Mathematics III
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Lecture - 26
ADI Method for Laplace and Poisson Equation

Welcome to the lecture series on differential equations for under graduate students. Today's lecture is on ADI Method for Laplace and Poisson Equations. We are learning the numerical methods to solve Laplace and Poisson equations. In the last lecture we had learned a numerical method to solve the Laplace equation del 2 u over del x 2 plus del 2 u over del y 2 is equal to 0.

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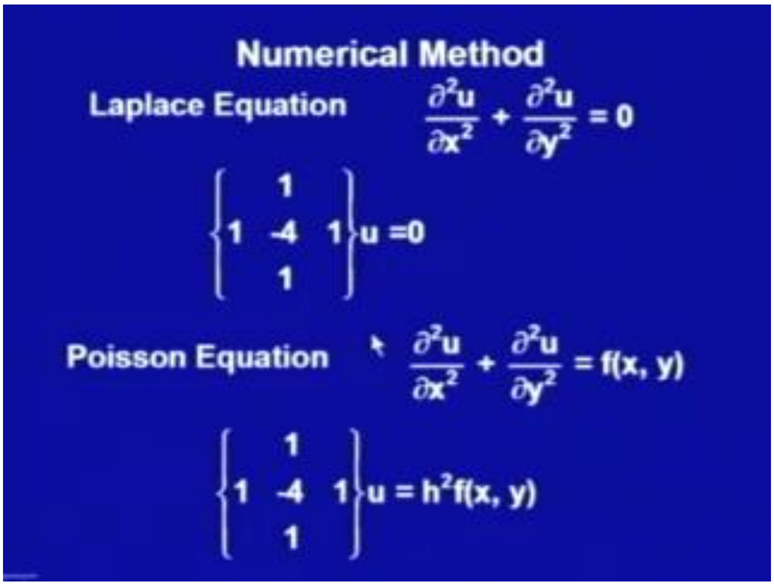

Using a Five point discretization formulation as 1 1 minus 4 1 1 u is equal to 0 and similarly for Poisson equation del 2 u over del x 2 plus del 2 u over del y 2 is equal to f x y, we can discretize again using the same kind of discretization 1 1 minus 4 1 1 u is equal to h square times f x y.

We do know that by this discretization method, when we have used this different quotient to change this partial derivatives to the different quotients we have got a matrix or we have actually arrived with algebraic equations, where the coefficient matrix was very sparse. And in that case we had find it out that the gauss elimination method was not a very suitable method for that one, because the matrix is very large and it is sparse.

So, we have learned that is iterative methods would be better and in the last lecture we had learned about one iterative method called Gauss-Seidel method, which at a older times was also known as Liebmann's method. Today will learn about one more iterative method.

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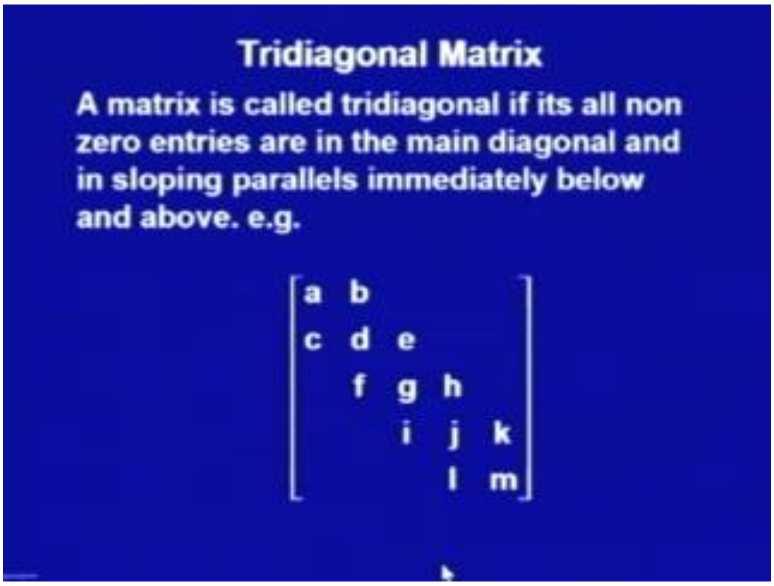

Let us see start with the same discretization scheme. We do have one more term, a matrix is called tridiagonal if it is all non zero entries are in the main diagonal and in the sloping parallels immediately below and above. Say for example, this matrix is a tri diagonal what we are having is one main diagonal then it is immediately below diagonal and then it is immediately above diagonal and all remaining entries in this matrix are zero.

Since, we are getting is that our coefficient matrix in this numerical method as sparse matrix, that is having many zeroes and we are finding it out, that we are placing those zeroes at different places and that is why we had used this Gauss-Seidel method rather than the gauss elimination method.

Now, the question is coming in this Laplace equation or in solving this Drichlet problem for Laplace and Poisson equation, when we are using this five point discretization formula for the different quotient; can we change our matrix coefficient matrix to the tridiagonal kind of thing, because what is happening is in the tridiagonal one.

If we can change our matrix to this form, and this form this Gauss elimination method can be used and it would be very simple since we have got that in iterative methods, we have to apply many iterations and then we have to come to the solution. So, in the gauss elimination method at one stage we could get the exact solution actually. Certainly if we are saying is that, if we could change it out of course we can change it out, but the method would be again iterative. So, the answer to this that can be change our coefficient matrix or can we formulate this Drichlet problem for Laplace equation in such a manner that...

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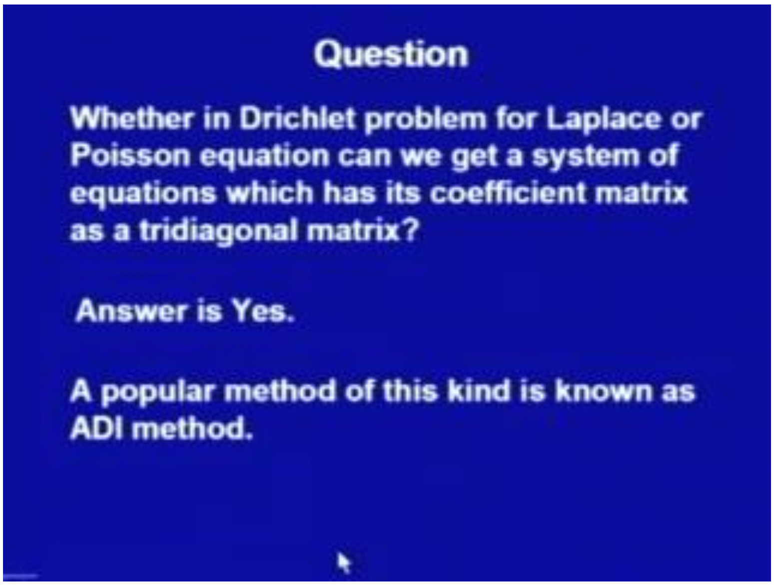

Coefficient matrix is a tridiagonal matrix, the answer is yes and a popular method of this kind is known as ADI method, now what is this ADI method?

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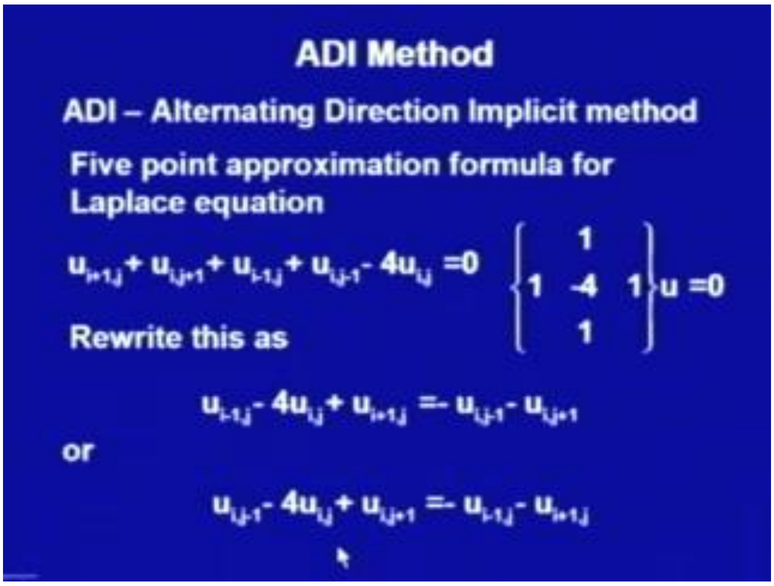

ADI method, it is name is actually Alternating Direction Implicit method, the name says is alternating direction implicit method that is, it is not very it is not explicit that is it would be using implicit means, that we have to use iteration, because both the way we would be getting the unknown ones.

