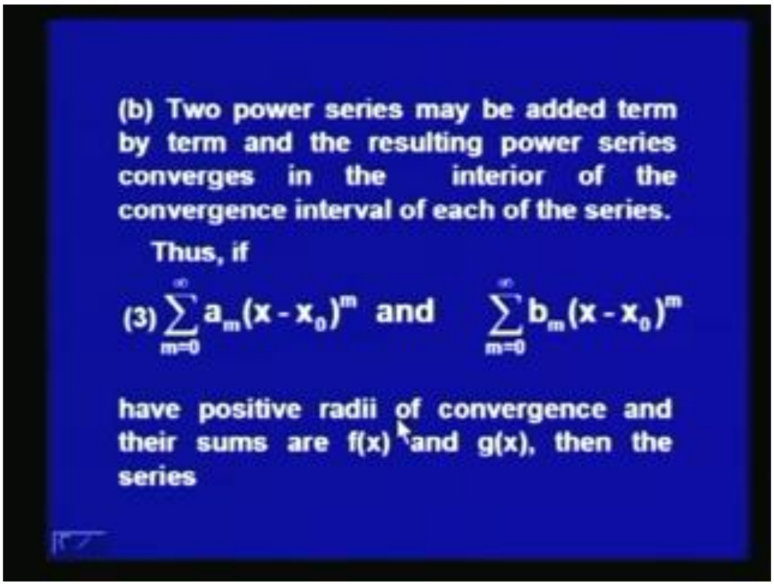

Next, we study when we get two power series, so two power series may be added term by term and the relative power series converges in the interior of the convergence interval of each of the series. Thus, if sigma m equal to 0 to infinity a m x minus x naught to the power m and sigma m equal to 0 to infinity b m x minus x naught to the power m have positive radii of convergence and their sums are f x and g x.

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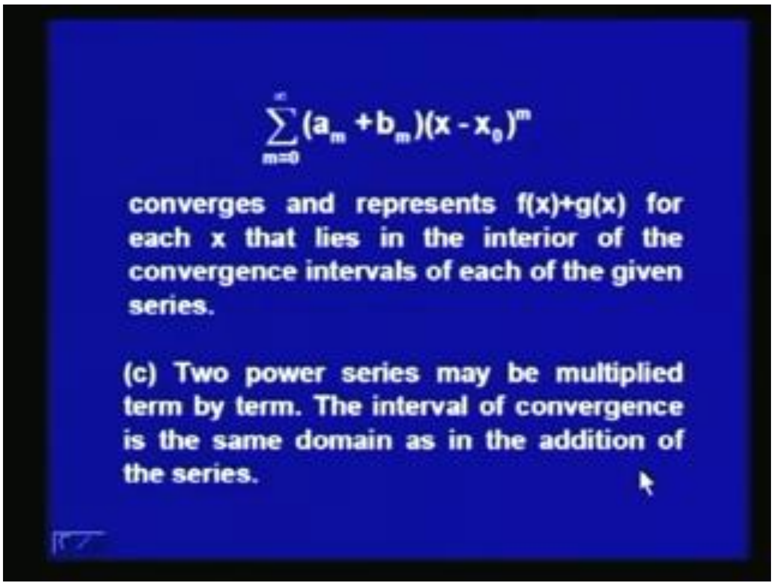

Then, the series sigma m equal to 0 to infinity a m plus b m x minus x naught to the power m converges and represents f x plus g x, for each x that lies in the interior of the convergence intervals of each of the given series. The third property is the two power series may be multiplied term by term; the interval of convergence is the same domain as in the case of the addition of the series, that is the interval of convergence is the interior of the convergence intervals of each of the given series.

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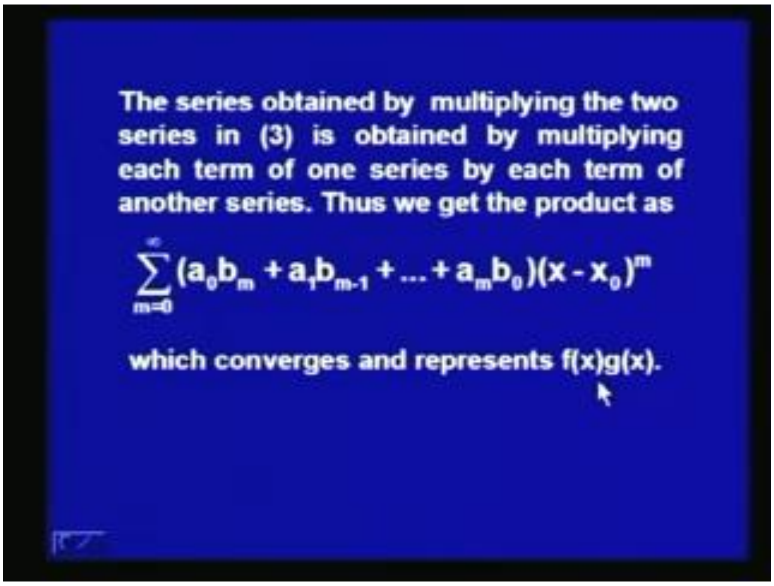

The series obtained by multiplying the two series in 3 is obtained by multiplying each term of one series by each term of the other series. And thus, we get the product as sigma equal to 0 to infinity a naught b m plus a 1 b m minus 1 and so on, a m b naught into x minus x naught to the power m, which converges and represents f x into g x.

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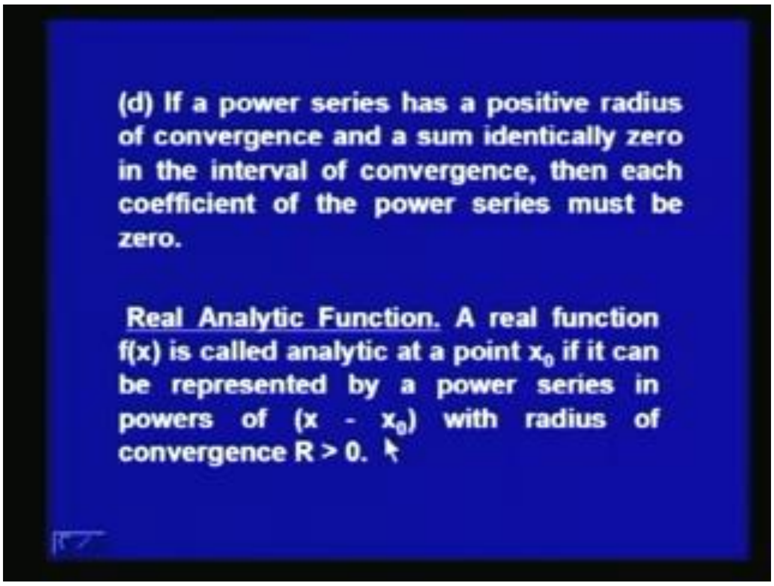

Now, lastly we discuss, if a power series has a positive radius of convergence and sum identically 0, in the interval of convergence, then each coefficient of the power series must be 0. Now, define a real analytic function a real function f x is called analytic at a point x equal to x naught, if it can be represented by a power series in the powers of x minus x naught with radius of convergence R greater than 0. So, that is to say that we will call a real valued function f x to be an analytic function at a point x equal to x naught if it can represented by a power series in the powers of x minus x naught with the positive radius of convergence. We next discuss, when can be finding a power series solution of a homogeneous linear of a homo linear differential equation.

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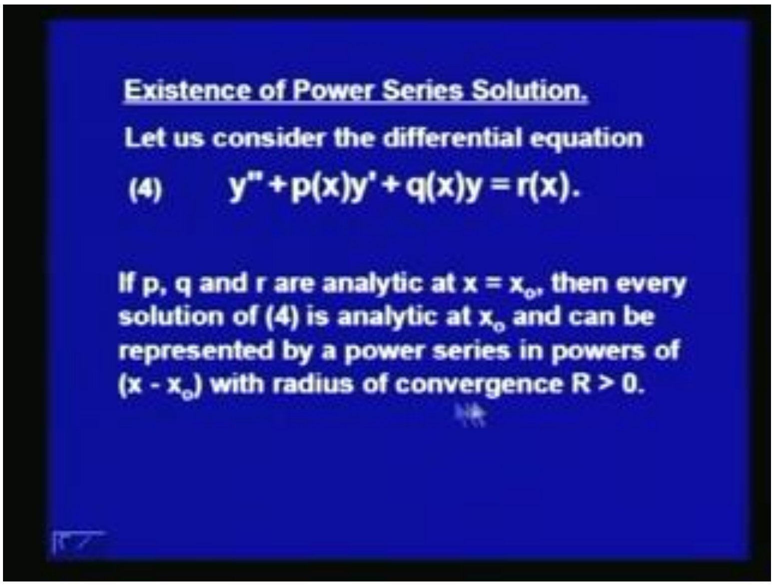

Let us, consider the differential equation y double dash plus p x into y dash plus q x into y equal to r x. If p q and R are analytic at x equal to x naught, then every solution of the differential equation 4 is also analytic at x equal x naught and can be represented by a power series in the powers of x minus x naught with a positive radius of convergence method of construction of power series.

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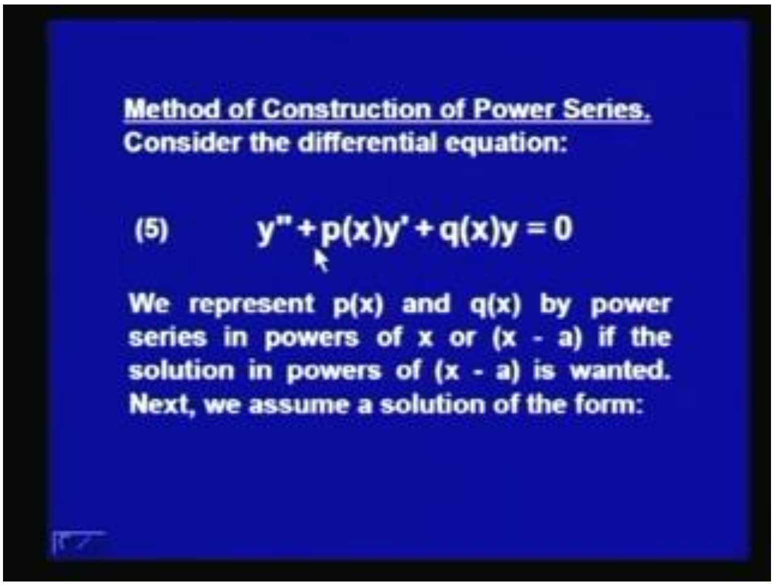

Let us, consider the differential equation y double dash plus p x into y dash plus q x into y equal to 0. So, here in this differential equation, we are taking r x equal to 0, because the Legendre’s equation and the Bessel’s equation, that we shall be studying later on or homogeneous linear differential equations. So, their equations of this type the functions p x and q x are represented in the powers of x or x minus a, if the solution in the powers of x minus a is wanted.

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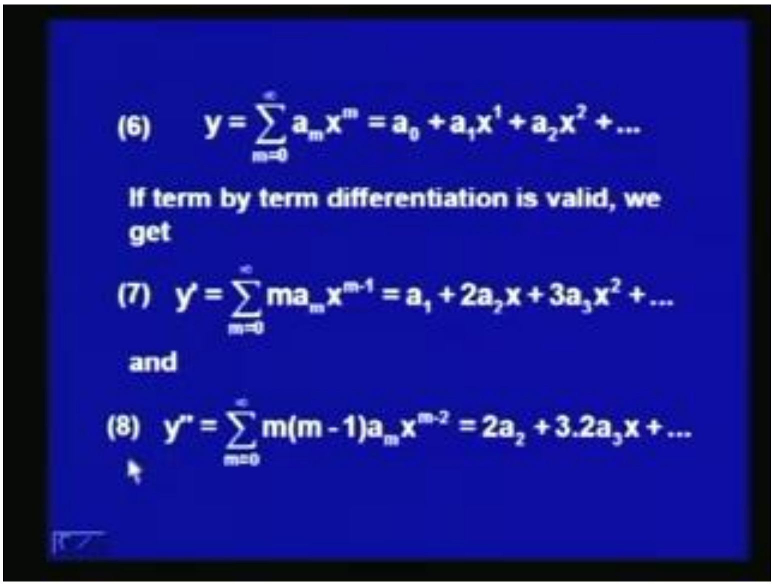

Now, let us assume a solution of this differential equation of the form y equal to sigma m equal to 0 to infinity a m x to the power m, that is a naught plus a 1 x plus a 2 x square and so on. If, we assume that this series can be differentiated term by term, then we get y dash equal to sigma m equal to 0 to infinity in infinity m a m x to the power m minus 1, that is a 1 plus 2 a 2 x plus 3 a 3 x square and so on. And, the second derivative y double dash is equal to sigma m equal to 0 to infinity m into m minus 1 a m x to the power m minus 2, which is 2 a 2 plus 3 into 2 a 3 x and so on.

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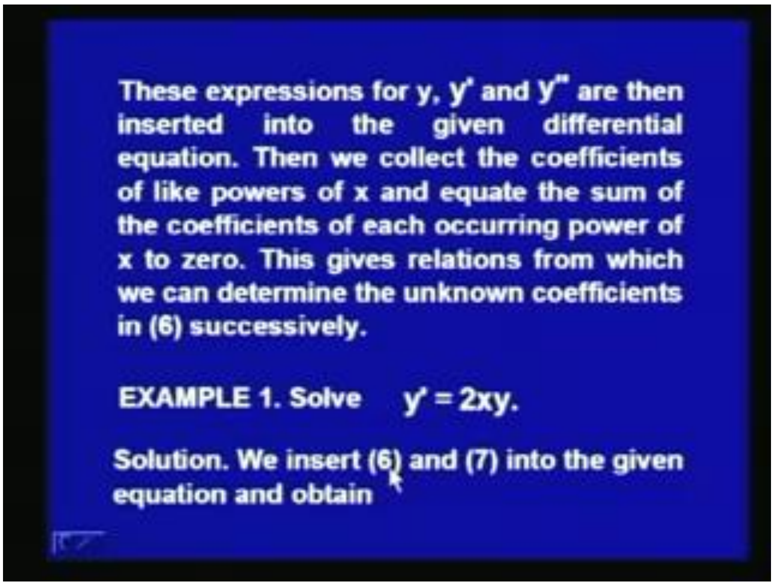

Now, when we put these values of y, y dash and y double dash x, these expressions for y, y dash and y double dash are then inserted in the given differential equation. We then collect the coefficients of like powers of x and equate the sum of the coefficients of each occurring power of x to 0. This will give us relations from which we can determine the unknown coefficients, which occur in the equation 6 successively. Say, let us take an example say y dash equal to 2 x y, so in this equation we will need the value of y and y dash y is equal to sigma m x to the power m given by equation 6 and y dash is given by equation 7. So, we insert the value of y and y dash given by equation 6 and 7 into the given equation.

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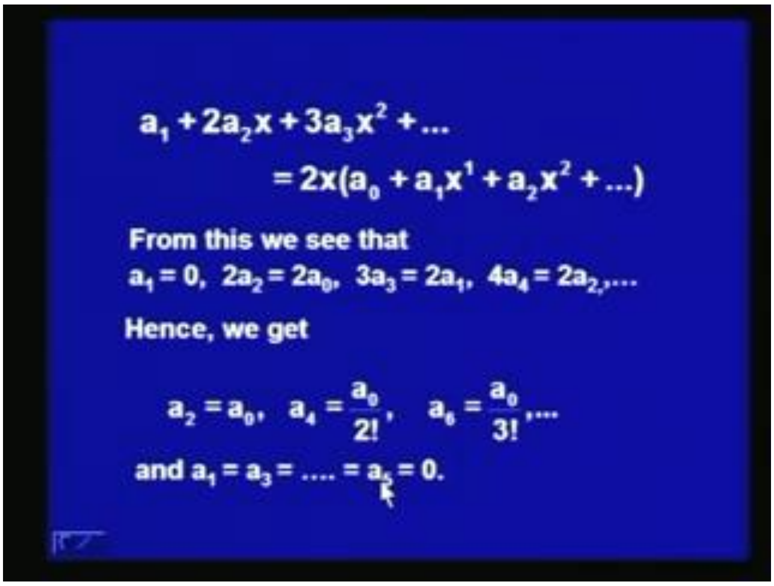

And, obtain a 1 plus 2 a 2 x plus 3 a 3 x square plus and so on equal to 2 x into a naught plus a 1 x plus a 2 x square and so on. We can then compare the coefficients of like powers of x on the two sides and see that a the must be on the lower cap a 1 plus 2 a 2 x plus 3 a 3 x square plus and so on equal to 2 x into a naught plus a 1 x plus a 2 x square and so on.

And, from this equation when we compare the coefficients of the like powers of x, we see that a 1 tells out to be 0, 2 a 2 is equal to 2 a naught, 3 a 3 is equal to 2 a 1 and then 4 a 4 is equal to 2 a 2 and so on. And thus, we get the values of a 2, a 4, a 6 as a naught a naught by 2 factorial a naught by 3 factorial and so on, while a 1 a 3 a 5 and so on, they are all 0s.

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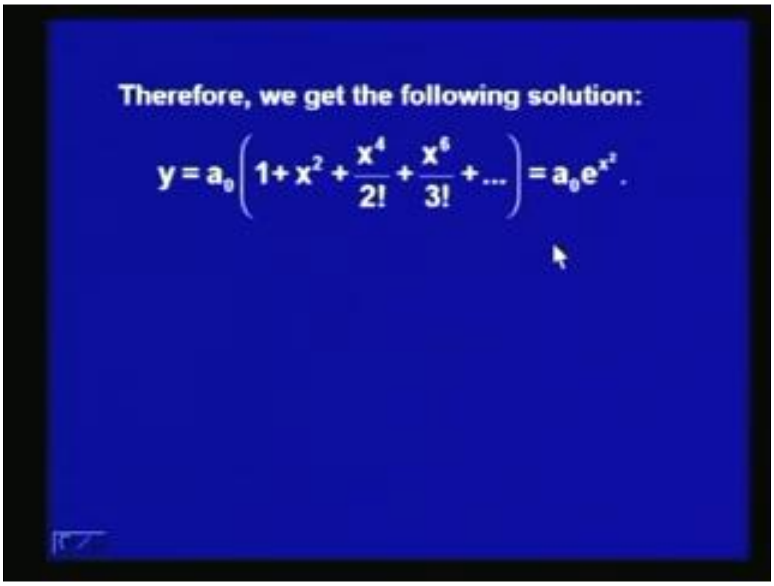

Thus, we get the solution of the differential equation as y equal to a naught into 1 plus x square plus x to the power 4 over 2 factorial plus x to the power 6 over 3 factorial and so on. And we can see that the series inside the brackets is the expression of differention e to the power x square and so we can write the right hand side as a naught into e to the power x square.

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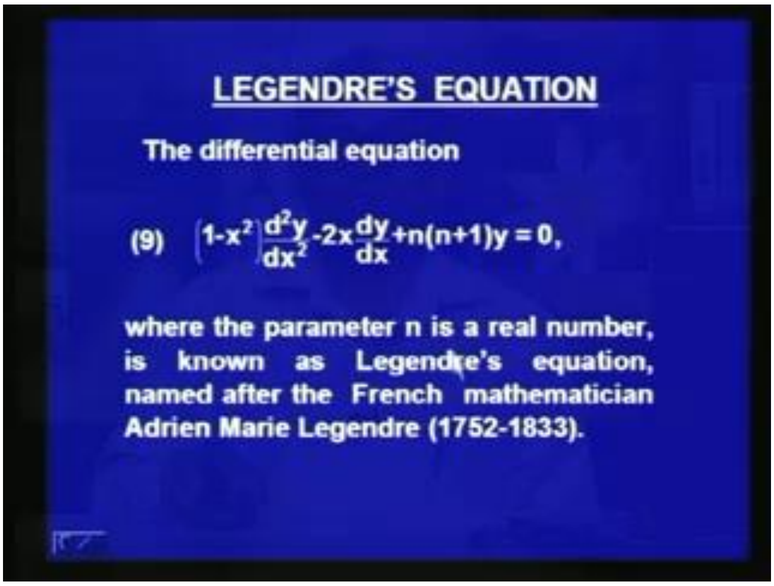

Now, next we study the Legendre's equation the differential equation 1 minus x square into d square y over d x square minus 2 x d y by d x plus n into n plus 1 y equal to 0, where the parameter n is a real number is known as Legendre's equation. It is named after the French mathematician Adrian Marie Legendre from 1752 to 1833.

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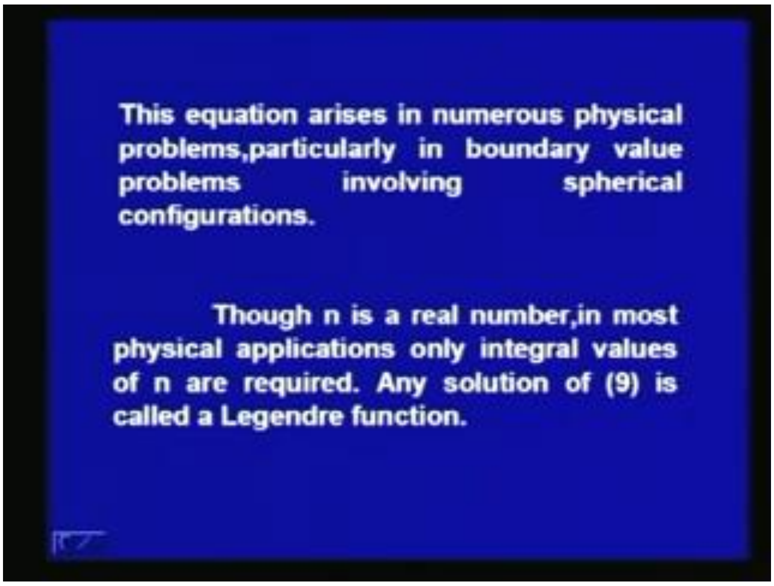

This equation arises in numerous physical problems, particularly in boundary value problems involving a spherical configurations, here though n is a real number in most physical applications. Only integral values of n are required, any solution of the Legendre's equation is called as a Legendre function.

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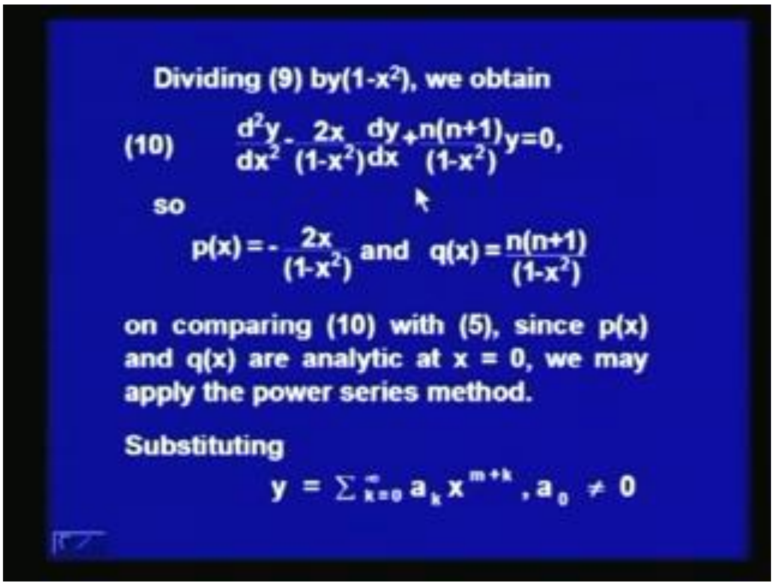

If we divide the Legendre's equation, that is equation number 9 by 1 minus x square, we will obtain d square y over d x square minus 2 x over 1 minus x square into d y by d x plus n into n plus 1 over 1 minus x square into y equal to 0. So, here start it tells out that p x is equal to minus 2 x over 1 minus x square and q x is equal to n into n plus 1 over 1 minus x square, when we compare the equation number 10 with the equation number 5.

Now, let us note that the functions p x and q x are analytic at the point x equal to 0, because they are rational functions of x and the denominator in the functions p x in p x and q x. When, we say that x equal to plus minus 1, so they are analytic at x equal to 0 and therefore, we can apply the power series method. Let us put y is equal to sigma k equal to 0 to infinity a k x to the power m plus k, where we assume that a naught is not equal to 0 in the equation number 9 and the values of the derivatives of y.