So, the five point approximation formula for Laplace equation we do is, this 1 1 minus 4 1 1 is u is equal to 0, that is if I am talking about this point u i j, then it would be u i minus 1 j, this would be u i plus 1 j, this would be u i j minus 1 and this would be u i j plus 1. So, this was the formula if you do remember that is in this manner we were moving, that is starting from here u i plus 1 j, plus u i minus u i j minus j minus 1 j, u i minus 1 j minus 4 u i j and plus u i j plus 1 u is equal to 0, that is what the Laplace equation formula we have got it.

So, this is what we were having this discretization formula at the point u i j and similarly for every u i j, we do write this equation that this differents coefficient and then we do get the discretization of our Laplace equation. Now, can we change it now that is we want it to be the tridiagonal you see tridiagonal means is I want diagonal elements over here, then one element over here and one element over here. Of course, certainly as I said is we could do it. Let us see how we are doing this, now let us rewrite this as u i minus 1 j, minus 4 u i j, plus u i plus 1 j.

So, what we are having is u i minus 1 j, u i j and u i plus 1 j this is at the left hand side and the right side is that answer u i j minus 1 minus u i j plus 1. So, now you see we are getting this row or this j th row this fixed row j, is common j is fixed one, so this j th row on the left hand side and the right hand side we are having is that i th column. Of course we can rewrite in another manner, that is we can rewrite this as that i th column on the left and j th row on the right. So, like this one if I am writing it u i j minus 1, minus 4 u i j, plus u i j plus 1 is equal to minus u i minus 1 j, minus u i plus 1 j.

So, what we are rewriting this equation this five point formula into this form, where either I am writing it as the left hand side as the j th row and the right hand side that remaining i th column or the left hand side is the i th column and the right hand side is the remaining j th row. Now, in this what we are getting is, that is we are getting u i j's on both the sides while here it we were having is the right hand side is the 0 or if it was it is a Poisson equation we would be having some known terms over here. While here we are having both the sides u i j that is why we are calling it ((Refer Time: 07:42)) and you are seeing is, that is either i can write it as a j th row or i can write as i th column kind of thing. So, this we would be alternating the method.

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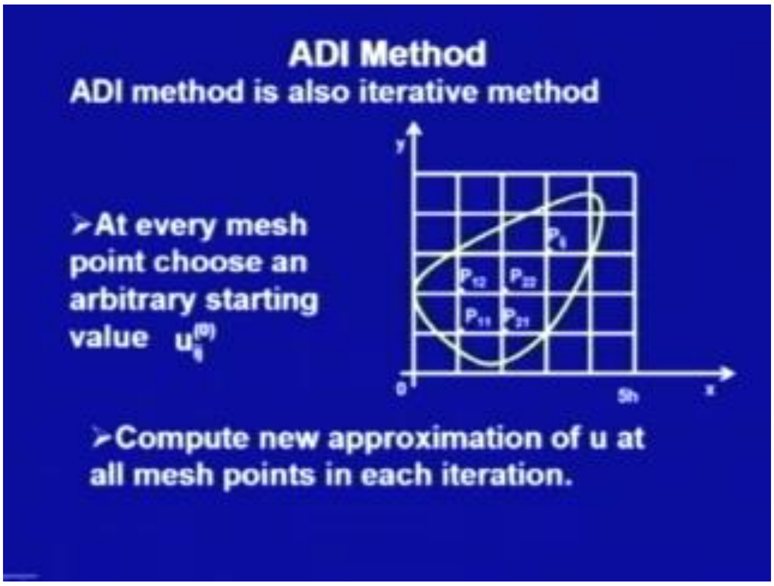

What is this method actually, method is also this iterative method what we do is given a Drichlet problem again we would discretize it, that is make the mesh and this get and get the mesh points and each mesh point we use that arbitrary starting value, that is because it is an iterative method we start with arbitrary starting value. So, u i j 0 you would take some values and then compute the new approximation of u at all mesh points in each iteration, that is again it is iterative at each iteration we would compute at each of this mesh points would calculate this new approximation.

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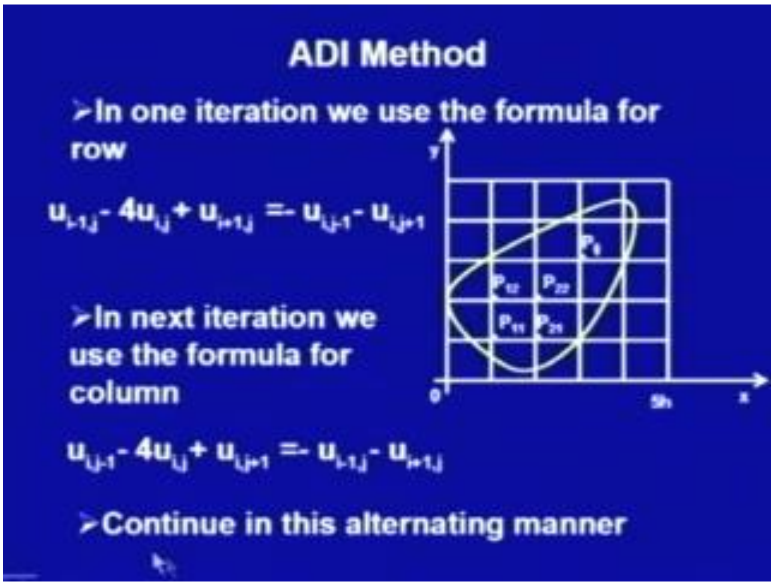

How we are approximating it. We would be doing this in the one iteration we use the formula for row, that is for fixed j th row we would use this formula and in the next iteration we will use the formula for that i th column, that is what we are altering the direction. Once, we are moving to the row and then we are moving to the column, so we are altering the direction how we are doing it.

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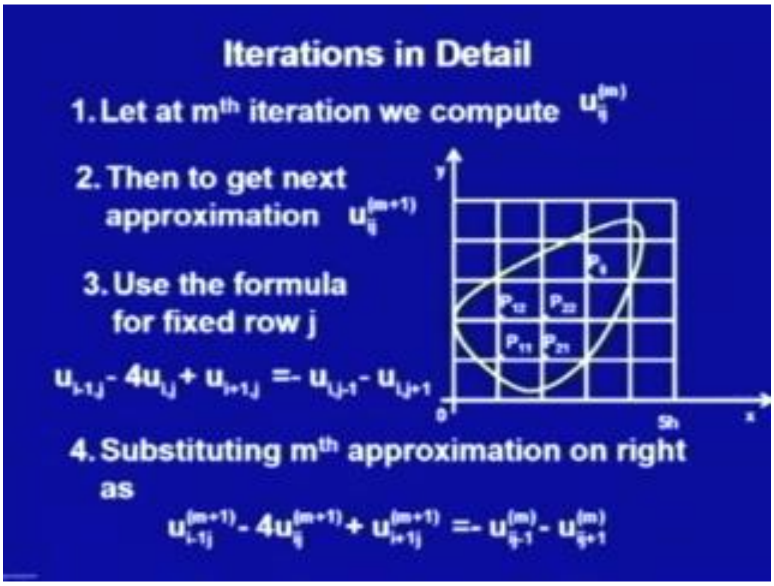

Let us say let at m th iteration we computed the approximation for i j th element as u i j m that is at i t h mesh point. We have calculated it as u i j m then what we will do, to get m plus first approximation, what we will do we will use the j th row formula that is for fixed row j we would use the formula. So, from here if you are moving we do says, that is we would get this formula, what we will do will substitute the m th approximation to the right side. And the left side we say is, that we would be getting the m plus 1 values, that is in this manner we would be doing substituting the m th approximation and write as this one you say.