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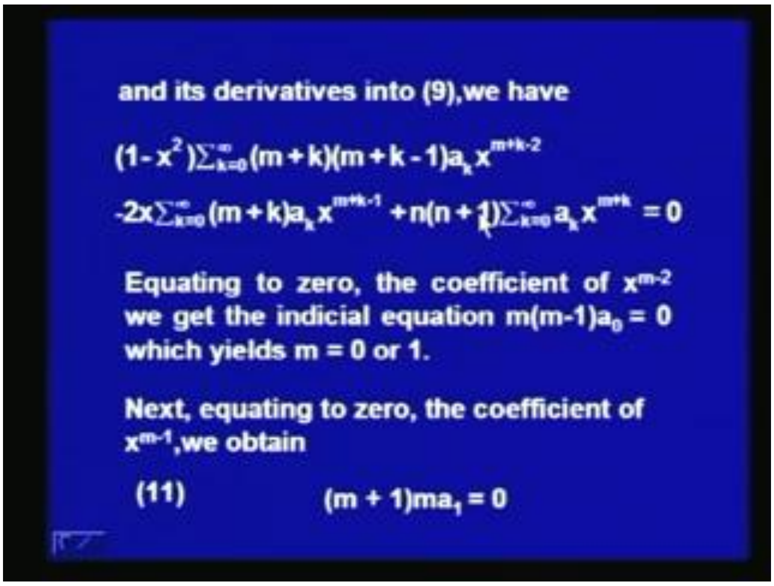

In the equation number 9, we will have 1 minus x square into sigma k equal to 0 to infinity m plus k, m plus k minus 1 a k x to the power m plus k, m plus k into m plus k minus 1, x to the power m plus k, a k x to the power m plus k minus 2, this we get after the differentiation 2 times. Then minus 2 x sigma k equal to 0 to infinity m plus k a k x to the power m plus k minus 1, this we get after we differentiate by once plus n into n plus 1 and then y, that is sigma k equal to 0 to infinity a k x to the power m plus k equal to 0. When, we equate the coefficient of x to the power m minus 2 that is the lowest power of x, which occurs in this equation.

We will get the equation m into m minus 1 a not equal to 0, this equation is known as the indicial equation, this equation tells us that the values of m are 0 and 1. Since, we have assumed that a naught is not equal to 0, when we equate to 0 the coefficient of next higher power of x, that is x to the power m minus 1. We can see that the coefficient of x to the power m minus 1 will be available only in the first term, when we take k equal to 1.

The indicial equation we had obtained from the first term also itself that by taking k equal to 0, when we take k equal to 1 in the first term, we get the coefficient of x to the power m minus 1 and it will be m plus 1 into m into a 1, so let us put that equal to 0. Now, we put the coefficient of x to the power the coefficient of x to the power m and higher powers are available in the first, second, and third terms.

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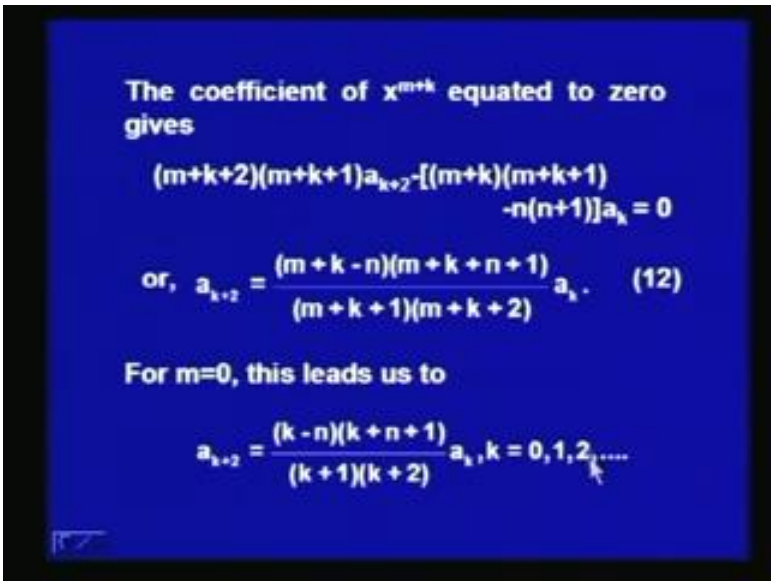

So, what we will do now is that, we will consider the coefficient of x to the power m plus k, where k can take values 0, 1, 2, 3 and so on and put that equal to 0. So, when we find the coefficient of x to the power m plus k in the first term, it will turn out to be m plus k plus 2 into m plus k plus 1 into a k plus 2. And from the second and third terms it will turn out that the coefficient of x to the power m plus k is m plus k into m plus k plus 1 minus n into n plus 1 a k.

So let us, put it equal to 0 and we get the recurrence relation a k plus 2 equal to m plus k minus n into m plus k plus 1, m plus k plus n plus 1 over m plus k plus 1 into m plus k plus 2. The numerator that is m plus k minus n, m plus k plus n plus 1 comes out from here, which this part can be written as m plus k whole square plus m plus k minus n square minus n and so m plus k whole square minus n square can be factorized into m plus k minus n into m plus k plus n. And then we can take m plus k minus n common the remaining thing will be m plus k plus n plus 1.

So, which the numerator here is m plus k minus n into m plus k plus n plus 1, this is the recurrence relation who connects a k plus 2 with a k. So, if we know the value of a k we can find the value of a k plus 2, when we take m equal to 0, here we get a k plus 2 equal to k minus n into k plus n plus 1 over k plus 1, k plus 2 into a k, where k takes values 0, 1, 2, 3, and so on.

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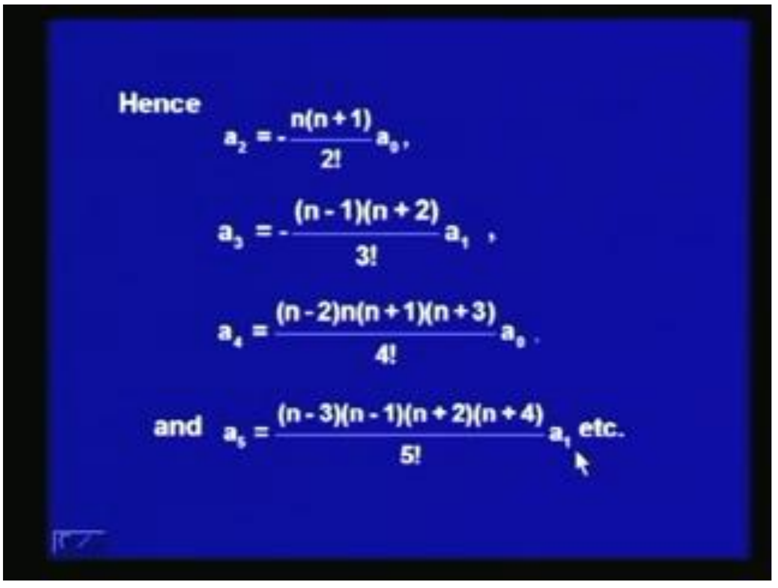

And therefore, we can find the values of a 2, a 3, a 4 and so on, a 2 is minus n into n plus 1 by 2 factorial a naught, a 3 is minus n minus 1 into n plus 2 over 3 factorial into a 1, a 4, after we put the value of a naught, after we put the value of a 2, we get a 4 equal to n minus 2 into n into n plus 1, n plus 3 up on 4 factorial into a naught. The value of an a 2 we have all ready found here, so we can make use of that to obtain a 4 in terms of a not.

Similarly, we can find a 5, a 5 is equal to n minus 3 into n minus 1 into n plus 2 into n plus 4 by 5 factorial into a 1, where we have made use of the value of a 3 in terms of a 1. To write a 5 in terms of a 1, for m equal to 0 from the indicial equation, it follows that 0 is equal to 0, so there a 1 may or may not be 0.

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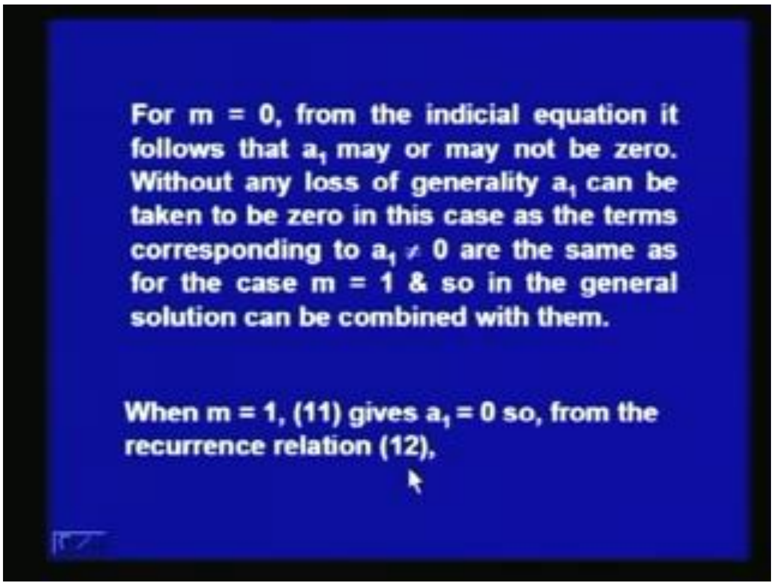

Now, without any loss of generality, we can assume that a 1 is equal to 0, in this case also. Because, the terms that correspond to the case where a 1 is not 0, they can be observed in the terms, which will get for the case m equal to 1, because they are the same case terms. So, they can be observed there and so the general solution will not be affected. And therefore, in the case m equal to 0. We can without any loss of generality we can assume that a 1 is equal to 0 and when m is equal to 1, the equation m into m plus 1 equal to 0 gives you a 1 equal to 0.

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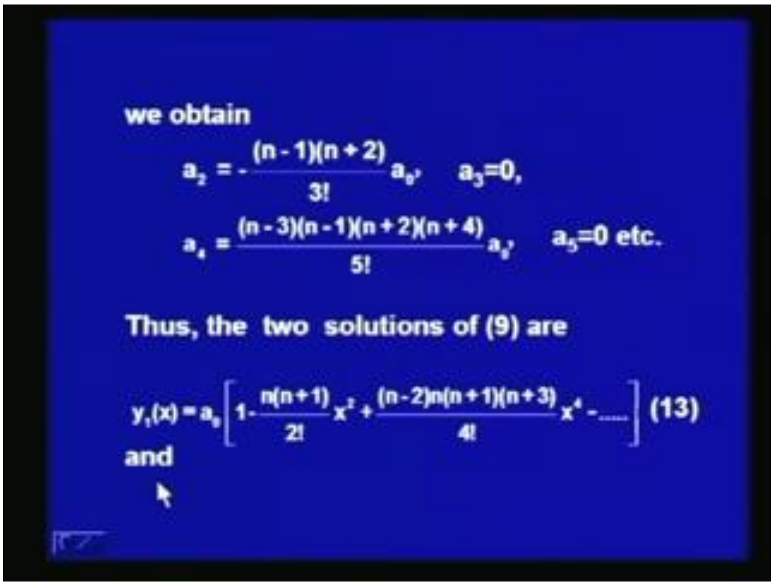

And, so from the recurrence relation 12, it follows that a 2 is equal to minus n into n minus 1 into n plus 2 by 3 factorial a naught, a 3 is equal to 0. Because, a 1 is 0, a 4 is n minus 3 into n minus 1, n plus 2 into n plus 4 by 5 factorial a naught, a 5 is 0, because a 3 0. So, thus we can say that a k is 0 for all odd k integers and hence we get the 2 solutions of the Legendre's equation for the values of m equal to 0 and m equal to 1 s, for m equal to 0 we get y 1 x. The solution y x, we call as y 1 x equal to a naught times 1 minus n into n plus 1 by 2 factorial x square plus n minus 2 into n into n plus 1 n plus 3 by 4 factorial x to the power 4 and so on.

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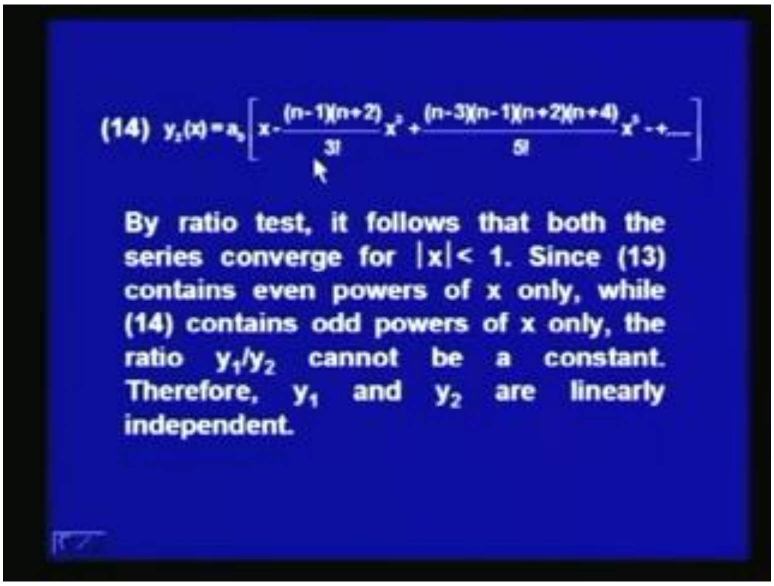

And, for m equal to 1 we get the solution y 2 x as a naught into x minus n minus 1 into n plus 2 by 3 factorial x cube plus n minus 3 into n minus 1 into n plus 2 into n plus 4 by 5 factorial x to the power 5 and so on. Now, when we apply ratio test to this infinite series, it follows that this they converge for mod of x less than 1. Now, the series that occurs in the function y 1 x, it that series contains only even powers of x, while the series that occurs in the expression for y 2 x in 14 contains only odd powers of x.

Therefore, if we find the ratio y 1 over y 2, it cannot be a constant and therefore, y 1 is not a scalar multiple of y 2 and so we can say that y 1, y 2 are y 1 and y 2 are linearly independent functions. So, a linear combination of the 2, that is we can say that y x equal to some constant c 1 times y 1 x plus c 2 times y 2 x is the general solution of the Legendre's equation.

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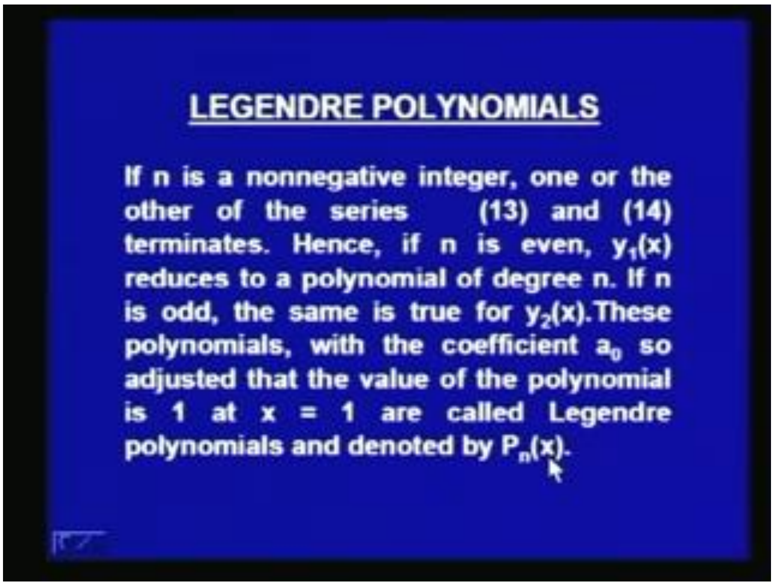

Now, let us assume that n is a non negative integer in the Legendre's equation, if we make this assumption, then one of the two series that is 13 or 14 will terminate in the 13nth. If n is even, it will terminate and y x, y 1 x will reduce to a polynomial of degree n, if n is an odd integer, then the series that occurs in the function y 2 x, that will terminate and will again give us a polynomial of degree n.

Now a naught the coefficient a naught that occurs in the functions y 1 and y 2 x is a tougher choice, so we can make a suitable choice for a not. Let us, choose it in such a way that the value of the polynomial, which occurs in y 1 or y 2 as n is even or odd has the value 1 at x equal to 1. Then, the polynomial will be called a Legendre polynomial and it will be denoted by P n X.

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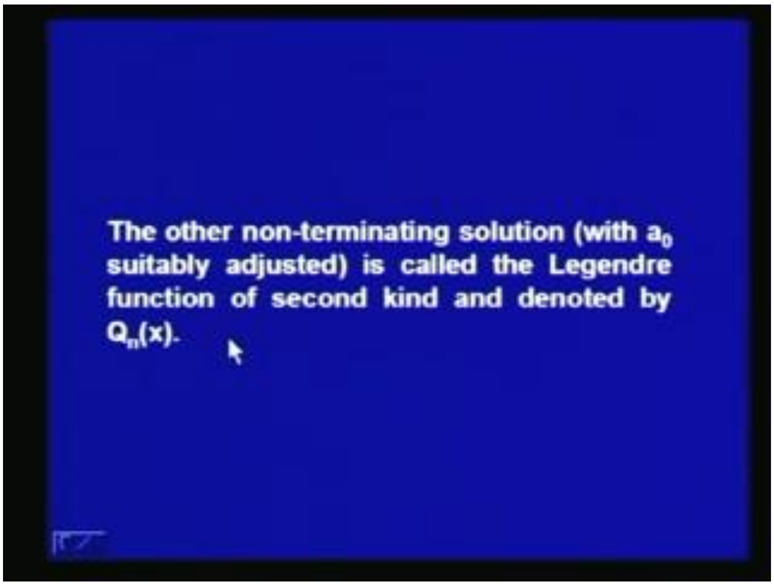

The other series or the other solution, which will be a non terminating infinite series, will be then called a Legendre function of second kind and we will denoting it by Q n X.

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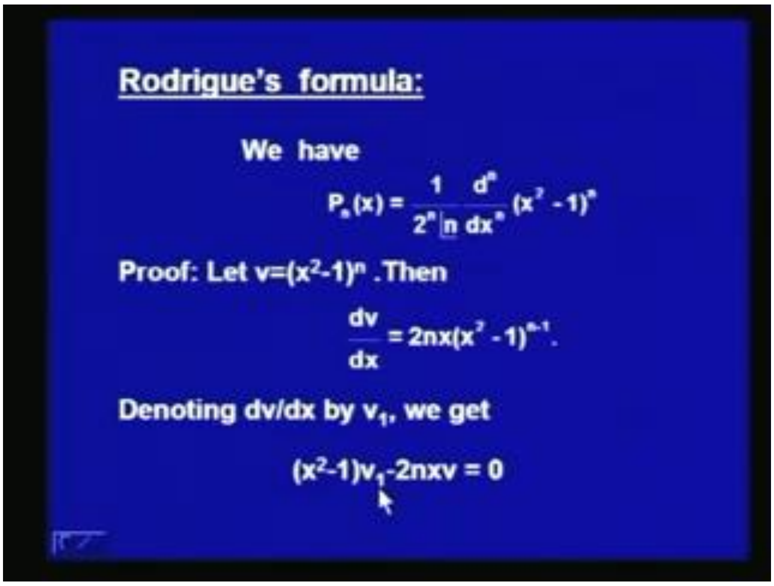

Now, we shall study the Rodriguez's formula which represents the Legendre's function of first kind that is P n X by this formula that is 1 over 2 to the power n into n factorial d n over d x n x square minus 1 to the power n. That is the finite series solution of the Legendre's equation for integral values of n can be expressed in a compact form which is given by this equation this known as Rodrigue's formula.

So, in order to prove this, let us assume that v is equal to x square minus 1 to the power n, when we differentiate it with respect to x, we find that d v over d x is 2 n x into x square minus 1 to the power n. Now, let us multiply this both sides of this equation by x square minus 1, we shall have x square minus 1 into v 1 minus to 2 n x into v equal to 0. If, we denote d v over d x by v 1, now we differentiate this equation n plus 1 times by Leibniz theorem.

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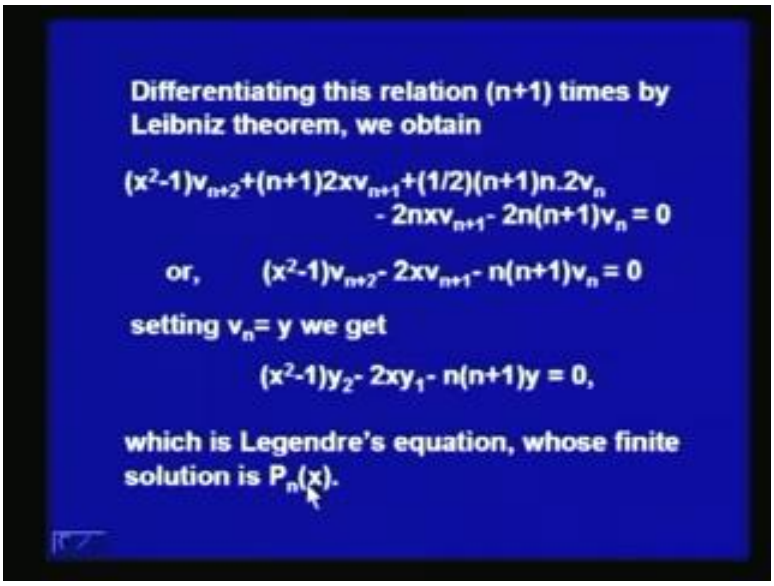

So, when we differentiate it n plus 1 times by Leibniz theorem, we shall have x square minus 1 into v n plus 2 plus n plus 1 into 2 x v n plus 1 plus half into n into n plus 1 2 v n minus 2 n x v n plus 1 minus 2 n into n plus 1 v n equal to 0; which after simplification gives us x square minus 1 into v n plus 2 minus 2 x v n plus 1 minus n into n plus 1 v n equal to 0.

If, we set here v n, that is the nth derivative of v with respect to x, d n v over d x n as y. Then, we shall have x square minus 1 into y 2 minus 2 x y 1 minus n into n plus 1 y equal to 0, which we can see is nothing but the Legendre's equation, whose finite solution we know is given by P n X and therefore, P n X is a constant multiple of P n X.

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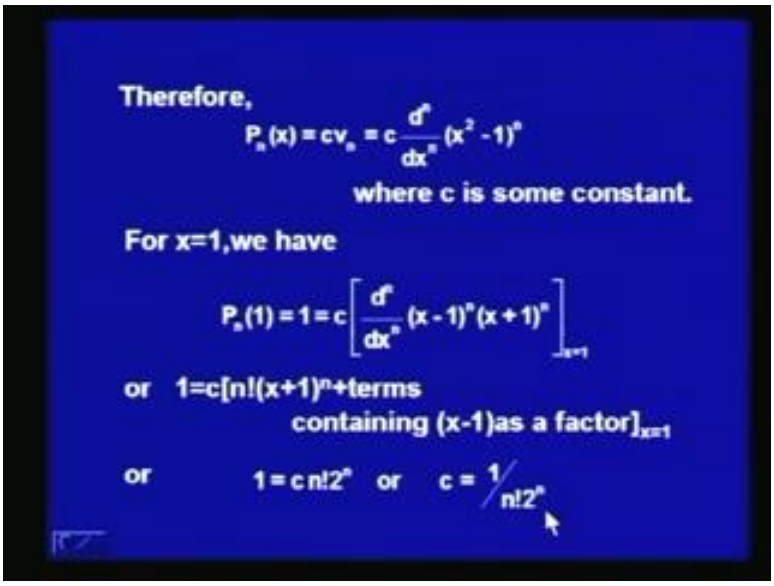

Because P n X is d n over d x n of x square minus 1 to the power n, so when you differentiate x square minus 1 to the power n, which is a polynomial of degree 2 n n x, n times you will get a polynomial in x of degree n. and P n X is also polynomial in x of degree n. So, there must be a constant multiple of each other. So, we have P n X equal to c times v n or we can write it as c times d n over d x n of x square minus 1 to the power n, where c is some constant.

The value of this constant c is found out using the fact that P n 1is equal to 1, when we add define Legendre's polynomial there, we had assumed that the constant suitable a not is so adjusted that the value of the polynomial is 1 at x equal to 1. So, making use of that we will have P n 1equal to 1 equal to c times d n over d x n of v n, v n is x square minus 1 to the power n.

We can factorize it as x minus 1 to the power n into x plus 1 to the power n, when we differentiate it n times and put x equal to 1, we will get v m at x equal to 1. Now, P n 1is equal to 1, right side is c times, now we can differentiate x minus 1 to the power n into x plus 1 to the power n times.

So, when you differentiate x minus 1 to the power n, n times you get n factorial into x plus 1 to the power n plus. Then, in the next term you will differentiate making use of the Leibniz theorem, you will differentiate x minus 1 to the power n, n minus 1 times x plus 1 to the power n, once and then multiply y and c 1. So, that term will contain x minus 1 as a factor in the next term you will differentiate x plus 1 twice, while x minus 1 to the power n will be differentiated n minus 2 times, so we get x minus 1 square as a factor and so on.

So, all the terms is starting from second term inside the bracket onwards will have x minus 1 as a factor therefore, when we evaluate the term inside the bracket at x equal to 1. It will turn out that it is value is 2 factorial into n factorial into 2 to the power n and thus we shall have 1 equal to c into n factorial into 2 to the power n given thus the value of the constant c as 1 over n factorial into 2 to the power n.