We defined out u i j m plus 1 as u i minus 1 j m plus 1, minus 4 u i j m plus 1, plus u i plus 1 j m plus 1 is equal to minus u i j minus 1 m. Now, we have used this whatever we have got in the m th iteration that approximation that we would be substituting on the right hand side and u i j plus 1 m, this we will do for all for we are using for fixed row j. So, if i am using this fixed row j we would go for all internal mesh points, all the mesh points because at the boundary condition this is the we are talking about the Drichlet problem, that is the boundary values are been given that boundary condition will give me the values at the mesh points at the boundary.

So, those point we will not be talking, we will be talking about all internal points and we will use, when we are using this formulation certainly at certain points we would get the boundary points as well and for that boundary points we will use the boundary conditions.

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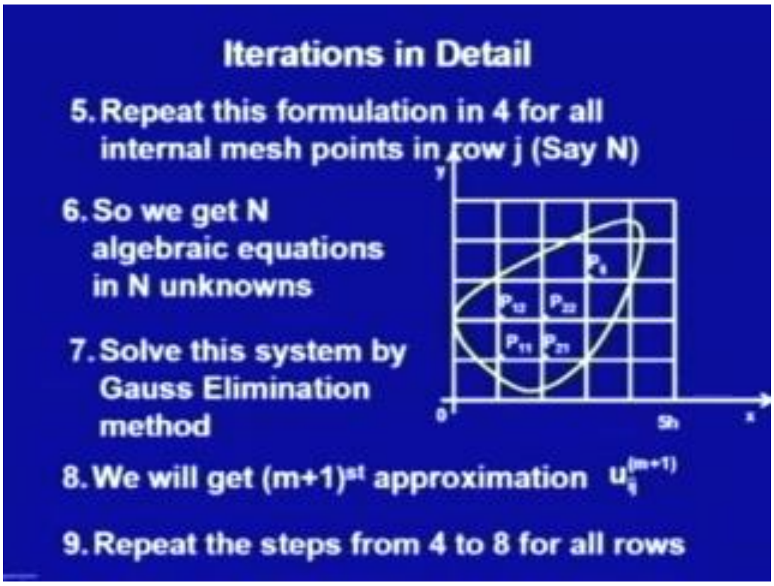

Thus what we would get, we will do it for each for all mesh points, internal mesh points for the row of j. So, let us say that is there are n points. In this manner when we use the boundary condition as well all this what we will get finally, will get N algebraic equation in N unknowns. We solve this system by gauss elimination method, this N algebraic equation N unknowns, which we will get they would be actually simple equations, that we could solve by gauss elimination method, so that we do get the solution exact solution at that time.

Now, in this manner we will go for all row j s that is I would go for starting from j is equal to 1 to j is equal to ((Refer Time:11:52)) in how many we this grid points, how many grids we have taken that is, how many rows we have taken after all those rows we would go. So, we will get m plus first approximation for all the internal points of the j th row and then repeat these steps, that is from the 4th step that is internal mesh points what we are getting is at approximation, we will repeat all those points for all rows. Thus we would get u i j m plus 1 for all u i j's inside this area of interest. Then, now we have got m plus first approximation in the next step what we do we alternate the direction that is now.

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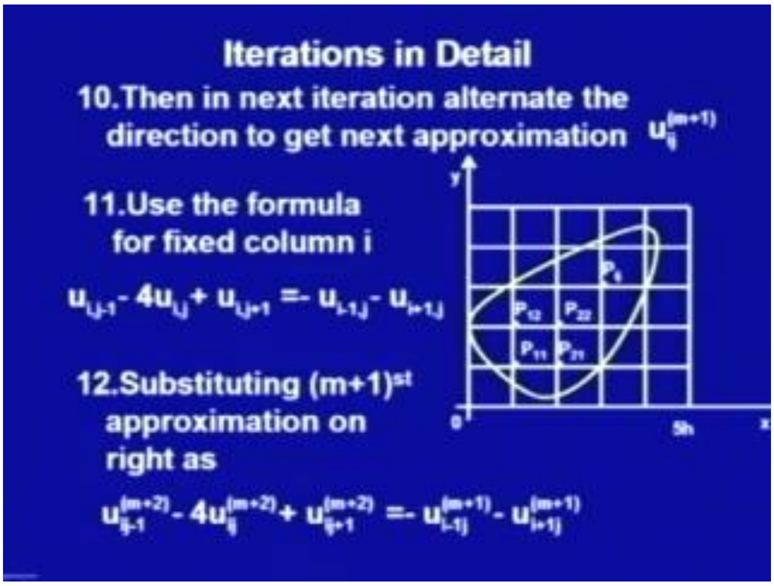

What we will go in the next iteration, we alternate the direction to get the next approximation u i j m plus 2. So, what we will use, we will use the formula for fixed column i, u i j minus 1 minus 4 u i j plus u i j plus 1 is equal to minus u i minus 1 j minus u i plus 1 j. What we will do again we substitute the m plus first approximate values, whatever we have got the values over in this last iteration in the right hand side. And thus we would get the equation as u i j minus 1 m plus minus 2 4 u i j m plus 2, plus u i j plus 1 m plus 2 is minus u i minus 1 j m plus 1, minus u i plus 1 j m plus 1. That is right hand side you had put the values at the m plus first approximate values and we are trying to find out the m plus second approximation for u i j 's Now, using this kind of one, here we the substitute those values and will get the equation over here, certainly with boundary condition and all those thing. So, what we would get.

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This we would do in all internal mesh points. So, let us say that is column fixed column i lets say there are m points, then what we would get we will get again that is m algebraic equations in m unknowns, we can again solve them using the gauss elimination because what equation here we would get that could be of this form of tridiagonal matrix actually. Because we would be at each mesh point if I am using this row, fixed row I would be getting only these 3 points, that is the point its left hand point and its right hand point. And if I am using the column one we would be using the we would be getting the point and its upper point and its lower point that is all these 3 points we would get.

So, like that if we are moving either in this manner or in this manner we would simply get a tridiagonal matrix and for that matrix we do get the solution using the gauss elimination method. So, again solve this system also by the gauss elimination method, I would get the approximation m plus second approximation u i j m plus 2 for all those internal points of the i th column.

Now, repeat this process of from 12 to 16th for all column. So, that I do get the approximation u i j m plus 2 for all. In this manner we would be go on repeating till we get a nice solution, So, let us what the nice solution we do mean is we do say is that when the two approximations are each other that is there is not much difference between the two approximations, that is say if I do have u i j m plus 1 and u i j m plus 2.

The absolute difference between u i j m plus 1 and u i j m plus 2, if this is less than some very small number which we have prefixed, then we say is that we can stop the iteration. In this manner we are going to find it out, let us do one example using this method, that is ADI method.

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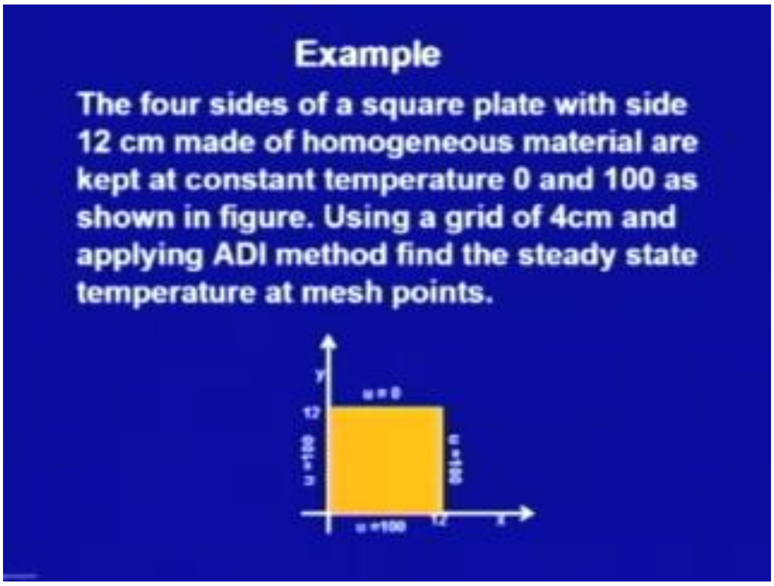

We are going to do the same example the 4 sides of a square plate with side 12 centimeter made of homogeneous material are kept at constant temperature 0 and 100 as shown in this figure. Using a grid of 4 centimeter and applying this ADI method alternating direction implicit method, find the steady state temperature at the mesh points. So, we are having this boundary condition is given that the temperature is being kept at the 3 sides at 100 degree centigrade and on the one side that is on the upper side is the 0 degree centigrade. We have to use this greater mesh point that is use this 4 centimeter, its total is 12 centimeter. So, we have to use this 3 cross 3 grid points. So, let us see what our problem is, you do remember that this is the same problem, which we have done using this Gauss-Seidel method.