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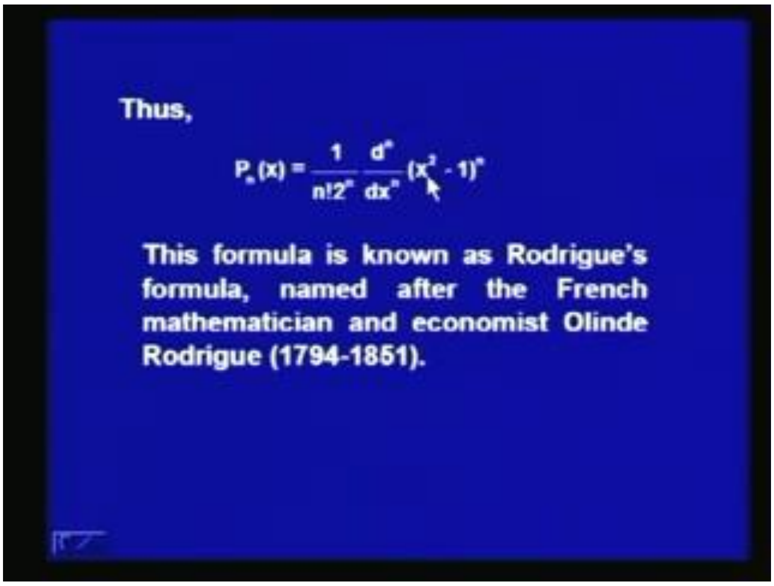

And thus we have the value of the Legendre’s polynomial that is P n X as 1 over n factorial 2 to the power n into d n over d x n, x square minus 1 to the power n. This formula is known as Rodrigue's formula and it was it is named after the French mathematician and economist Olinde Rodrigue 1794 to 1851.

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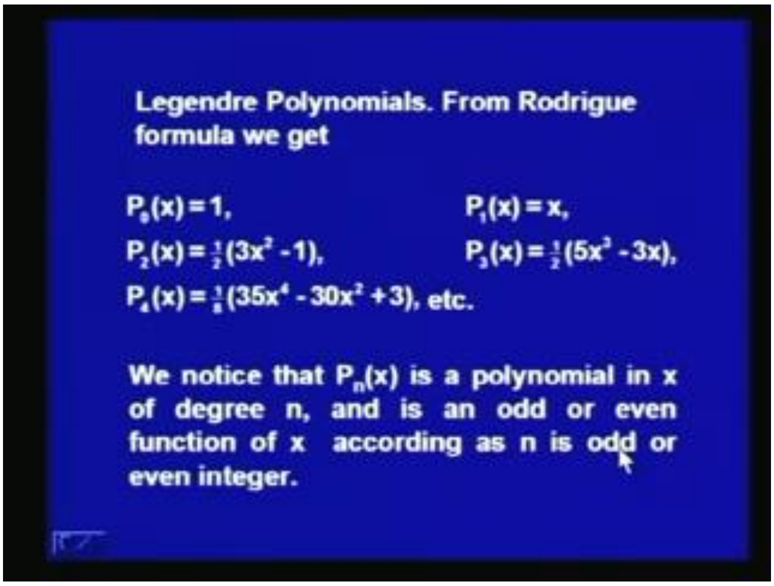

Now from the Rodrigue's formula, if we find the values of p naught x, it tells out that p naught x is 1, P 1 x is x, P 2 x is 1 by 2 into 3 x square minus 1, P 3 x is 1 by 2 into 5 x cube minus 3 x, P 4 x is 1 by 8 into 335, x 4 minus 30 x square plus 3 and so on. Now, let us note that P naught x is a polynomial of degree 0, P 1 x is a polynomial of degree 1, P 2 x is a polynomial of degree 2, P 3 x is a polynomial of degree 3 P n X p 4 x is a polynomial of degree 4.

So, P n X in general is a polynomial in x of degree n and is an even or odd function of x, according as n is an odd integer or an even integer, we can see here that, when n is odd that is n is equal to 1 or 3. We have P 1 x equal to x, which is an odd function, P 3 x is equal to 1 by 2 5 x cube minus 3 x, which is again an odd function and so on for P 5 x also is an odd function and so on. While P naught P 2, P 4 they are all even functions, so P n X is an odd or even function of x according as n is an odd or even integer.

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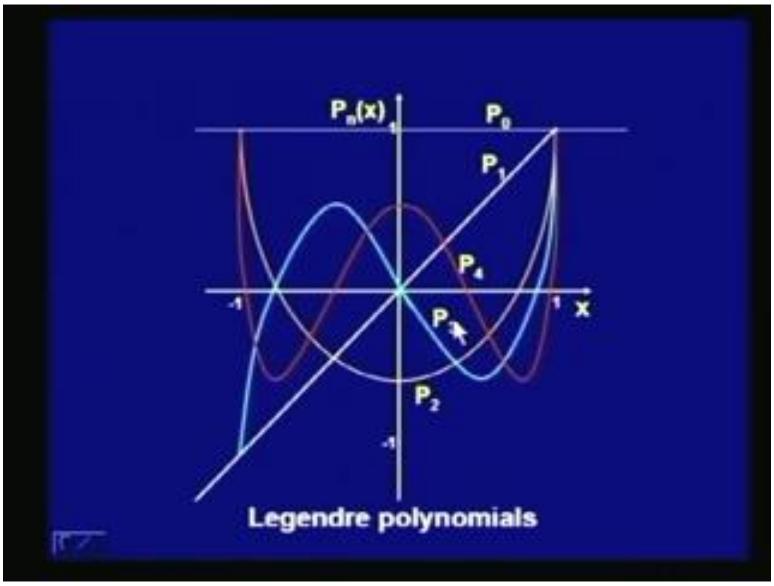

In this picture, we see that the P naught x is equal to 1, this is the straight line P naught x equal to 1 and then we have P 1 x equal to x, this is the line P 1 x equal to x. Then, we have P 2 x, this is the parabola given by P 2 x and then we have P 3 x is this function which we can see easily that is an odd function of x. And then p 4 x is this 1, which we can see is an even function of x, so they are the graphs of the Legendre's polynomials for n equal to 0, 1, 2, 3, 4, etcetera.

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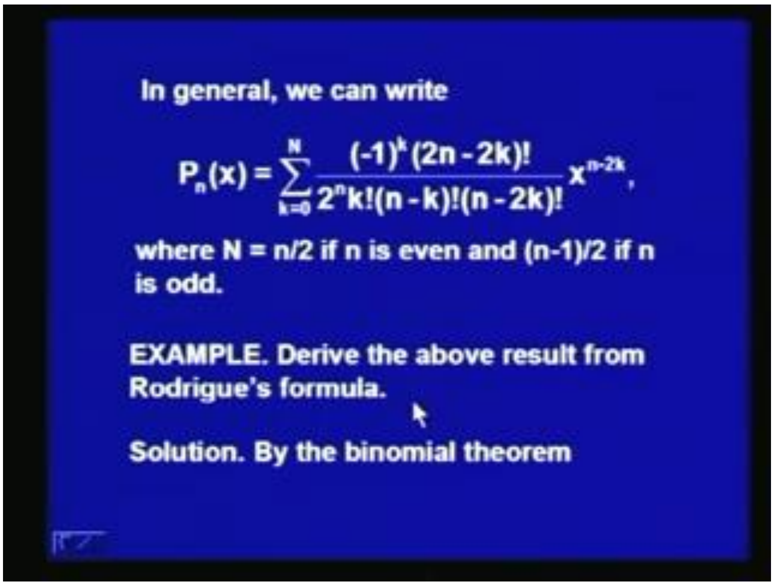

Now, in general we can write P n X given by the Legendre’s given by the Rodrigue's formula as sigma k equal to 0 to m minus 1 to the power k, 2 n minus 2 k factorial over 2 to the power n into k factorial into n minus k factorial into n minus 2 k factorial, x to the power n minus 2 k, where this capital N is equal to n by 2. If n is an even integer and n minus 1 by 2 if n is an odd integer.

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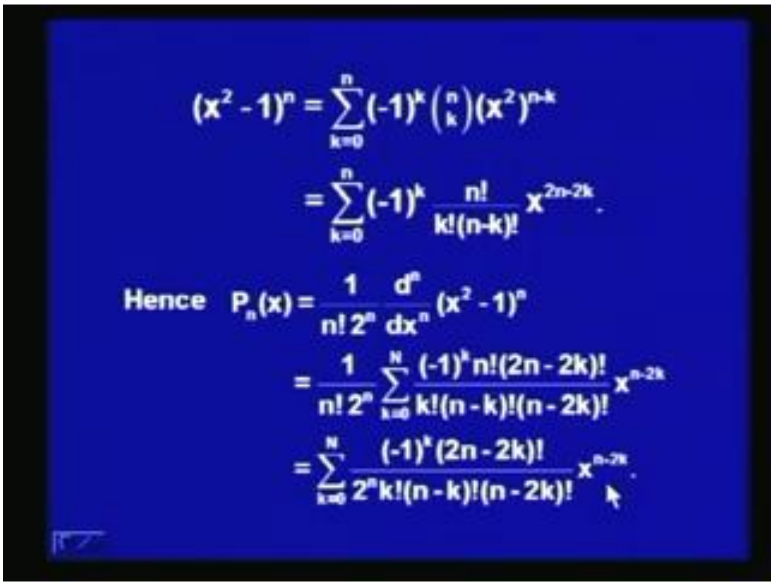

Let us, derive this result by using the Rodrigue's formula by mean by binomial theorem. we can write x square minus 1 to the power n as sigma k equal to 0 to m minus 1 to the power k into n c k, x square to the power n minus k, which can be written as sigma k equal to 0 to n minus 1 to the power k, n factorial over k factorial n minus k factorial x to the power 2 n minus 2 k. Now, put this value of x square minus 1 to the power n in the Rodrigue's formula, P n X equal to 1 over n factorial 2 to the power n, d n over d x n x square minus 1 to the power n.

After, differentiating n times with respect to x, the expression for x square minus 1 to the power n, will have 1 over n factorial, 2 to the power n sigma k equal to 0 to n, 1 minus 1 to the power k, n factorial 2 n minus 2 k over k factorial n minus k factorial n minus 2 k factorial. When, you differentiate this with respect to x, n times the power of x, which is 2 n minus 2 k, will reduce by m and will become n minus 2 k, but the power of x is either 0 or it is 1.

So, when n is an even integer, this n will be equal to n by 2 and when n is an odd integer, this n will be n minus 1 by 2. And thus, we have sigma k equal to 0 to n minus 1 to the power k into 2 n minus 2 k factorial n factorial will cancel and we have 2 to the power n k factorial n minus k factorial n minus 2 k factorial x to the power n minus 2 k.

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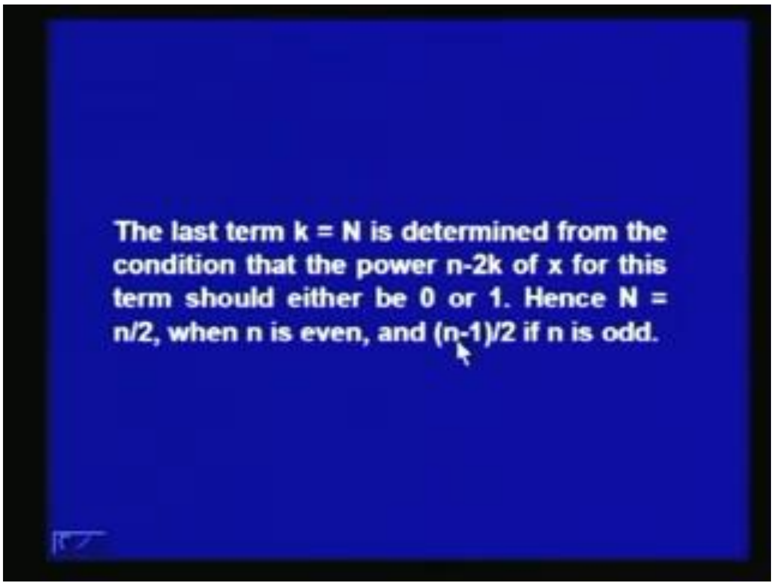

The last term is k equal to n depends on n, if it is even integer n will have value n by 2, if is it is an odd integer will have value n minus 1 by 2, because the power of x has to be either 0 or 1.

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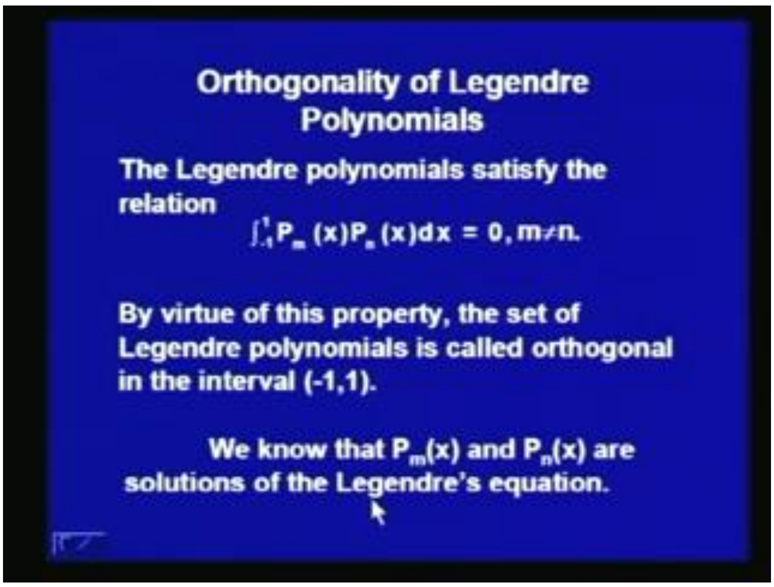

Next, we discuss the Orthogonality of Legendre's polynomial Legendre's polynomial satisfy an very important condition, that is integral over minus 1 to 1, P m Xinto P n X d x is equal to 0, where m and n are integers and m is not equal to n. By virtue of this property, the set of Legendre's polynomial is called orthogonal in the interval minus 1 to 1.