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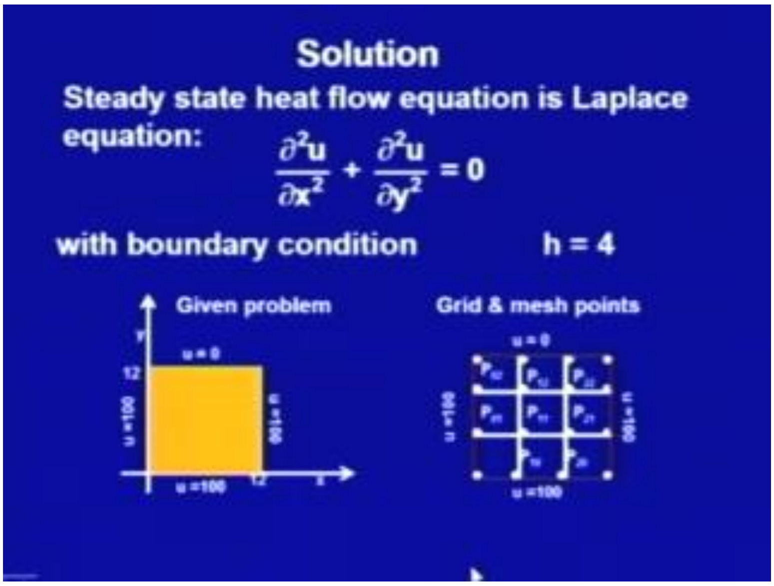

So, their steady state heat flow equation is the Laplace equation, that we do know del 2 u over del x 2 plus del 2 u over del y two is equal to 0. Now, with the boundary conditions is given as this, using this h is equal to 4 we have to find out the grid and mesh points. So, grid and mesh points will meet on this region of interest, that is whatever the region we have been given this is the square region.

So, given problem is this, so we do get it a 3 cross 3 grid points, so we say this is for each h as 4 from this side as well as from this side. All these mesh points of course, it is a 0 0 mesh points this is one 0 this is two 0, because x is increasing in this way and y is increasing in this way. So, we are taking x as the first one and y as the second one. So, we do get a here p 0 0 this is the point p 0 1, this is the point p 0 2 this point could be p 1 1 and so on and the boundaries we do not, that is at u 0 0 it is 100, u 1 0 would be 100, u 2 0 would be 100 and so on.

In this side we do have is that is u 1 3 is a 0, u 2 3 is 0 and say u 3 3 is 0, so like this we do have this one, now we have to formulate our problem that is we have to find the total different, that is apply this alternating direction method.

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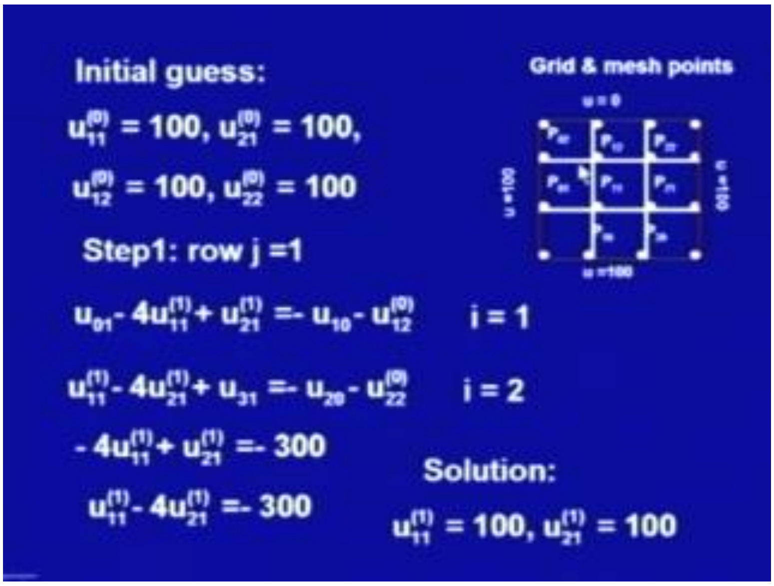

So, start from the step one, we would have to have a initial guess. So, let us say the initial guess again as a in the last one, we had the 3 sides of the boundary condition is 100, so again we are taking this initial guess as 100. Since we are having in this one this 4 internal points, all these other points are on the boundary and for the boundary, the boundary condition is give, so we do know that u 0 0 has to be 100 here, u 1 0 has to be 100 here. Similarly, u 0 3 has to be 0 u 1 3 has to be 0 and so on.

So, we do meet at that is the 4 internal points only, so u 1 1, u 2 1, u 1 2 and u 2 2 that is u 1 1, u 2 1, u 1 2 and u 2 2. These 4 points we are taking and so we would actually two rows and two columns, that is the points and each row and each column will have only two and two, this is the problem we have taken. So, that we can solve it here by hand without using any calculator or any calculating machines.

So, this is a initial guess we have taken as 100, now the first step is that is use the fixed row, so first I am using the j row j that is equal to on the first row, the first row the first point is p 1 1, that is point 1 1, here we would like to write the equation. So, we would get u 1 1, so u 0 1 do you remember that is u i minus 1 j minus 4 u i j plus u i plus 1 j. So, my j is fixed as 1 and i is varying from 1 to, so i is also 1, i is also 1. So, Ii would get u 0 1 minus 4 u 1 1 at 1 plus u 2 1 at 1 is equal to minus u 1 j minus u 1 2 0.

Now, you see it here in the screen when you have described you had used all the places m plus 1 and all the places as m, here at certain places I am not using this m or that is 0 or 1 why, let us see here this is the point u 1 1, it is before point that is we are talking about this fixed row, so this 3 points we have to talk about.

This is the point u 0 1, u 0 1 is the point at the boundary that is it is known boundary condition, this is not an internal point, so we are not using any initial is this we would use exactly what is the boundary condition, so that is why this one we are not using. Similarly, when I go to the right hand side this point and this point we have to consider, for this point of course, we are taken the initial guess for this point we have to use the boundary condition, that is why this is not been used.

Similarly, if I go for the second point that is i is equal to 2, u 2 1, so I do get it from here u 1 1 see this is the internal point, so 1 minus 4 u 2 1 this is also internal point. Plus u 3 1 this is boundary condition boundary point, so we are taking it u 3 1 as such, minus u 2 0 this point, this is also a boundary point minus u 2 2 this is internal point, so initial guess is 0. Now, this is for second i is equal to 2. Now, substitute the values, since we are taking this u 1 2 0, u 2 2 0 these are 100, u 1 2 0 and u 2 2 0 they are 100.

The boundary condition u 0 1, u 0 1 means this point u 0 1 means 100, then the other point is u 1 0 this one, this is also 100 and here the boundary point is u 3 1, u 3 1 means this point this is also 100 and then u 2 0 this is the point this is also 100. So, let us go one by one, the first 1 minus 4 u 1 1 plus u 2 1 at one is equal to minus 100 minus 100 and this point also taken to this side minus 100, so I have got minus 300.

Similarly, here u 1 1 minus 4 u 2 1 is equal to this is minus 100, this is minus 100, this point is also u 3 1 is this point this is also 100. So, this side we are taking it as minus 300. So, we have got this equation, now minus 4 u 1 1 plus u 2 1 1 minus 300 upon u u 1 1 minus 4 u 2 1 is equal to minus 300. This is because we are having only two internal points here, so we do get system of two equations in two ((Refer Time: 23:12)).

The simple one as I said is you are doing it without any computing help, so we are taking it by small example we can solve it using simple method of gauss elimination, that says is that the solution I would get in this manner. The first approximation u 1 1 at one is 100 and similarly u 2 1 at one is 100, you can solve it by yourself and see it its very simple equation, so we are getting it as a 100.

Now move to the row two, now we have started that is this process has to be repeated for all rows, so the row one I have taken, now move to the row two.

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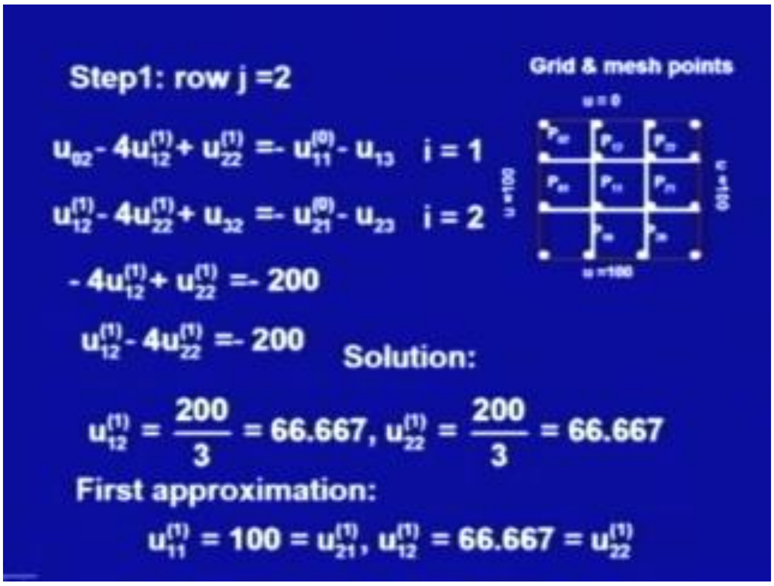

Row two if I write this equation, so the first point that is i is equal to 1 if I have to take the point is p one two, that is we have to take u 1, so that is here we would get it u 0 minus 2 4 u 1 2 plus u 2 2 is equal to minus u 1 1 minus u 1 3. Now, you see u 0 two that is boundary point, so we are not using any approximation and u 1 3 is also boundary point. So, we are not using any point over there also and u 1 2 we want the first approximation for u 1 2 in the right hand side u 1 1 we will use it initial approximation.

Certainly here we are not going to use any other thing, so we have got ((Refer Time: 24:41)). Now, similarly if I go for i is equal to 2, that means, this point u 2 2 point p 2 2 point, so for that what we would get u 1 2 at 1 minus 4 times u 2 2 at 1, plus u 3 two this is the boundary point is equal to minus u 2 1 this is the internal point so 0, minus u 2 3 this is a boundary point this is for i is equal to 2.

Now, put the values for the first one u 0 2, u 0 2 means this point this is 100. u 1 3, u 1 3 means is this is the point which is 0, so we are getting it 100 this is 0 and this is also 100, so we do get is minus 4 u 1 2 1 plus u 2 2 one is equal to minus 200. Similarly, in this 1 u 3 we would will be getting it the point u 3 2, u 3 2 point means this is the point, yes u 3 2 is this point that is 100 and then u we would be getting it as the point u 2 3, u 2 3 is this point which is 0.

So, again we would be getting is u 1 1, u 2 at 1 minus 4 u 2 2 at 1 is equal to minus 200, again we have got because we are having two internal points only, the system of two equations in two unknown again simple one. The solution using the gauss elimination method we do get is as u 1 2 as 200 by 3 and u 2 2 also as 200 by 3. So, we have got now the first approximation u 1 1 as 100 u 2 1 as 100, u 1 2 as 66.667, u 2 2 at one is 66.667 this is my first approximation. Now, we will move to the next step, so move to the second step.

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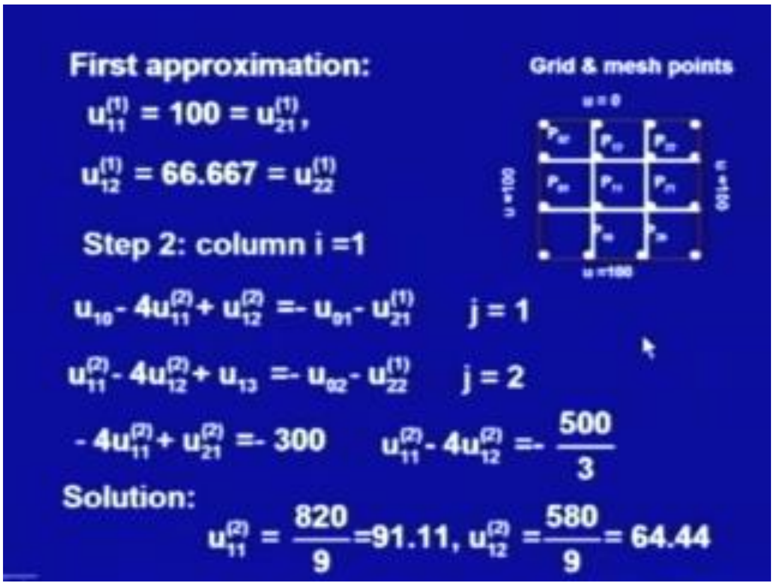

So, first approximation is this one, from here we would move to the second step, first the second step again what we have to do, we have to alternate the direction. Since we have done in the first step using the row, now we will move to the using the columns. So, first I will take the column i is equal to 1, in this column I do have two internal points and two boundary points. So, I would start with the first point that is j is equal to 1. Here if I write because we are going to write it using the columns, so we start from here u 1 0 minus 4 u 1 1 plus u 1 2 is equal to minus u 0 1 minus u 2 1.

Now, the point u 1 0 and u 0 1 are the boundary points and the right hand side we do have u 2 1 the point this point this is internal points, so the first approximation and we want the second approximations to the left hand side, we will take all those points as the internal points as the second approximation, so this is my first point.

Now, similarly move for the second point in this column that is, this u 1 2 that is p 1 2 point, so for this we have to write using the column one, so we will go for u 1 1 minus 4 u 1 2 plus u 1 3 minus u 0 2 minus 2 u 2 2. Again these two points are the boundary points, so those points we are writing as such and this right hand side u 2 2 this will take the first approximation and others we would write as the second approximation. So, this is for j is equal to 2.

This one, now let us just write the points, since we are getting is here u 1 0, u 1 0 means this point, this points says is 100 then u 0 1, u 0 1 means is this point, this is also 100 and u 2 1 and initial one that is first approximation we had got it as 100, so what we have got again as minus 300. Similarly, if I move here u 1 3, u 1 3 means is we would be getting this point this is 0, and then we do half it our point here u 0 2, u 0 2 means this point this point is 100 and the initial approximation u 2 2 one is 66 point this one, so what we do get is, that is I am writing it as 200 by 3 actually.

So, u 1 1 minus 4 u 1 2 is equal to minus 500 by 3 that is minus 200 and minus 200 by 100 minus 200 by 3. Now, if I solve this system approximation what we have got two internal two points only on the column i this first column, so we have got two equations in 200 ((Refer Time 30:05)). It is simple system we can solve it using again the simple gauss elimination method, we do get is u 1 1 as 820 by 9 or 91.11 and u 1 2 as 580 by 9 is equal to 64.44. This is what we have done for the first column. Now, move to this know we have to repeat this process for remaining columns, so the next column is second column.