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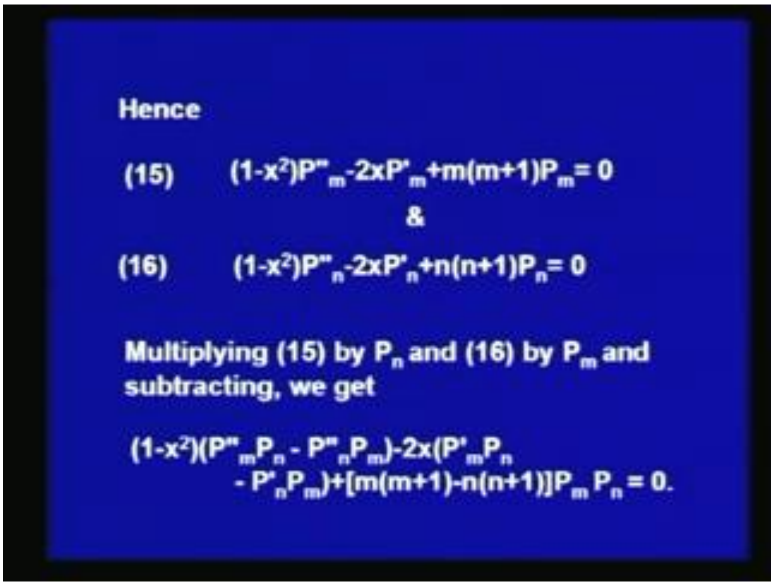

We know that P m X and P n X are solutions of the Legendre's equation, 1 minus x square into p m double dash minus 2 x, p m dash plus m into m plus 1 p m equal to 0, because p m is the solution of the Legendre's equation. So, we can write 1 minus x square into p m double dash minus 2 x, p m dash plus m into m plus 1 p m equal to 0 and P n X is also solution of the Legendre's equation. So, we will have 1 minus x square p m double dash minus 2 x p n dash plus n into n plus 1 p n equal to 0.

Well, let us multiply the equation 15 by p n and 16 by p m and then subtract, we will have this. 1 minus x square p m double dash into p n minus p n double dash into p m minus 2 x into p m dash p n minus p n dash p m plus m into m plus 1 minus n into n plus 1 p m into p n equal to 0.

The first two term, 1 minus x square into p m double dash p n minus p m, p n double dash p m minus 2 x into p m dash p n minus p n dash p m can be combined to give the following. We have 1 minus x square into p m double dash into p n minus p m double dash into p m minus 2 x into p m dash into p m minus p n dash into p m can be combined to give the following d over d x of 1 minus x square into p m dash into p n minus p n dash into p m.

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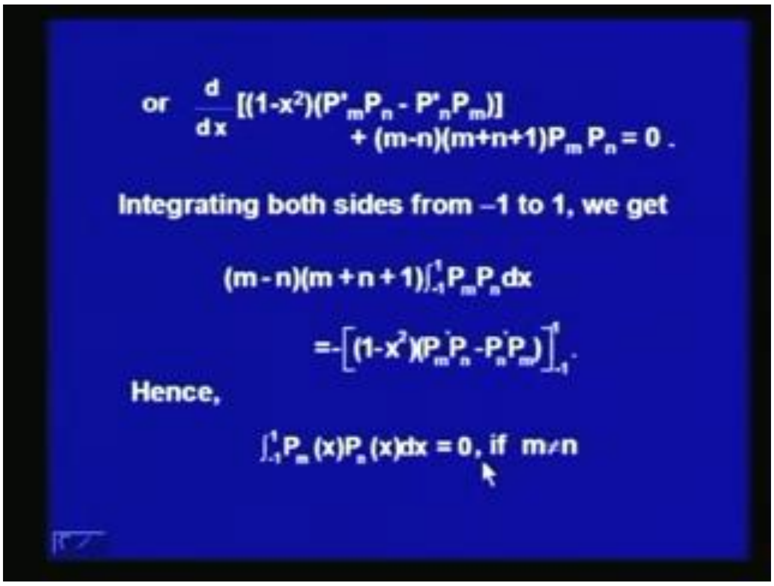

The third term can be factorized to get m minus n into m plus n plus 1 into p m into p n equal to 0. Now, let us integrate both sides over the interval minus 1 to 1, we shall get m minus n into m plus n plus 1, integral over minus 1 to 1, p m into P n X, d x equal to minus 1 minus x square into p m dash into p n minus p n into dash into p m evaluated at x equal to minus 1.

When, we put x equal to 1 and minus 1 in the expression inside the bracket, the value of this is 0. So, we shall have m minus n into m plus n plus 1 integral over minus 1 to 1, P m X into P n X d x equal to 0. Now, if we assume that m is not equal to n, then we can divide by m minus n into m plus n plus 1 and we will have we write integral over minus 1 to 1 P m X into P n X, d x equal to 0; however, when n is equal to m is equal to n.

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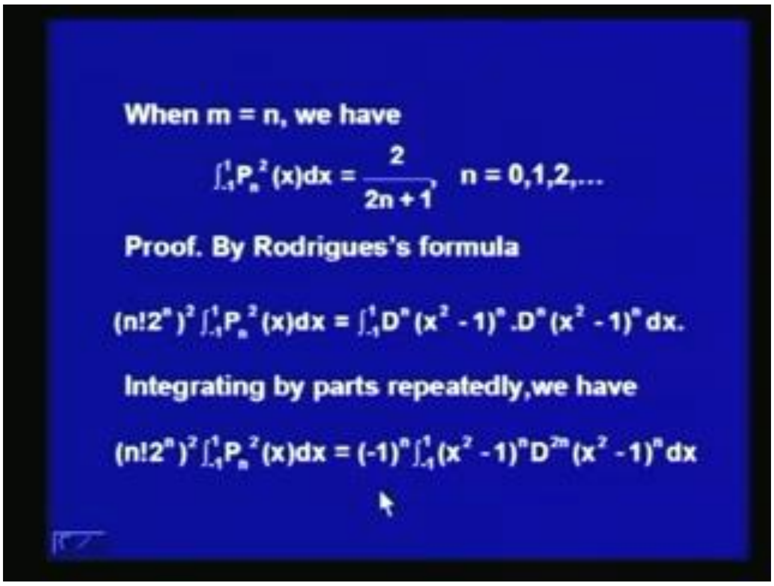

Now, when m is equal to n will not be able to decide divide by m minus n in the previous equation. Therefore, we will have to find it is value separately, it tells out that when m is equal to n, the value of the integral minus 1 to 1 p m into P n X, d x that is minus 1 to 1, p n square x d x is equal to 2 over 2 n plus 1, where n takes values 0 1 2 3 and so on.

Let us, derive this result by Rodrigues's formula, we know that we 2 n into n factorial, P n X is equal to nth derivative of x square minus 1 to the power n. So, making use of that n factorial into 2 to the power n whole square integral over minus 1 to 1, p n square x d x is equal to minus 1 to 1, d n x square minus 1 to the power n into d n x square minus 1 to the power n d x.

Now, when we integrate the right hand side, by parts if we integrate it once, what will happen is that you will have integral of d n x square minus 1 to the power n, which will give us d n minus 1 x square minus 1 to the power n into d n of x square minus 1 to the power n. Then, we will evaluate it is value at minus 1 and 1 it will be 0, because d n minus 1 into x square minus 1 to the power n will contain x square minus 1 as the factor.

So, after integrating it once we will have, it is value as minus integral over minus 1 to 1 d n minus 1 x square minus 1 to the power n into d n plus 1 x square minus 1 to the power n d x. When, we do this integration by parts n times, the right hand side will become minus 1 to the power n, integral over minus 1 to 1 x square minus 1 to the power n into d 2 n x square minus 1 to the power n d x. Will have 2 nth derivative of x square minus 1 to the power n here and here there will be no derivative of x square minus 1 to the power n, which is equal to minus 1 to the power n integral over minus 1 to 1.

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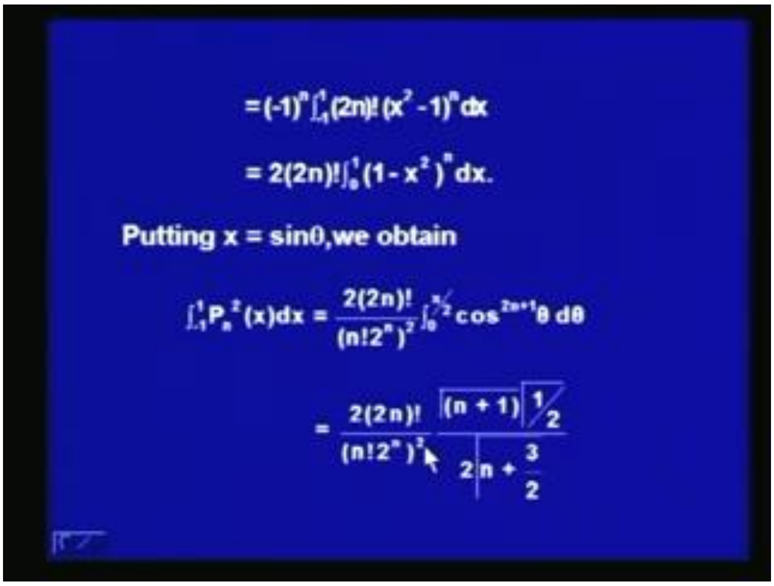

And then d 2 n of x square minus 1 to the power n, will give us 2 n factorial, because x square minus 1 to the power n is a polynomial n x of degree 2 n. So, when we differentiate, it 2 n times, we get 2 n factorial into x square minus 1 to the power n d x. Now, this is equal to 2 times 2 n factorial integral over 0 to 1, 1 minus x square raise to the power n d x.

This is obtained on multiplying x square minus 1 to the power n by minus 1 to the power n, which gives us 1 minus x square to the power n and then since 1 minus x square to the power n is an even function of x. So, using the property of does it integrals, integral over minus 1 to 1 can be written as 2 times integral over 0 to 1, 1 minus x square d x, x square raised to the power n d x.

Now, in this we put x equal to sin theta and then we see that integral over minus 1 to 1, p n square x d x becomes 2 into 2 n factorial, over n factorial into 2 to the power n. Whole square into integral over 0 to pi by 2 will get some integration change from 0 to 1 to 0 to pi by 2 1 minus x square to the power becomes cos theta raise to the power 2 n and d x is cos theta d theta. So, we will have cos theta raise to the power 2 n plus 1 d theta.

Now, we make use of the gamma function, in order to evaluate this integral, here the power of sin theta is 0, where the power of cos theta is 2 n plus 1. So, using the formula integral 0 to pi by 2 cos sin theta raise to the power n, cos theta raise to the power n d theta equal to gamma n plus 1 by 2, over 2 times gamma n plus n plus 2 by 2.

You have evaluated the value of integral 0 to pi by 2 cos theta raise to the power 2 n plus 1, d theta s gamma n plus 1, gamma half over 2 times, gamma n plus 3 by 2. Could you multiply this coefficient 2 into 2 n factorial, over n factorial into 2 to the power n whole square.

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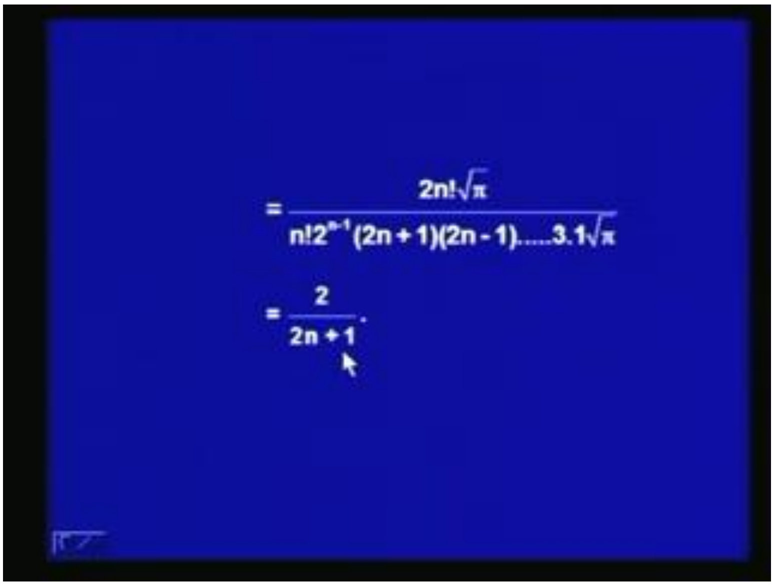

Now, it comes out after we expand the gamma functions, it comes out to be equal to 2 n factorial, gamma half is root pi n factorial 2 to the power n minus 1 into 2 n plus 1, 2 n minus 1 and so on. 3, 1 into root pi root pi can be cancelled and then we can expand 2 n 2 n factorial functions and will have 2 over 2 n plus 1 here. Next we consider the generating function, the function we will see that 1 minus 2 x z plus z square raise to the power minus half can be expanded as sigma n equal to 0 to infinity P n X into z to the power n.