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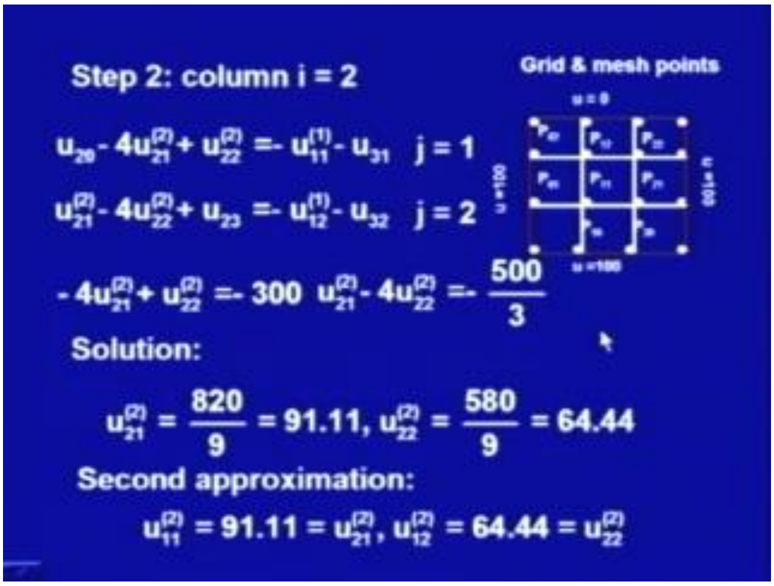

Now, for column two, now for this again column two we do have two internal points first will move for i is equal to j is equal to 1 that is, the first point the point p 2 1, that is corresponding to u 2 1, we are moving in the direction and of the column. So, will take u 2 0 minus 4 u 2 1 plus u 2 2 is equal to minus u 1 1 minus u 3, u 3 1 and u 2 0 these are your boundary points, so we are writing them as such and the left hand side the second approximation on the right hand side where u 1 1 the point, so this is the first approximation. This is for u j is equal to 1 for j is equal to 2 the point is p 2 2 that is why u 2 2, for this again we are using this column method. So, we would get u 2 1 minus 4 u 2 2 plus u 2 3 is equal to minus u 1 2 minus 2 u 3 2.

Now, again u 3 2 and your u 2 3 they are the boundary points, so they are being kept as the right hand side u 1 2 this is the first approximation left hand side the internal points u 2 1 and u 2 2 we want second approximation. Now, substitute the values u 2 0, u 2 0 is this boundary points which is 100, u 3 1, u 3 1 is this boundary points which is also 100, so it is minus 200, u 1 1 we at the first state we have already got as minus 100.

So, the first equation would give me minus u 2 1 plus u 2 2 is equal to minus 300 then u 2 3, u 2 3 is this point which is 0 and u 3 2 is 100, so minus 100, u 1 2 at one we had already got minus 200 by 3, so minus 200 by 3 minus 100 we would get it minus 500 by 3. So, my second equation in gives me u 2 1 minus 4 u 2 2 is equal to minus 500 by 3, since there are two internal points only to internal mesh points we will get two equation in two unknown. So, this first equation this second equation two unknowns.

Now, simplify it the solution we would get the simplification you can do yourself, so this would be u 2 1 we are getting is 820 by 9 as or as 91.11, u 2 2 we are getting as 580 by 9 or 64.44. So, what we have got second approximation now that is for all four internal points we have got the approximations, the second approximation we have got u 1 1 as well as u 2 1 is 91.11 and u 1 2 as well as u 2 2 is your 64.4 4. So, we are getting is that, we are moving in the row wise manner actually we are getting the same ones.

Now, we are getting is that is they are internal points, I am not going beyond this second approximation this is just to explain with the typical step, so I had started with the initial guess and gone with the first step, that is the fixed row step and then alternating the direction then to the columns step. Now, in this manner you can move on doing it till you do get that approximation stabilized.

Now, here in this, when we are doing this process this approximation are depending upon what the initial guess we are starting and of certainly about that method, what this method is saying is the convergence of these methods, we are not discussing in this first course actually. Now, if I start with any other initial guess how does it affects. Let’s say the same example, here we had started with initial guess as 100, 100 and 100 for all four points now if I says, I started initial guess without knowing anything let us say that initially all the points are 0, then what will happen. So, let us you this same example with another initial guess.

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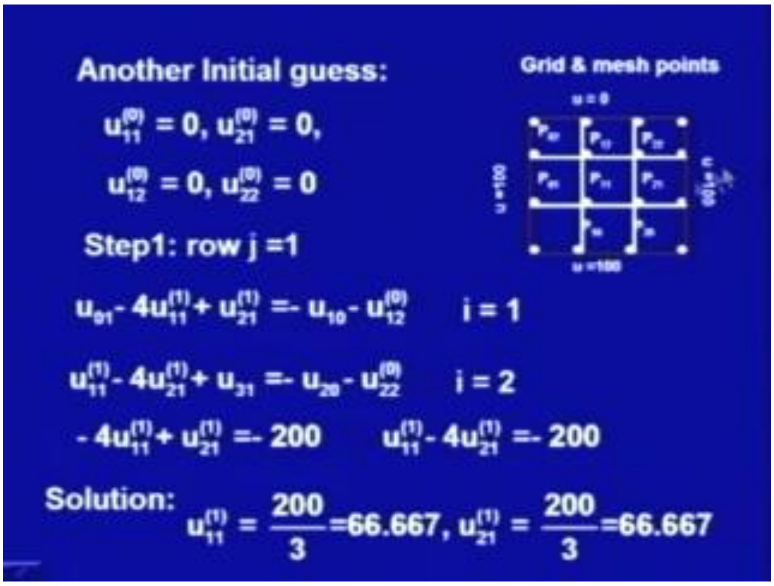

As u 1 1 is 0, u 2 1 is 0, u 1 2 is 0 1, u 2 2 is 0. So, what this is my initial guess first I will go with the fixed row, that is again I would like to go with the row one, row one means is I would have two internal points i is equal to 1 that u 1 1 and i is equal to two that u to 1, so first for u 1 1, I am going with the fixed row, so for i is equal to 1, I would get u 0 1 minus 4 u 1 1 plus u 2 1 is equal to minus u 1 0 minus u 1 2.

That is again using the similar manner of this one for the last as we have just note down, so for this is for i is equal to 1 and for u 2 1, I would get it u 1 1 minus 4 u 2 1 plus u 3 1 is equal to minus u 2 0 minus u 2 2 0. Now, we have the boundary conditions are same, so here also we do get is 100 and here also 100 that is my u 0 1 and u 1 0 both are 100, but now my u 1 2 0 this is we have take initial guess as 0. So, my equation now would be first equation would be minus 4 u 1 1 plus u 2 1 is equal to minus 200.

Similarly, if I say in the second equation my boundary points u 3 1 and u 2 0 both are 100, but the initial guess u 2 2 is 0, so again I would get the equation as u 1 1 minus 4 u 2 1 is equal to minus 200. Now, solve these two again we are getting is two internal points, so two equation is ((Refer Time: 36:46)) unknown, the solution would be u 1 1 is 200 by 3, u 2 1 is also 200 by 3. So, we have got the different solution now and now move to the second row.

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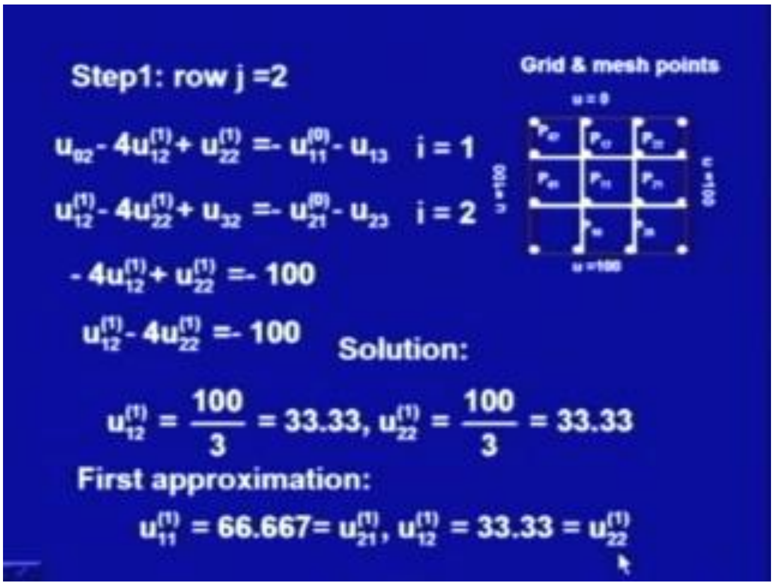

Second row means the points I would have u 1 2 and u 2 2, so first part u 1 2 we are moving in the row manner, so for j is equal to 1, i is equal to 1 we do get it u 0 2 minus 4 u 1 2 plus u 2 2 is equal to minus u 1 1 minus u 1 3. So, these two points are our boundary points and remaining right hand side, this point we are having is u 1 1 that is initial guess is 0. u 0 2, u 0 2 means is your this point which is 100 and u 1 3 this point which is 0. So, you are getting is minus 100 and this internal, this initial guess u 1 1 was 0, so what I would get the equation would be actually right hand side would be 100.

But, let us first write with the second equation as such that is for the point u 2 2. So, for the point u 2 2 you would get u 1 2 minus 4 u 2 2 plus u 3 2 is equal to minus u 2 1 minus u 2 3, u 2 3 and u 3 2 they are boundary points one is 0 another is 100 and initial guess u 2 1 is 0, so for here for the first equations since our initial guess is 0, u 0 2 is 100 and u 1 3 is 0, I do get the first equation is minus 4 u 1 2 plus u 2 2 is equal to minus 100.

Similarly, from the second equation again I would get u 1 2 minus 4 u 2 2 is equal to minus 100. So, again we have got because the two equations, two unknown and two equations that is because we are having only two internal points, solve this system of equations we do get u 1 2 as 100 by 3 and u 2 2 of also as 100 by 3. So, finally, what we have got our first approximation in this case as u 1 1 is same as u 2 1 is 66.667 and u 1 2 and u 2 2 as the first approximation as 33.33. Now, move to the, now alternate the direction that is now move to the column one using this as the now first approximation.

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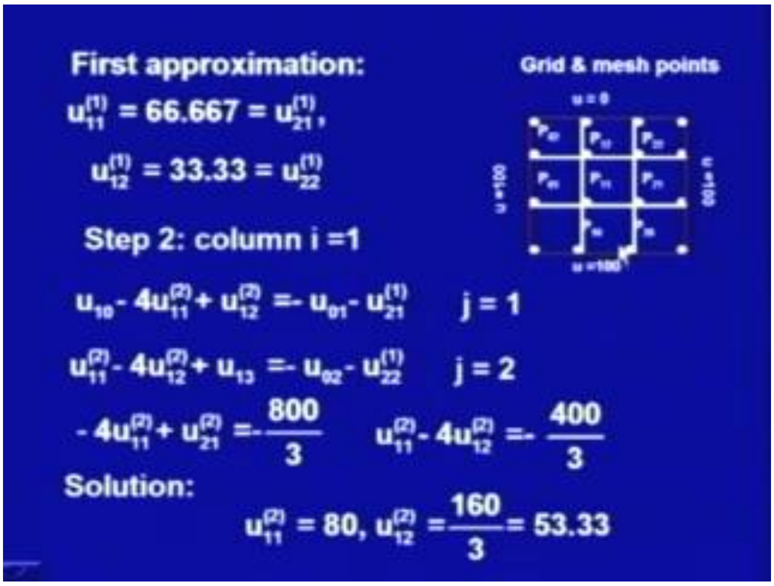

So, we will move to the second approximation, the first approximation is u 1 1 and u 2 1 as 66.667 at 200 by 3, u 1 2 and u 2 2 as 100 by 3. Move to the first column because will now move to the column first fixed column, the first column we do have two internal point, so again we will start with the first point j is equal to 1, what we would get is u 1 0 minus 4 u 1 1 plus u 1 2, this is the points is equal to minus u 0 1 minus u 2 1. So, this is the point you are getting over here, so right hand side you would write the first guess and second and the left hand side you want the second approximation.

The boundary points u 1 0 and u 0 1, u 1 0 and u 0 1 both are 100. Now, first approximation we have got u 2 1 as a 200 by 3, so we are getting is minus 200 minus 200 by 3, so this is for j is equal to 1 now will move to the j is equal to 2, that is u 1 2 point what we would get, we would get u 1 1 minus 4 u 1 2 plus u 1 3 is equal to minus u 0 2 and minus u 2 2 again u 0 2 and u 1 3 this is u 0 2 u 1 3 they are the boundary points. So, we are noting keeping anything, u 2 2 is the internal point for this the first approximation and the left hand side we want the second approximation.

U 2 1, now come to this our first equations u 1 0 and u 0 1, u 1 0 and u 0 1 both are 100. So, you would get it here this side minus 200, minus u 2 1 at first approximation that is minus that is 200 by 3, so minus 200 minus 200 by 3 what you would get is minus 4 u 1 1 plus u 2 1 is equal to minus 800 by 3. Similarly, here u 1 3, u 1 3 is 0 and your u 0 2 is 100, so will get minus 100 and u 2 2 at one is 100 by 3.

So, what you would get the second one is u 1 1 minus 4 u 1 2 is minus 100 minus 100 by 3 that is minus 400 by 3, again because of two internal points only we will get a system of two equations in two unknowns, solve it and we would get the solution as u 1 1 as 80 and u 1 2 as 160 by 3 or 53.33. Now, move to the second column...

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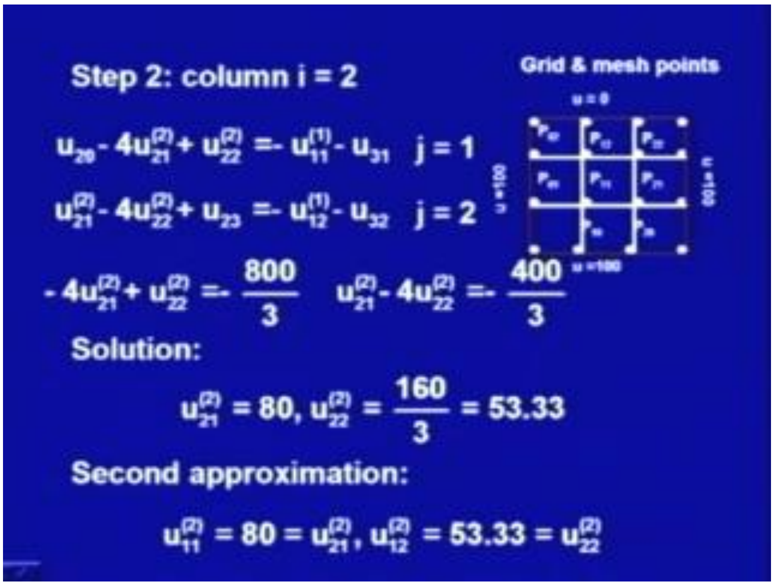

We do again have two internals points u 2 1 and u 2 2, so first start with first internal point, we are moving in the column manner, so we get u 2 0 minus 4 u 2 1 plus u 2 2 is equal to minus u 1 1 minus u 3 1. Now, u 3 1 and u 2 0 these are your boundary points and u 1 1 on the right hand side this is from the first approximation.

Similarly, if I move to the second point u 2 2 what I would get because we are moving in the column direction, it should be u 2 1 minus 4 u 2 2 plus u 2 3 is equal to minus u 1 2 minus u 3 2, u 3 2 and u 2 3 they are boundary points. Moreover this right hand side u 1 2 is the point that is for the first approximation and here we would get on the left hand side the second approximation.

Now, if I substitute it boundary points here in the first one are giving you u 2 0 and u 3 1, u 2 0 means this point and u 3 1 is this point both are 100, so we are getting is minus 200 and the first approximation u 1 1 at one is 200 by 3, so what we would be getting is minus 4 u 2 1 plus u 2 2 is equal to minus 200 minus 200 by 3 that is minus 800 by 3. Similarly, if I move to u 2 3 and u 3 2, u 2 3 would be the point which is 100 and u 3 2 is the point which is 0, so you would get is minus 100 and the first approximation for u 2 1 is 100 by 3, so we would get the second equation as u 2 1 minus 4 u 2 2 is equal to minus 400 by 3.

We have got two points, so two equations in two unknowns solution we will get u 2 1 as 80 and u 2 2 as 53.33. So, what we have got the second approximation, second approximation we have got as u 1 1 and u 2 1 both as 80 and u 1 2 and u 2 2 both as 53. So, we had got that is, now the same example we have solve with two methods.

First we have done in the last lecture as the Gauss-Seidel methods, then we have a solved it using this ADI method with the two initial guesses, one guess we had started as 100 and 100 another guess we had started with 0 and 0. And moreover because this was a simple problem we could actually make it as a very simple equations, so we had also find out after this discretization, the gauss elimination method, the exact solution of those system of equations, which initially we had got before applying this one our Gauss-Seidel one. So, let us see compare all these methods what the answers we have got.