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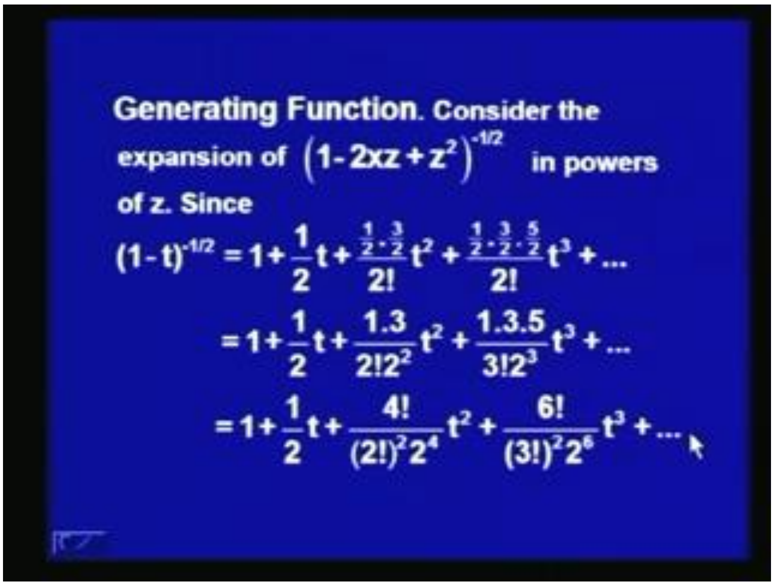

So, the coefficient of z to the power n in the expansion of 1 minus 2 x z plus z square raise to the power minus half gives us the Legendre polynomial P n X and thus we can call this function as the generating function. When, we expand this function in the powers of z the power of z to the power n gives us the Legendre's polynomial P n X or you can say the Legendre's function of first kind, so it may be called as a Legendre's generating function.

Let us consider the expansion of this in the powers of z, let us first write the expansion of 1 minus t, raise to the power minus half by binomial expansion. It can be written as 1 plus 1 by 2 t, 1 by 2 into 3 by 2 over 2 factorial, t square 1 by 2, 3 by 2, 5 by 2 over 2 factorial t cube and so on, where use binomial theorem here. And then 1 plus half t 1 into 3 by 2 factorial 2 square, t square plus 1, 3, 5 over 3 factorial 2 to the power 3 t cube and so on. We can expand write it like this and which can be further written in a convenient form as 1 plus half t plus 4 factorial over 2 factorial square 2 to the power 4 t square, then 6 factorial over 3 factorial square 2 to the power 6 into t cube and so on.

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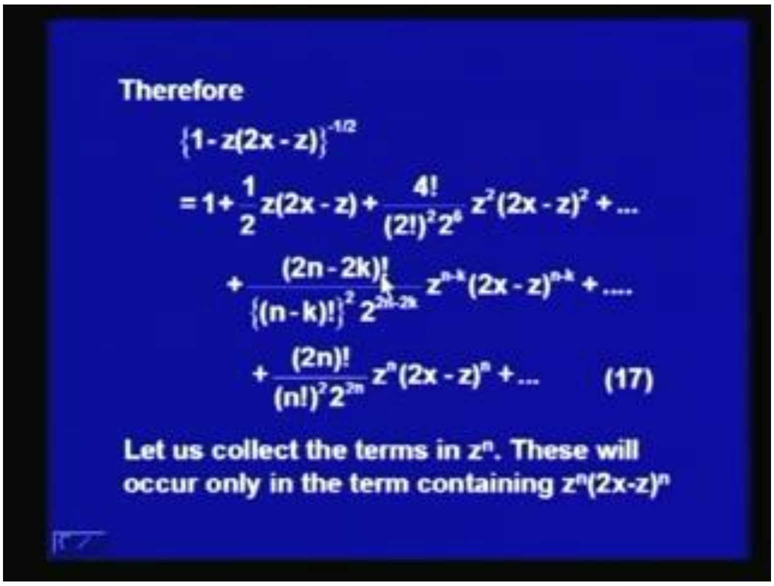

And therefore, let us put now t equal to z into 2 x minus z, we will get the expansion of 1 minus 2 x, z plus z square raise to the power minus half as 1 plus half z into 2 x minus z plus 4 factorial over, 2 factorial square 2 to the power 6 z square 2 x minus z whole square plus this n minus k plus nth term and then we have this n plus 1th term here. Let us collect, the coefficient of z to the power, let us collect the terms in z to the power n in this expansion.

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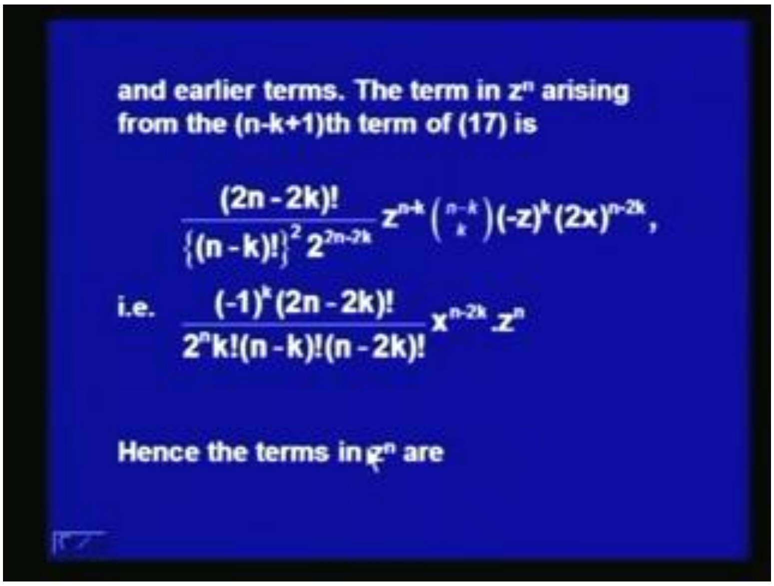

Now, the terms in z to the power n, will occur only in the term containing z to the power n, 2 x minus z to the power n and the previous terms, because after this term we will have the power of z as n plus 1 and the power of 2 x minus z as n plus 1. So, the least power of z, will be n plus 1, so the coefficients of z to the power n will occur only in this term that is z can the term containing z to the power n into 2 x minus z to the power n and the previous terms.

The term in z to the power n arising from the term in the second line of equation 12, in this equation, if you find the coefficient of z to the power n, it turns out to be 2 n. Now, that the terms in z to the power n, which arises from the n minus k plus 1th term of the equation 17 is given by 2 n minus 2 k factorial over n minus k factorial square into 2 to the power 2 n minus 2 k into z to the power n minus k.

And then n minus k c k into minus z to the power k into 2 x to the power n minus 2 k, which after simplification gives us minus 1 to the power k into 2 n minus 2 k factorial. Then, we have 2 to the power n into k factorial, n minus k factorial, n minus 2 k factorial x to the power n minus 2 k into z to the power n.

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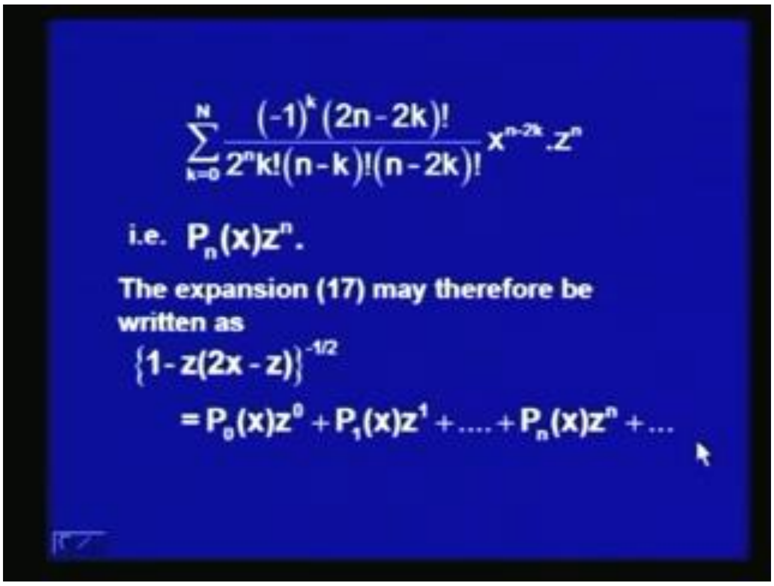

Hence the terms in z to the power n are sigma k equal to 0 to n, capital N minus 1 to the power k, 2 n minus 2 k factorial over 2 to the power n into k factorial, n minus k factorial into n minus 2 k factorial, x to the power n minus 2 k into z to the power n. Now, this coefficient of z to the power n, as we have seen earlier is nothing but the legendry's function of first kind that is legendry's polynomial P n X, so we get P n X into z to the power n.

Here, we may recall that the value of this capital n is small n, y 2, if n is an even integer and the value of capital n is n minus 1 by 2, if n is an odd integer. And thus, we can see that the expansion 17 may be written as 1 minus z into 2 x minus z to the power minus half equal to p naught x into z to the power 0 plus p 1 x into z to the power 1 and so on, P n X into z to the power n.

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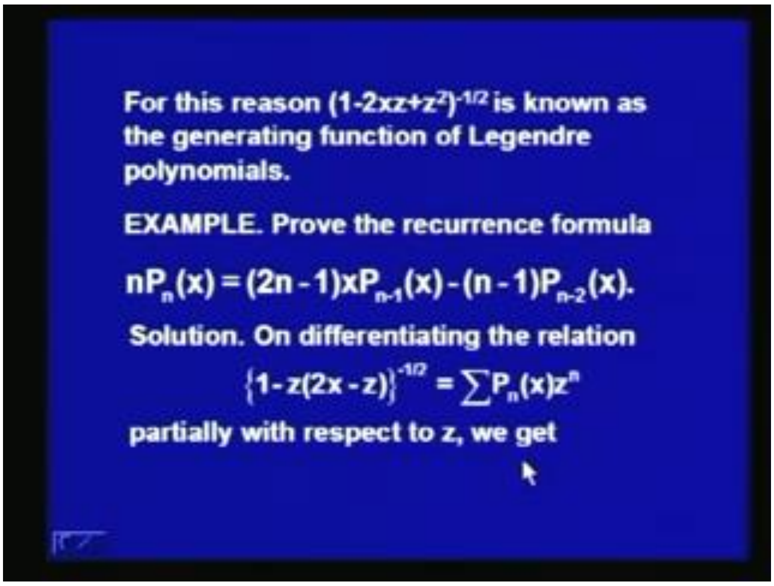

And so on which can be written as sigma n equal to 0 to infinity P n X into z to the power n and y. For this reason we can call 1 minus 2 x z plus z square to the power minus half as the generating function of the legendry's polynomial. Now, let us take an example on this, let us prove the recurrence relation n into P n X is equal to 2 n minus 1 into x p n minus 1 x minus n minus on p n minus 2 x, this result is known as the recurrence formula. Because, from here if we know the Legendre's polynomial p n minus 1 and p n minus 2, we can obtain the value of P n X.

So, in order to establish this result, we shall make use of the Legendre generating function of Legendre's polynomial, that is 1 minus z into 2 x minus z to the power minus half. We know that, this is equal to sigma P n X into z to the power n, when we differentiate this with respect to z partially we will get minus half into 1 minus z into 2 x minus z to the power minus 3 by 2 into minus 2 x plus 2 z.