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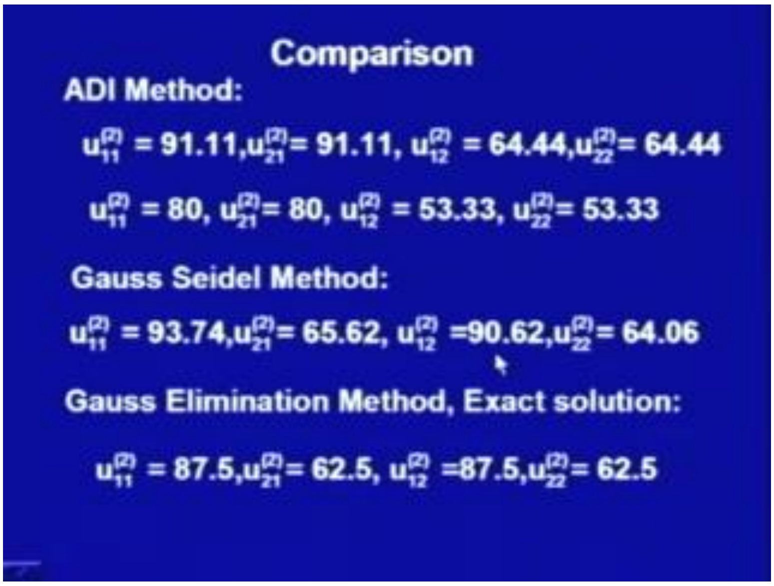

With the ADI method in the second approximations using initial guess as 100, we have got u 1 1 and u 2 1 both as 91.11, u 1 2 and u 2 2 both are 64.44. When I have changed my initial guess from 100 to 0, we had reached to this u 1 1 and u 2 1 as 80, u 1 2 and u 2 2 as 53.33. In the Gauss-Seidel method in the last lecture, we have got the solutions u 1 1 as your 93.74, u 2 1 as 65.62, u 1 2 as 90.62 and u 2 2 as 64.06. While is the exact solution, this was starting with initial guess as 100,100,100 for all.

If, do remember that is we have done is gauss elimination method that the exact solution of the system of equations after the changing our partial derivatives to the difference quotient. We have got say algebraic equations for the four unknowns and four equation, that we had solved and we got that is the solution was u 1 1, u 2 1 was 87.5, u 2 1 was your 62.5, u 1 2 was 87.5 and u 2 2 was 62. 5.

Now, you see them of course, the solutions are coming as different here in ADI method we are getting is in one solutions we are getting in the row manner, they are equal for the first row and they are equal for the second row. In the Gauss-Seidel method we are getting as they are coming as a the column manner or the first column the points we are getting it rather they are not equal they are all the four points are different while as in the exact solution, we have that in the column manner that is u 1 1 and u 1 2 are same and u 2 1 and u 2 2 are same.

Of course, that such a smaller problem and with only two steps and is not easy to check about the convergence in more general, but still we do have an idea that these methods, iterative methods they would depend upon what is initial guess. So, we have to make a very cautious initial guess at each point. Moreover the problem is that my solution is moving depending on the methods also.

The Gauss-Seidel method we are getting different way in the ADI method we are getting different manner. So, we have to chose this method also very cautiously, that is in which problem we are talking about here only actually one problem that is the Drichlet problem for the Laplace equation we have talked about. We had find it out that is difference method are giving our different approximations, if I solve this same problem you can do this as an exercise.

Usual method that is we have done it the Drichlet problem for the Laplace equation using the four way method. There of course you will get the Fourier series and that Fourier series again to find out this solution, that is approximate that at each point that is your temperature at each mesh point if you have to do. That is you have to find out rather at the internal point that is Fourier series you have to approximate or you have to find out the sum of the series. So, that there also the approximation, but the Fourier series you can use any of those known software's or any of those series.

So, you can find out that approximate value over there and then from there, you can compare these approximation and see because this is a very simple examples, so you can do this exercise by yourself.

See, which of these three solutions is more near to the Fourier approximation thus Fourier approximation, first we have got using the analytically and then solution, final solution we had approximated. Here what we are doing is we are solving the problem numerically, that is we are actually modeling the problem rather than analytical solution, we are finding out the numerical solutions.

So, see which one is better one or which one is approximating with better level at the second approximation itself, you think go ahead with the exercises, you can use your calculator or computers you can make the program and can go on checking it, that is there they are converging at convergent state whether they are giving you same solution or similar kind of solution with less actual solution, or what you could check it, that is the gauss elimination method is that exact solution which I had over here.

This is a little bit go near to the Fourier series of approximation than this one, so these this can move if I can this can also converge to this kind of solutions or to the Fourier solutions. If I had gone with the more iterations, so that process you can do by yourself because their step is very easy you can just go on using the calculators or if you are little bit more interested you can make a computer program for these methods, because they very simple solving algebraic equations.

And you can check, that is where they are converging how much time they are taking and which method you would prefer all these things you can do at the first stage, that is just to get understanding. Certainly there are many people who have worked on these ones and they had suggested in which kind of equations which kind of method is more suitable, with what kind of boundary conditions, which kind of methods more suitable.

So, we are lot of numerical method available in the text books as well as you can find it out more further proper solution of any equation. This method, numerical method I had discussed only for the Laplace equation, I have not gone for the Poisson equation. The Poisson equation that difference would be that is the right hand side, when I was getting it say if I am our talking about this first equation which we have got that is Four equations and Four unknowns, we have got on the right hand side all the things 0.

Right hand side is 0, there what you would get is h square f x y, that is at all those mesh points x y points means, that is whatever the mesh point that you are getting is. So, if is this same problem I am talking about is the first mesh point is our u 1 1 that is at the point 4 4, that is x was 4 and y was 4. So, we find out whatever the function f x y is given at x is equal to 4 and y is equal to 4, what will be the value of that function f x y. Suppose f x y is the x square, then the value of the f x y at x is equal to 4 and y is equal to 4 would be simply 4 square that is your 60. So, rather than having on the right hand side as 0 in the first equation, we would get 60 into h square. Now h was your also for, so you would get it 16 into16 that is 256.

So, rather than getting the right hand side as 0, you would get it the right hand side as some numbers and then using the boundary conditions, since you have got all the right hand side is negatives the minus 200, minus 200, if do you remember your gauss elimination this example minus 200, minus 200 and minus 100, minus 100, you would get it actually 256 minus 200 that is you would get 56 and. So, on. So, that is you would be getting some right hand side different column.

The method would remain same, either you have to use and the simple one, here if you do you can find out the gauss elimination method or Gauss-Seidel or ADI. Certainly gauss elimination directly we are not using, because it takes too much time you can check it with yourself. Also that this is gauss elimination actually what it does, it does is that it changes your non 0 entries to 0 entries in a we could say is in a fixed manner or in a fixed manner that is in the first going in the first column, then the second column like that one and then you are getting a solution.

Since, our matrix coefficient matrix comes out to this sparse, what it is doing is it is never guaranteeing that is in a 0 entry it will not make non 0, this is never been guaranteed in gauss elimination. So, what it will do is it will do again 0 entries to non 0 and non 0 and then say again you have to make it 0 entry. So, it would take too much time, so that is why this is not be preferred. Of course it get the exact solution for the coefficient matrix for that our algebraic equations.

So, we are going to use this any of this methods taking less of a your computation time, but here, you have to be very cautious about or very judicious what is initial guess must be and which method you should use in which kind of boundary condition and in which kind of equations.

So, we had learned about this numerical method to solve this partial differential equations, that is all for today's lecture and we would be ending up here our this course for the differential equations on under graduate one. We have learned in the partial differential equations, linear equations, the first order equations, the second order equations and the second order equations.

We have learned about certain practical problems, that is the heat equation and wave equation, we solve them using the analytical methods, say is the Fourier methods and then we had done some insight into the numerical methods. So, this is what the partial differential equations, in ordinary differential equations. Certainly we had learnt many analytical methods for different kind of differential equations.

So, thank you.