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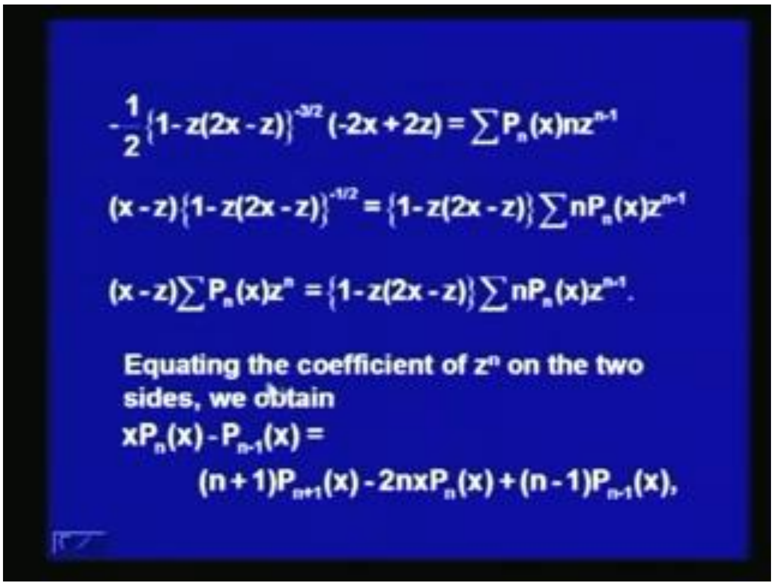

While, the right hand side will become sigma P n X into n z to the power n minus 1, after simplification this equation becomes x minus z into 1 minus z into 2 x minus z to the power minus half equal to 1 minus z into 2 x minus z into sigma n P n X into z minus z to the power n minus 1. We have multiplied after simplification, this equation by 1 minus z into 2 x minus z also to make the power of 1 minus z into 2 x minus z as minus half.

And then we make use of this generating function for Legendre’s polynomial, we can write this equation as x minus z into sigma P n X into z to the power n and then 1 minus z into 2 x minus z into sigma P n X into z to the power n minus 1. Now, let us equate the coefficients of z to the power n on both sides, so left hand side the coefficient of z to the power n will be x minus z into p n that is x P n X.

The coefficient of z to the power n will be x into P n X and then on the right side when we take the coefficient of z to the power n, we will have n P n X, z minus 2 x into n P n X. And here, we shall have the coefficient of z to the power n will be n plus 1 into p n plus 1, because when you multiply this by 1, you get n the coefficient of z to the power n plus 1 into p n plus 1.

So, we get n plus 1 into p n plus 1 and left side the coefficient of z to the power n will be x into P n X and then here the power of z becomes n plus 1. So, z to the power n coefficient will be p n minus 1. So, we get x into P n X minus p n minus 1 and p n minus 1 x as the right hand side gives us n plus 1 into p n plus 1 x and then minus 2 n x into P n

X, this we get from here minus 2 x into n P n X and then z to the power n. The last term is plus z square multiplied to this, so that makes the power of z as n plus 1. So, the coefficient of z to the power n will be n minus 1 into p n minus 1 x.

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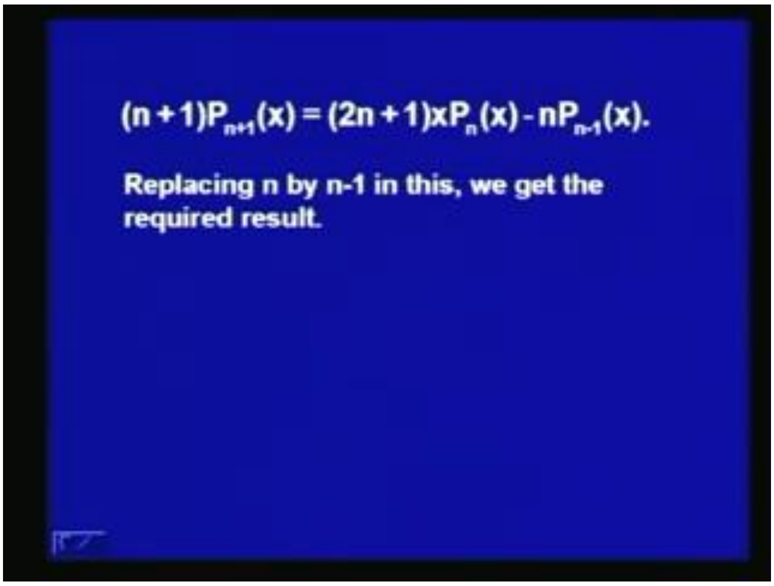

And, after simplification this gives us n plus 1 into p n plus 1 x equal to 2 n plus 1 into x P n X minus n into p n minus 1 x. Now, when we replace n by n minus 1 in this equation we will get the required result, now we discuss the associated Legendre’s equation, the associated Legendre’s equation had applications in potential theory. Let us, see how we arrive at the associated Legendre’s equation.

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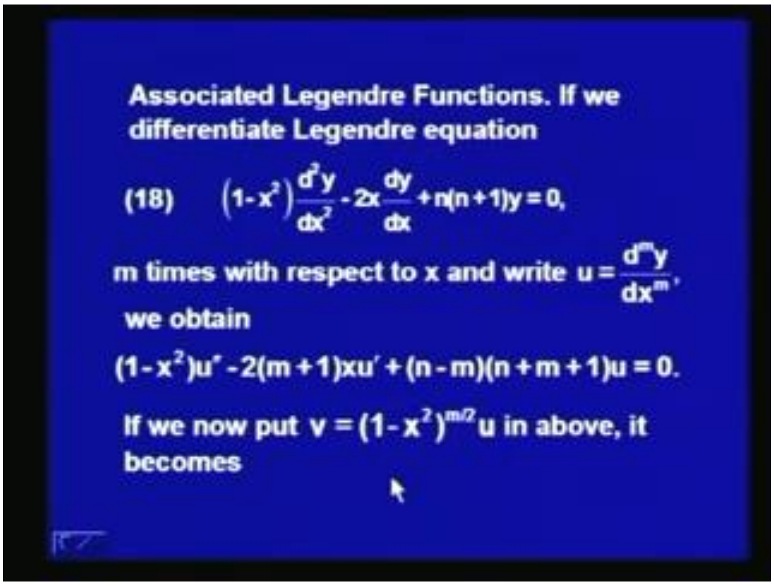

If, we differentiate the Legendre's equation 1 minus x square into d square y over d x square minus 2 x, d y by d x plus n into n plus 1 y equal to 0, m times with respect to x and write u equal to d m y over d x m, that is mth derivative of y with respect to x. We shall have this equation, 1 minus x square u double dash minus 2 m plus 1 x u dash plus n minus m into n plus m plus 1 into u equal to 0. We arrive at this differential equation, after we differentiate the equation a team m times by Leibniz theorem and make use of u equal to d m y over d x m. Now, if we put, v equal to 1 minus x square to the power m by 2 into u in this equation.

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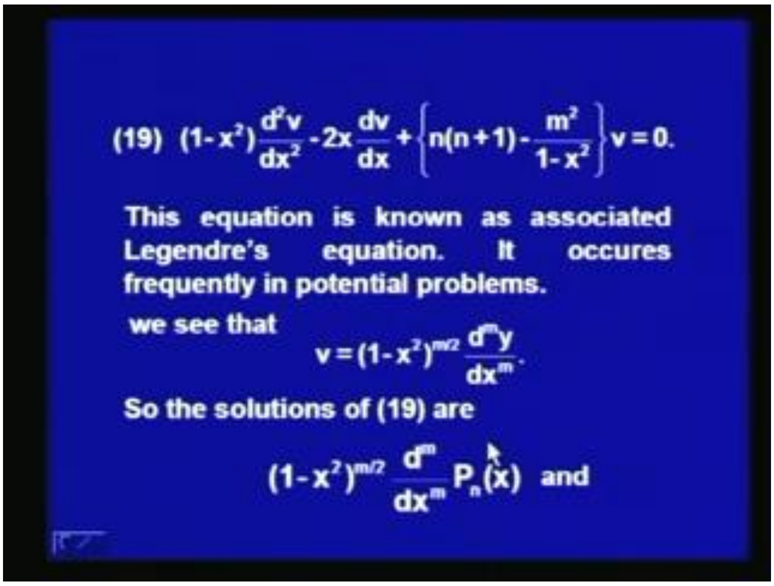

Then, it becomes the following 1 minus x square into d square v over d x square minus 2 x, d v by d x plus n into n plus 1 minus m square over 1 minus x square into v equal to 0. This equation is known as associated Legendre's equation and it has applications in the potential theory, it occurs frequently in potential problems.

And, now we see that v is equal to 1 minus x square raise to the power m by 2 into u, so m u is m d m y over d x m, so v is equal to 1 minus x square to the power m by 2, d m y over d x d x m, And therefore, the solutions of the equation 19 are 1 minus x square to the power m by 2 d m over d x m, P n X and 1 minus x square to the power m by 2, d m over d x m, Q n X, P m X.

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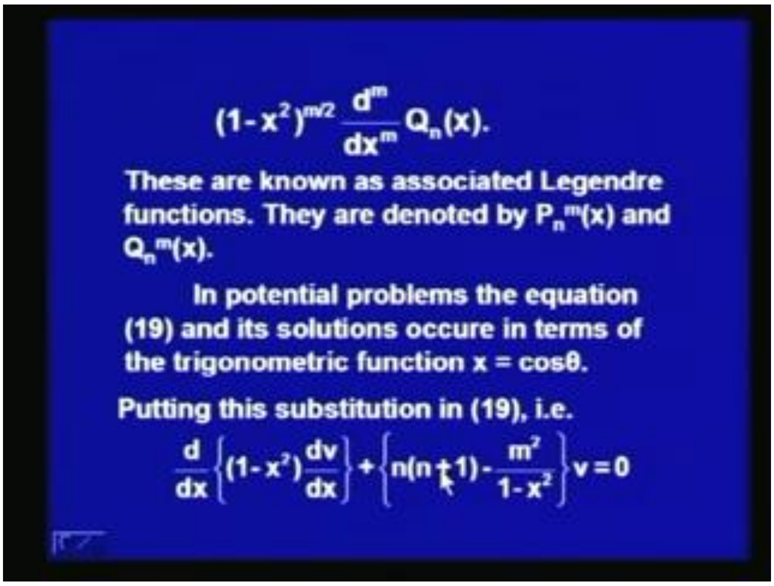

We know that P n X and q x are the solutions of the Legendre's equation, these are known as associated Legendre's functions they are denoted by P n m X and Q n m X. In the potential problems the equation 19 and it is solutions occur in terms of the trigonometric function x equal to cos theta. So, let us put x equal to cos theta in the equation 19, we can write alternately in this form d over d x of 1 minus x square into d v by d x plus n into n plus 1 minus m square over 1 minus x square into v equal to 0.

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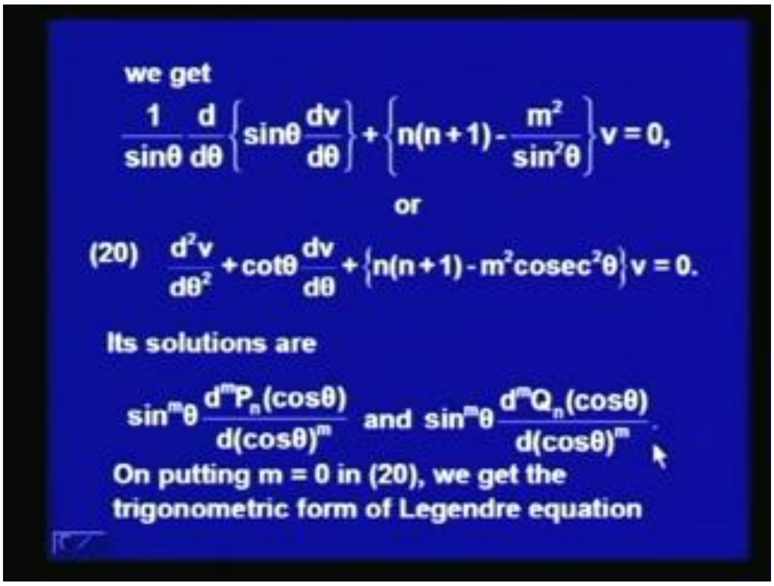

So, now, if we put x equal to cos theta in this equation, we shall see that, we get the following 1 over sin theta, d over d theta, sin theta d v over d theta plus n into n plus 1 minus m square over sin square theta into v equal to 0 or we will have this, d square v over d, theta square plus cot theta d v over d theta plus n into n plus 1 minus m square cosec square theta into v equal to 0.

And hence, it is solutions are sin theta raise to the power m into mth derivative of p n cos theta over d cos theta raise to the power m and sin m theta raise to the power m, d m of cos q, q n cos q theta over d cos theta raise to the power m. If we put m equal to 0 in this equation, if we put m equal to 0 in this equation 20, then we shall get the trigonometric form of the Legendre's equation. In our next lecture, we shall discuss the extension of the power series method, that is the Frobenius method and with the help of the Frobenius method. We shall find the solution of the Bessel's equation, which again has lots of applications in the applied mathematics.

Thank you.